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<th>GENERALIZATION OF YOUNG DIAGRAMS AND HOOK FORMULA (Algebraic Combinatorics related to Young diagram and statistical physics)</th>
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京都大学
GENERALIZATION OF YOUNG DIAGRAMS AND HOOK FORMULA

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1. Preliminaries

First, we give several notations for root systes. We always fix a root datum \((\mathcal{A}, h, h^*, \Pi, \Pi^\vee)\):

\[ A = (a_{i,j})_{i,j \in I} : \text{a generalized Cartan matrix}. \]

\[ h : \mathbb{R} - \text{vector space}, \]

\[ h^* : \text{the dual space of } h, \]

\[ \langle , \rangle : h^* \times h \to \mathbb{R} : \text{the canonical bilinear form}. \]

\[ \Pi := \{ \alpha_i | i \in I \} \subset h^* : \text{linearly independent subset} \]

\[ \Pi^\vee := \{ \alpha_i^\vee | i \in I \} \subset h : \text{linearly independent subset such that } \langle \alpha_j, \alpha_i^\vee \rangle = a_{i,j}. \]

For each \(i \in I\), we define the simple reflection \(s_i \in \text{GL}(h^*)\) by:

\[ s_i : \lambda \mapsto \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_j, \lambda \in h^*. \]

equivalently, \( s_i : h \mapsto h - \langle \alpha_j, h \rangle \alpha_i^\vee, \ h \in h. \)

\[ W := \langle s_i | i \in I \rangle : \text{the Weyl group} \]

We define a (real) root system and a (real) coroot system:

\[ \Phi := W \Pi \left( \bigoplus_{i \in I} \mathbb{Z} \alpha_i \right) : \text{(real) root system} \]

\[ \Phi_+ := \Phi \cap \bigoplus_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i : \text{(real) positive root system} \]

\[ \Phi_- := \Phi \cap \bigoplus_{i \in I} \mathbb{Z}_{\leq 0} \alpha_i : \text{(real) negative root system} \]

\[ \Phi = \Phi_+ \coprod \Phi_- : \text{(disjoint union)} \]

\[ \Phi^\vee := W \Pi^\vee \left( \bigoplus_{i \in I} \mathbb{Z} \alpha_i^\vee \right) : \text{(real) coroot system} \]

\[ \Phi^\vee_+ := \Phi^\vee \cap \bigoplus_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i^\vee : \text{(real) positive coroot system} \]

\[ \Phi^\vee_- := \Phi^\vee \cap \bigoplus_{i \in I} \mathbb{Z}_{\leq 0} \alpha_i^\vee : \text{(real) negative coroot system} \]

\[ \Phi^\vee = \Phi^\vee_+ \coprod \Phi^\vee_- : \text{(disjoint union)} \]

For a real root \(\beta = w(\alpha_i) \in \Phi\), we define the dual coroot \(\beta^\vee \in \Phi^\vee\) of \(\beta\) by:

\[ \beta^\vee = w(\alpha_i^\vee). \]

Remark 1. This is independent from the choice of \(w \in W\) and \(\alpha_i \in \Pi\).
The map $\phi: \beta \mapsto \beta^\vee \in \Phi^\vee$ is a bijection.

For each $\beta \in \Phi$, we define the reflection $s_\beta \in W$ by:

\[ s_\beta(\lambda) = \lambda - \langle \lambda, \beta^\vee \rangle \beta, \quad \lambda \in \mathfrak{h}^*, \]
\[ s_\beta(h) = h - \langle \beta, h \rangle \beta^\vee, \quad h \in \mathfrak{h}. \]

**Definition 1.** Let $w \in W$. We define the inversion set $\Phi(w)$ of $w$ by:

\[ \Phi(w) := \{ \gamma \in \Phi_+ | w^{-1}(\gamma) < 0 \}. \]

**Definition 2.** Let $w \in W$. We denote by $\text{Red}(w)$ the set of reduced decompositions of $w$:

\[ \text{Red}(w) := \{ s_{i_1} s_{i_2} \cdots s_{i_d} | \text{reduced decompositions of } w \}. \]

**Definition 3.** An element $\lambda \in \mathfrak{h}^*$ is said to be an integral weight if:

\[ \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}, \quad i \in I. \]

The set of integral weights is denoted by $P$.

**Definition 4.** An integral weight $\lambda \in P$ is said to be dominant if:

\[ \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geq 0} = \mathbb{N}, \quad i \in I. \]

The set of dominant integral weights is denoted by $P_{\geq 0}$.

2. **MINUSCULE ELEMENTS AND PETERSON-PROCTOR HOOK FORMULA**

**Definition 5** (Peterson (see [1])). Let $\Lambda \in P_{\geq 0}$. An element $w \in W$ is said to be $\Lambda$-minuscule if there exists a reduced decomposition $s_{i_1} s_{i_2} \cdots s_{i_d} \in \text{Red}(w)$ of $w$ such that

\[ \langle s_{i_{k+1}} \cdots s_{i_d}(\Lambda), \alpha_i^\vee \rangle = 1, \quad k = 1, 2, \ldots, d. \]

**Remark 2.** This definition is independent from the choice of reduced decompositions of $w$.

**Example 1.** A Grassmannian permutation is a $\Lambda$-minuscule element in the Weyl group of type $A$ (symmetric group).

**Theorem 2.1** (Proctor (see e.g. [7])). Suppose that the underlying generalized Cartan matrix is simply-laced. Then there exists a one-to-one correspondence between $\{(\Lambda, w)\}$ and $d$-complete posets.

**Theorem 2.2** (Peterson-Proctor (see [1])). Let $\Lambda \in P_{\geq 0}$ and $w \in W$ a $\Lambda$-minuscule element. Then we have:

\[ \#\text{Red}(w) = \frac{\ell(w)!}{\prod_{\beta \in \Phi(w)} h_t(\beta)}. \]

This hook formula is, of course, a generalization of hook length formula for a Young diagram due to Frame-Robinson-Thrall [2], and a shifted Young diagram due to Thrall [9].

In terms of $d$-complete posets, this counts the number of linear extensions of the $d$-complete posets.

Now, we have three approaches to prove Peterson-Proctor hook formula.

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<th>colored hook formula</th>
<th>probabilistic algorithm</th>
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<td>N. (preprint)</td>
<td></td>
<td>N.-Okamura (preprint)</td>
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3. Finite predominant integral weights

Definition 6. An integral weight $\lambda \in P$ is said to be pre-dominant if:
$$\langle \lambda, \beta^\vee \rangle \geq -1, \ \beta \in \Phi_+. $$

The set of pre-dominant integral weights is denoted by $P_{\geq -1}$.

Definition 7. Let $\lambda \in P_{\geq -1}$. We define a set $D(\lambda)$ by:
$$D(\lambda) := \{ \beta \in \Phi_+ \mid \langle \lambda, \beta^\vee \rangle = -1 \}.$$ 

The set $D(\lambda)$ is called a diagram of $\lambda$. A pre-dominant integral weight $\lambda$ is said to be finite if $\#D(\lambda) < \infty$. The set of finite pre-dominant integral weights is denoted by $P_{\geq -1}^{\text{fin}}$.

Example 2. As an example, we consider how Young diagram \[ \begin{array}{cccc}
0 & 1 & 2 & 3 \\
-1 & & & + \\
-2 & + & & - \\
-3 & & 0 & - \\
0 & & & 
\end{array} \] is realized as $D(\lambda)$.

According to the above picture, we put $\lambda := 1\Lambda_{-2} + (-1)\Lambda_0 + 1\Lambda_1 + (-1)\Lambda_2 + 1\Lambda_3$, in the root system of type $A_6$ with index $I = \{-2, -1, 0, 1, 2, 3\}$, where $\Lambda_i$ denotes $i$-th fundamental weight. Then we have $\lambda \in P_{\geq -1}^{\text{fin}}$ such that $(D(\lambda); <)$ is order-isomorphic to the original Young diagram.

Thus, we recover the original Young diagram.

Theorem 3.1. Let $\Lambda \in P_{\geq 0}$ and $w \in W$ a $\Lambda$-minuscule element. Then we have $w(\Lambda) \in P_{\geq -1}^{\text{fin}}$. Furthermore, this correspondence is bijective between $P_{\geq -1}^{\text{fin}}$ and the set of such pairs $(\Lambda, w)$.

\[ \begin{array}{ccc}
\alpha_{-1} + \alpha_0 + \alpha_1 + \alpha_2 & \alpha_0 + \alpha_1 + \alpha_2 & \alpha_2 \\
\alpha_{-1} + \alpha_0 & \alpha_0 & \\
\end{array} \]

Thus, we recover the original Young diagram.
Put $\lambda := w(\Lambda)$. Then we have

$$\Phi(w) = D(\lambda).$$

**Definition 8.** Let $\lambda \in P_{\geq -1}^{\text{fin}}$ and $\beta \in D(\lambda)$. We define a set $H_\lambda(\beta)$ by:

$$H_\lambda(\beta) := \{ \gamma \in D(\lambda) \mid s_\beta(\gamma) < 0 \} = D(\lambda) \cap \Phi(s_\beta).$$

We call the set $H_\lambda(\beta)$ the hook at $\beta$.

**Proposition 3.2.** Let $\lambda \in P_{\geq -1}^{\text{fin}}$ and $\beta \in D(\lambda)$. Then we have:

1. $\#H_\lambda(\beta) = ht(\beta)$.
2. $s_\beta(\lambda) \in P_{\geq -1}^{\text{fin}}$.
3. $D(s_\beta(D(\lambda) \setminus H_\lambda)) = s_\beta(D(\lambda) \setminus H_\lambda(\beta))$.

**Definition 9.** Let $\lambda \in P_{\geq -1}^{\text{fin}}$. A sequence $(\beta_1, \beta_2, \cdots, \beta_l)$ $(l \geq 0)$ of positive real roots is said to be a $\lambda$-path if:

$$\beta_k \in D(s_{\beta_{k-1}} \cdots s_{\beta_1}(\lambda)), \quad (k = 1, 2, \cdots, l).$$

The set of $\lambda$-paths is denoted by $\text{Path}(\lambda)$.

**Definition 10.** Let $\lambda \in P_{\geq -1}^{\text{fin}}$. A $\lambda$-path of maximal length is called a maximal $\lambda$-path. The set of maximal $\lambda$-paths is denoted by $\text{MPath}(\lambda)$.

Note that if $\#D(\lambda) = d$ then length of maximal $\lambda$-path is $d$, and hence that maximal $\lambda$-path is of a form $(\alpha_{i_1}, \alpha_{i_2}, \cdots, \alpha_{i_d})$.

**Example 3.** Back to Example 2, put $\lambda := \Lambda_{-1} - \Lambda_0 + \Lambda_1 - \Lambda_2 + \Lambda_3$. Then we have 5 maximal $\lambda$-paths below:

\[
\begin{align*}
& (\alpha_0, \alpha_{-1}, \alpha_2, \alpha_1, \alpha_0) \cdots \begin{array}{c} 5 \ 4 \ 3 \\ 2 \ 1 \end{array} \\
& (\alpha_0, \alpha_2, \alpha_{-1}, \alpha_1, \alpha_0) \cdots \begin{array}{c} 5 \ 4 \ 2 \\ 3 \ 1 \end{array} \\
& (\alpha_2, \alpha_0, \alpha_{-1}, \alpha_1, \alpha_0) \cdots \begin{array}{c} 5 \ 4 \ 1 \\ 3 \ 2 \end{array} \\
& (\alpha_0, \alpha_2, \alpha_1, \alpha_{-1}, \alpha_0) \cdots \begin{array}{c} 5 \ 3 \ 2 \\ 4 \ 1 \end{array} \\
& (\alpha_2, \alpha_0, \alpha_1, \alpha_{-1}, \alpha_0) \cdots \begin{array}{c} 5 \ 3 \ 1 \\ 4 \ 2 \end{array}
\end{align*}
\]

Now we restate the Peterson-Proctor hook formula:

**Theorem 3.3.** Let $\lambda \in P_{\geq -1}^{\text{fin}}$. Put $d := \#D(\lambda)$. Then we have:

$$\#\text{MPath}(\lambda) = \frac{d^!}{\prod_{\beta \in D(\lambda)} \text{ht(\beta)}}.$$

We give two of three approaches to prove the above theorem in section 4 and 5.
4. Colored Hook Formula

Let $\lambda \in P_{\geq -1}^{\text{fin}}$, and put $d = D(\lambda)$. Then we have:

Theorem 4.1 ([4]).

$$\sum_{(\beta_1, \beta_2, \ldots, \beta_d) \in \text{Path}(\lambda)} \frac{1}{\beta_1} \frac{1}{\beta_1 + \beta_2} \cdots \frac{1}{\beta_1 + \cdots + \beta_d} = \prod_{\beta \in D(\lambda)} \left(1 + \frac{1}{\beta}\right).$$

Taking the lowest degree, we get:

Corollary 4.2.

$$\sum_{(\alpha_1, \alpha_2, \ldots, \alpha_d) \in \text{MPath}(\lambda)} \frac{1}{\alpha_1} \frac{1}{\alpha_1 + \alpha_2} \cdots \frac{1}{\alpha_1 + \cdots + \alpha_d} = \prod_{\beta \in D(\lambda)} \frac{1}{\beta}.$$

Taking the specialization $\alpha_i \mapsto 1$, we get:

Corollary 4.3 (Peterson-Proctor hook formula).

$$\#\text{MPath}(\lambda) = \frac{d!}{\prod_{\beta \in D(\lambda)} \text{ht}(\beta)}.$$

5. Probabilistic Algorithm

For simplicity of description, we assume that the underlying root datum is simply-laced.

We call the following algorithm the algorithm $A$ for $\Gamma$:

GNW1.: Set $k := 0$ and set $\lambda_0 := \lambda$.

GNW2.: (Now $D(\lambda_k)$ has $d - k$ roots.) Pick a root $\beta \in D(\lambda_k)$ with the probability $1/(d - k)$.

GNW3.: If $\#H_{\lambda_k}(\beta) - \{\beta\} \neq 0$, then pick a $\gamma \in H_{\lambda_k}(\beta) - \{\beta\}$ with the probability $1/\#(H_{\lambda_k}(\beta) - \{\beta\})$, put $\beta := \gamma$ and repeat GNW3.

GNW4.: (Now $\#(H_{\lambda_k}(\beta) - \{\beta\}) = 0$.) Let $\alpha_{i+1} := \alpha_i$ and set $\lambda_{k+1} := s_i(\lambda_k)$.

GNW5.: Set $k := k + 1$. If $k < d$, return to GNW2; if $k = d$, terminate.

Then, by the definition of the algorithm A for $\lambda$, the sequence $(\beta =) (\alpha_1, \ldots, \alpha_d)$ generated above is a maximal $\lambda$-path. We denote by $\text{Prob}_A(\mathcal{B})$ the probability we get $\mathcal{B} \in \text{MPath}(\lambda)$ by the algorithm A. The algorithm A for $\lambda$ gives a probability measure $\text{Prob}_A()$ over a finite set $\text{MPath}(\lambda)$.

Theorem 5.1 (S. Okamura [6], N-S. Okamura [5]). Let $\mathcal{B} \in \text{MPath}(\lambda)$. Then we have:

$$(5.1) \quad \text{Prob}_A(\mathcal{B}) = \frac{\prod_{\beta \in D(\lambda)} \text{ht}(\beta)}{d!}. $$

Since the right-hand side of (5.1) is independent from the choice of $\mathcal{B} \in \text{MPath}(\lambda)$, the probability measure is uniform. Hence, taking the inverse, we get:

Corollary 5.2 (Peterson-Proctor hook formula).

$$\#\text{MPath}(\lambda) = \frac{d!}{\prod_{\beta \in D(\lambda)} \text{ht}(\beta)}. $$

REFERENCES


