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Lazy Clones and Essentially Minimal Clones

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Abstract

Over a finite set A, a *lazy function* on A is a multi-variable function defined on A that can produce practically no new functions by composition. A *lazy clone* is a clone generated by a lazy function. In this paper we present some examples of lazy clones and relate lazy clones to essentially minimal clones.

Keywords: clone; essentially minimal clone; lazy clone

1 Preliminaries

Let A be a non-empty set. For $n>0$ denote by $\mathcal{O}_A^{(n)}$ the set of n-variable functions on A, i.e., $\mathcal{O}_A^{(n)}=A^{A^n}$. Let $\mathcal{O}_A$ be the union of $\mathcal{O}_A^{(n)}$ over $1\leq n<\omega$, i.e., $\mathcal{O}_A=\bigcup_{n=1}^{\infty}\mathcal{O}_A^{(n)}$. $\mathcal{J}_A$ denotes the set of *projections* e_i^n on A $(1\leq i\leq n)$ where $e_i^n(x_1,\ldots,x_i,\ldots,x_n)=x_i$ for all $(x_1,\ldots,x_n)\in A^n$.

A subset C of $\mathcal{O}_A$ is a *clone* on A if (i) $C\supseteq\mathcal{J}_A$ and (ii) C is closed under (functional) composition. The set of all clones on A is denoted by $\mathcal{L}_A$ and called the *lattice of clones* on A. It is well-known that, while $\mathcal{L}_A$ is countable for $|A|=2$, $\mathcal{L}_A$ for each A with $|A|>2$ has the cardinality of continuum and the structure of $\mathcal{L}_A$ is extremely complex. An atom of $\mathcal{L}_A$ is called a minimal clone. In other words, a minimal clone is a clone which sits just above $\mathcal{J}_A$ in the lattice $\mathcal{L}_A$. Minimal clones are known only for A with $|A|=2$, 3 or 4. To characterize all minimal clones for A where $|A|>4$ seems to be a highly challenging task.

For $F\subseteq\mathcal{O}_A$ we denote by $\langle F\rangle$ the clone generated by F, i.e., $\langle F\rangle$ is the smallest clone containing F. In particular, when $F=\{f\}$ for some $f\in\mathcal{O}_A$ we write $\langle f\rangle$ instead of $\langle F\rangle$.

A function $f\in\mathcal{O}_A^{(n)}$ *depends* on its i-th variable if there exist $a_1,\ldots,a_n,b_i\in A$ which satisfy

$$f(a_1,\ldots,a_i,\ldots,a_n)\neq f(a_1,\ldots,a_{i-1},b_i,a_{i+1},\ldots,a_n).$$

In this case, we say that the i-th variable is an *essential variable* of f. For $f\in\mathcal{O}_A^{(n)}$, the *essential arity* of f (denoted by *ess-arity* (f)) is the number of essential variables of f. A function f in $\mathcal{O}_A$ is said to be *essential* if it depends on at least two variables, i.e., ess-arity$(f)\geq 2$. A clone C in $\mathcal{L}_A$ is *essential* (or, *non-unary*) if it contains at least one essential function. Let us denote by $\mathcal{L}_A^{ess}$ the set of essential clones in $\mathcal{L}_A$.

Definition 1.1 *A clone C on A is an* essentially minimal clone *if* (1) *C is a minimal element in $\mathcal{L}_A^{ess}$ and* (2) *C is not a minimal clone* (*in $\mathcal{L}_A$*).

In the following section we shall introduce a new kind of clones, called lazy clones, which share a similar property with both minimal clones and essentially minimal clones in a sense that all of these clones are weakly productive by means of composition. Then we shall relate lazy clones to essentially minimal clones.

2 Lazy Clones

For $t > 0$ put $[t] = \{1, \dots, t\}$. For $m, n > 0$, a map $s : [n] \longrightarrow [m]$ and $f \in \mathcal{O}_A^{(n)}$, a function $w \in \mathcal{O}_A^{(m)}$ defined by

$$w(x_1, \dots, x_m) = f(x_{s(1)}, \dots, x_{s(n)})$$

will be called an *s-minor* of f. Let W_f be the set of s-minors of f for all $m > 0$ and all $s \in [m]^{[n]}$.

Consider the set W_f for $f \in \mathcal{O}_A$ together with all projections on A, that is, the union $\mathcal{J}_A \cup W_f$. It is obvious that $\mathcal{J}_A \cup W_f \subseteq \langle f \rangle$. In general, the union $\mathcal{J}_A \cup W_f$ is not a clone. However, there are cases where $f \in \mathcal{O}_A$ is so weakly productive and $\mathcal{J}_A \cup W_f$ does form a clone. A function f is called *lazy* if $\mathcal{J}_A \cup W_f$ is a clone and, in this case, $\mathcal{J}_A \cup W_f$ is called a *lazy clone.*

Trivial example of a lazy clone is $\mathcal{J}_A$. (Take any projection for f.) Less trivial examples of lazy clones can be found on the three-element set $A = \{0, 1, 2\}$. We shall present two such examples.

The first example is a clone generated by $f \in \mathcal{O}_A^{(2)}$ having the following Cayley table.

$f(x, y) =$

$x \backslash y$	0	1	2
0	0	0	0
1	2	1	1
2	2	1	1

It is easy to see that f is a lazy function and a clone $\langle f \rangle$ generated by f is a lazy clone.

The second example is a clone generated by $g \in \mathcal{O}_A^{(3)}$ whose Cayley table is the following.

$g(x, y, z) =$

$x \backslash y$	0	1	2	0	1	2	0	1	2
0	0	0	0	0	0	0	0	0	0
1	1	1	1	2	1	1	2	1	1
2	1	1	1	2	1	1	2	1	1
	$z = 0$			$z = 1$			$z = 2$		

Again, it is not hard to check that g generates a lazy clone $\mathcal{J}_A \cup W_g$.

Fact For functions f, g given above, $\langle f \rangle$ is strictly included in $\langle g \rangle$, i.e., $\langle f \rangle \subset \langle g \rangle$.

Proof The inclusion follows from $g(x, y, x) = f(x, y)$. The strictness is due to the fact that f is a 2-variable lazy function while the essential arity of g is 3. □

As f is an essential function, a lazy clone $\langle g \rangle$ cannot be essentially minimal. Hence, we have:

Corollary: There exists a lazy clone which is not essentially minimal.

It is interesting to know which lazy clones are minimal (resp. essentially minimal) and which lazy clones are not minimal (resp. essentially minimal). At the moment we do not have solutions to these problems. In the next section we present a result which relates lazy clones to essentially minimal clones.

Before we proceed, we give a sufficient condition for $f \in \mathcal{O}_A$ to be lazy. For $f \in \mathcal{O}_A$, the *diagonal* f^* of f is a unary function on A defined by $f^*(x) = f(x, \dots, x)$ for all $x \in A$.

Lemma 2.1 *Let $f \in \mathcal{O}_A^{(n)}$ for $n > 0$. If f satisfies the following three conditions then f is a lazy function.*

(1) $(f^*)^2 = f^*$

(2) $f(x_1, \dots, x_n) = f(f^*(x_1), \dots, f^*(x_n))$ *for all* $x_1, \dots, x_n \in A$

(3) $f^*(f(x_1, \dots, x_n))$ *is not essential.*

Proof It suffices to show, w.o.l.g., that a composition $f((f(x_1, \dots, x_n), x_{n+1}, \dots, x_{2n-1})$ belongs to W_f. Let us put $w(x_1, \dots, x_{2n-1}) := f((f(x_1, \dots, x_n), x_{n+1}, \dots, x_{2n-1})$.

First note that, due to condition (3), there exist i, $1 \le i \le n$, and $u \in \mathcal{O}_A^{(1)}$ such that $f^*(f(x_1, \dots, x_n)) = u(x_i)$ for all $x_1, \dots, x_n \in A$. It then follows that $u(x_i) = f^*(f(x_i, \dots, x_i)) = (f^*)^2(x_i) = f^*(x_i)$ by condition (1).

Now, condition (2) and the above remark imply the following:

$$\begin{aligned} w(x_1, \dots, x_{2n-1}) &= f(f^*(f(x_1, \dots, x_n)), f^*(x_{n+1}), \dots, f^*(x_{2n-1})) \\ &= f(u(x_i), f^*(x_{n+1}), \dots, f^*(x_{2n-1})) \\ &= f(f^*(x_i), f^*(x_{n+1}), \dots, f^*(x_{2n-1})) \end{aligned}$$

By applying condition (2) again, we obtain

$$w(x_1, \dots, x_{2n-1}) = f(x_i, x_{n+1}, \dots, x_{2n-1})$$

as desired. □

Remark Laziness of two functions $f(x, y)$ and $g(x, y, z)$ given in the preceding page can be verified by Lemma 2.1.

3 Lazy Clones related to Essentially Minimal Clones

For each $f \in \mathcal{O}_A$, we define a subset $\Gamma(f)$ of A as follows. First, for a unary function $h \in \mathcal{O}_A^{(1)}$,

$$\Gamma(h) = \{x \in A \mid \exists i > 0,\ h^i(x) = x\}$$

i.e., $\Gamma(h)$ is the set of elements on cycles of the graph G_h of h. Next, for an n-ary function $f \in \mathcal{O}_A^{(n)}$, $n \ge 1$, we define $\Gamma(f)$ by $\Gamma(f) = \Gamma(f^*)$.

In Proposition 3.1, assuming the knowledge of lazy clones, we give a characterization of those essentially minimal clones C for which there exists a generator $f \in \mathcal{O}_A^{(n)}$ such that the restriction $f|_{\Gamma(f)^n}$ of f to $\Gamma(f)^n$ is essential.

The original version of the following proposition appeared in [MR84]. However, the description given there was not fully accurate. Here we present a modified version. (Proof is omitted.)

Proposition 3.1 *Let* $f \in \mathcal{O}_A^{(n)}$ *be a function whose restriction* $f|_{\Gamma(f)^n}$ *to* $\Gamma(f)^n$ *is essential. Then* $\langle f \rangle$ *is an essentially minimal clone if and only if the following conditions are satisfied.*

(i) $\Gamma(f) \subset A$ *(proper inclusion)*

(ii) $(f^*)^2 = f^*$

(iii) $f(x_1, \dots, x_n) = f(f^*(x_1), \dots, f^*(x_n))$ *for all* $x_1, \dots, x_n \in A$

(iv) (a) $f(\Gamma(f)^n) \subseteq \Gamma(f)$ *and* $f|_{\Gamma(f)^n}$ *generates a minimal clone on* $\Gamma(f)$, *or*

(b) $f(\Gamma(f)^n) \not\subseteq \Gamma(f)$, $f^*(f(x_1, \dots, x_n))$ *is non-essential and* $\langle f \rangle$ *is minimal among (non-trivial) lazy clones on* A.

4 Supplementary Remark

A more detailed description of lazy clones is currently in preparation and, hopefully, will appear elsewhere in the near future.

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