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The proportion of numerical semigroups with no descendant or an infinite number of descendants [1]

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Abstract

Let p be the map between the sets of numerical semigroups sending a numerical semigroup to the one whose genus is decreased by 1. We give many examples of numerical semigroups H with $p^{-1}(H)=\emptyset$. We investigate the density of some kinds of numerical semigroups H with $p^{-1}(H)=\emptyset$ in the whole set of numerical semigroups. Moreover, we determine the numerical semigroups H with $p^{-n}(H)\neq\emptyset$ for any n.

1 The conductor and descendants

Let $\mathbb{N}_0$ be the additive monoid of non-negative integers. A submonoid H of $\mathbb{N}_0$ is called a *numerical semigroup* if the complement $\mathbb{N}_0\backslash H$ is finite. The cardinality of $\mathbb{N}_0\backslash H$ is called the *genus* of H, denoted by $g(H)$. In this section H stands for a numerical semigroup of genus g. We set

$$m(H)=\min\{h\in H\mid h>0\},$$

which is called the *multiplicity* of H. We set

$$c(H)=\min\{c\in\mathbb{N}_0\mid c+\mathbb{N}_0\subseteqq H\},$$

which is called the *conductor* of H. Then we have $g+1\leqq c(H)\leqq 2g$. We note that $c(H)-1\notin H$. We set $p(H)=H\cup\{c(H)-1\}$, which is a numerical semigroup of genus $g-1$. The numerical semigroup $p(H)$ is called the *parent* of H. The numerical semigroup H is called a *child* of $p(H)$. Let $M(H)$ be the minimal set of generators for H. For $\mu\in M(H)$ with $\mu>f(H)$, which is called an *effective generator* of H, we set $H_\mu=H\backslash\{\mu\}$, which is a child of H, and vice versa. A numerical semigroup H' is called a descendant of H if there exists $i\geqq 1$ such that $p^i(H')=H$. A child of H is a descendant of H. In this paper we are interested in numerical semigroups H which have either no descendant, i.e., no child or an infinite number of descendants.

Proposition 1.1. *Suppose that $c(H)=g+1$. Then we have $H=\langle g+1\longrightarrow 2g+1\rangle$, which has an infinite number of descendants. In fact, for any $i\geqq 1$ we have*

$$p^i(\langle g+1+i\longrightarrow 2g+1+i\rangle)=H.$$

[1]This paper is an extended abstract and the details will appear elsewhere.

Proposition 1.2. *Suppose that $c(H) = g + 2$. Then H has an infinite number of descendants.*

Proof. We have $c(H) - 1 - g = 1$. Since we have $g.c.m.(\lambda_0, \lambda_1) = \lambda_1 > 1$, by Theorem 10 in [1] H has an infinite number of descendants. □

We set $\alpha_i = l_i - i$ for $i = 1, \dots, g$ where $\mathbb{N}_0 \backslash H = \{l_1 < \cdots < l_g\}$. We call $\alpha(H) = (\alpha_1, \dots, \alpha_g)$ the *Schubert index* of H. Then we have $\alpha(p(H)) = (\alpha_1, \dots, \alpha_{g-1})$.

Proposition 1.3. *Assume that $c(H) = 2g$. If $H \neq \langle 2, 2g+1 \rangle$, then H has no child.*

Proof. Assume that H has a child $\tilde{H}$, i.e., $p(\tilde{H}) = H$. Since H is symmetric, i.e., $c(H) = 2g$, we have $\alpha(\tilde{H}) = (\alpha_1, \dots, \alpha_{g-1}, g-1, \alpha_{g+1})$. Hence we get $\alpha_{g+1} = g - 1$ or g.
Case 1: $\alpha_{g+1} = g$. Then $\tilde{H}$ is symmetric. Since $2g - 1 \notin \tilde{H}$, we have $\tilde{H} \ni 2(g+1) - 1 - (2g-1) = 2$, which implies that $\tilde{H} = \langle 2, 2(g+1)+1 \rangle$. Hence, we get $H = p(\tilde{H}) = \langle 2, 2g+1 \rangle$.
Case 2: $\alpha_{g+1} = g - 1$. Then $\tilde{H}$ is quasi-symmetric. Since $2g - 1 \notin \tilde{H}$, we have $\tilde{H} \ni 2(g+1) - 2 - (2g-1) = 1$, which implies that $\tilde{H} = \mathbb{N}_0$. □

Proposition 1.4. *Assume that $c(H) = 2g - 1$. If H is different from $\langle 3, g+2, 2g+1 \rangle$ with $g \not\equiv 1$ mod 3 and $\langle g \longrightarrow 2g-3, 2g-1 \rangle$, then H has no child.*

Proof. For the proof see Theorem 3.9 in [4]. □

2 The proportion of certain kinds of numerical semigroups

Let ϵ be a fixed positive number. Let $\gamma = \dfrac{5+\sqrt{5}}{10} = \dfrac{\phi}{\sqrt{5}}$ where ϕ is the golden ratio. For a non-negative integer g let $NS(g)$ be the set of numerical semigroups of genus g. We set $\Phi S_\epsilon(g) = \{H \in NS(g) \mid (\gamma - \epsilon)g < m(H) < (\gamma + \epsilon)g\}$.

Remark 2.1. ([3])) We have $\lim\limits_{g \to \infty} \dfrac{\sharp \Phi S_\epsilon(g)}{\sharp NS(g)} = 1$.

For any positive integer $n \geqq 2$ we set $L_n(H) = \{l_1 + \cdots + l_n \mid l_i \in \mathbb{N}_0 \backslash H, \text{ all } i\}$.

Key Lemma 2.2. *Let $0 < \epsilon < \dfrac{1}{21}$ and $m \geqq 420$. Assume that $m = m(H)$ and $(2-\epsilon)m < c(H) - 1 < (2+\epsilon)m$. If $\sharp L_n(H) \geqq (2n-1)(g-1) - 19$ with some $n \geqq 2$, then we have $g < 1.38175m$.*

For the proof see [5].

Theorem 2.3. *We set*

$$BS(-19, g) = \{H \in NS(g) \mid \sharp L_n(H) \geqq (2n-1)(g-1) - 19 \text{ for some } n \geqq 2\}.$$

Then we obtain $\lim\limits_{g \to \infty} \dfrac{\sharp BS(-19, g)}{\sharp NS(g)} = 0$.

For the proof see [5].

Remark 2.4. ([7])) Assume that $c(H) \neq 2g$. Then we have $L_2(H) \supseteqq \{2,3,4,5,\ldots,2g\}$.

Using Remark 2.4 we get the following:

Key Lemma 2.5. *Assume that* $c(H) = 2g - i$ *with* $1 \leqq i \leqq g-1$. *Then we have* $\sharp L_2(H) \geqq 3g - 3 - (i-1)$.

For the proof see [5].

Main Theorem 2.6. *We set*

$$CS(20, g) = \{H \in NS(g) \mid 2g - 20 \leqq c(H)\}.$$

Then we obtain $\lim_{g\to\infty} \dfrac{\sharp CS(20,g)}{\sharp NS(g)} = 0$.

For the proof see [5].

Corollary 2.7. *We have* $\lim_{g\to\infty} \dfrac{\sharp\{H \in NS(g) \mid c(H) = 2g\}}{\sharp NS(g)} = 0$.

Corollary 2.8. *We have* $\lim_{g\to\infty} \dfrac{\sharp\{H \in NS(g) \mid c(H) = 2g-1\}}{\sharp NS(g)} = 0$.

Problem 1. Assume that $c(H) \leqq 2g - 21$. What kind of numerical semigroup H has a child?

Problem 2.

$$\lim_{g\to\infty} \frac{\sharp\{H \in NS(g) \mid H \text{ has no child}\}}{\sharp NS(g)} = 0\ ?$$

3 Numerical semigroups with an infinite number of descendants

We are interested in numerical semigroups which have infinite numbers of descendants. Such a numerical semigroup is said to be *IND.* We set $d_2(H) = \{h' \in \mathbb{N}_0 \mid 2h' \in H\}$, which is also a numerical semigroup. $n(H)$ stands for the minimum odd number in H.

Theorem 3.1. *Assume that* $n(H) \geqq 2c(d_2(H)) + 1$. *Then the following are equivalent:*
i) *H is IND.*
ii) $H = 2d_2(H) + \langle n, n+2, \ldots, n+2(m'-1)\rangle$ *where* $n = n(H)$ *and* $m' = m(d_2(H))$.

For the proof see [6].

Example 3.1. Let $t \geqq 1$. We set $H = 2\langle 2, 2t+1\rangle + \langle 4t+1, 4t+3\rangle$. Then we have $n(H) = 4t+1$, $d_2(H) = \langle 2, 2t+1\rangle$ and $c(d_2(H)) = 2t$. Hence, H is IND. In fact, when we set $H_i = 2\langle 2, 2t+1\rangle + \langle 4t+1+2i, 4t+3+2i\rangle$, we obtain $p^i(H_i) = H$ for $i \geqq 1$.

Theorem 3.2. *Let H be a numerical semigroup and $m' = m(d_2(H))$. For an odd number n we set $H = 2d_2(H) + \langle n, n+2, \dots, n+2(m'-1)\rangle$.*
i) *If $n \geqq 2c(d_2(H)) + 1$, then H is IND.*
ii) *If $g(d_2(H)) \geqq 1$ and $n = 2c(d_2(H)) - 1$, then H is IND.*
iii) *If $n = n(H)$ and $n \leqq 2c(d_2(H)) - 5$, then H is not IND.*

For the proof see [6].

Theorem 3.3. *Let H be a numerical semigroup, $m' = m(d_2(H))$, $g' = g(d_2(H)) \geqq 2$ and $c' = c(d_2(H))$. We set $H = 2d_2(H) + \langle 2c'-3, 2c'-3+2, \dots, 2c'-3+2(m'-1)\rangle$.*
i) *If $d_2(H)$ is not IND, then neither is H.*
ii) *Assume that $d_2(H)$ is IND. Then H is IND if and only if we have*

$$(\lambda'_0, \lambda'_1, \dots, \lambda'_{c'-1-g'}, 2c'-3) > 1$$

where $d_2(H) = \{\lambda'_0 < \lambda'_1 < \cdots < \lambda'_{c'-1-g'} < \cdots\}$.

For the proof see [6].

Theorem 3.4. *Assume that $n(H) \leqq 2c' - 1$ where $c' = c(d_2(H))$. If H is IND, then there exists $i \geqq 0$ such that $p^i(H)$ is one of the following:*
i) *$2d_2(p^i(H)) + \langle 2c^{(i)}-1, 2c^{(i)}+1, \dots, 2c^{(i)}+2m^{(i)}-3\rangle$ where $c^{(i)} = c(d_2(p^i(H)))$ and $m^{(i)} = m(d_2(p^i(H)))$*
ii) *$2d_2(p^i(H)) + \langle 2c^{(i)}-3, 2c^{(i)}-1, \dots, 2c^{(i)}+2m^{(i)}-5\rangle$ with $(\lambda^{(i)}_0, \lambda^{(i)}_1, \dots, \lambda^{(i)}_{c^{(i)}-1-g^{(i)}}, 2c^{(i)}-3) > 1$ where $g^{(i)} = g(d_2(p^i(H)))$ and $d_2(p^i(H)) = \{\lambda^{(i)}_0 < \lambda^{(i)}_1 < \cdots\}$.*

For the proof see [6].

Remark 3.5. The converse of Theorem 3.4 does not hold. In fact, let

$$H = \langle 10, 15, 17, 18, 21, 22, 23, 24, 26, 29\rangle.$$

Then we have $c(H) = 20$, $g(H) = 15$ and $c(H) - 1 - g(H) = 4$. It follows from $H = \{0 < 10 < 15 < 17 < 18 < \cdots\}$ and $(0, 10, 15, 17, 18) = 1$ that H is not IND. Moreover, we have $d_2(H) = \langle 5, 9, 11, 12, 13\rangle$. Then we obtain $2c(d_2(H)) - 3 = 2 \times 9 - 3 = 15$, $m(d_2(H)) = 5$, $2c(d_2(H)) + 2m(d_2(H)) - 5 = 23$ and

$$p(H) = 2\langle 5, 9, 11, 12, 13\rangle + \langle 15, 17, 19, 21, 23\rangle.$$

We note that $d_2(p(H)) = \langle 5, 9, 11, 12, 13\rangle$, $c(d_2(H)) - 1 - g(d_2(H)) = 9 - 1 - 7 = 1$ and $(0, 5) = 5 > 1$.

On the other hand we consider

$$H' = \langle 10, 15, 18, 19, 21, 22, 23, 24, 26, 27\rangle.$$

Since $g(H') = 15$ and $c(H') = 18$, we obtain $c(H') - g(H') - 1 = 2$. It follows from $(0, 10, 15) = 5 > 1$ that H' is IND. Moreover, we have $p(H) = p(H')$.

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