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Kyoto University
SUMMARY OF STUDIES OF CLOSED/OPEN MIRROR SYMMETRY FOR QUINTIC THREEFOLDS THROUGH LOG MIXED HODGE THEORY

SAMPEI USUI

0. Introduction and Statements

This is a summary of [U14p].

We correct the definitions and descriptions of the integral structures in our previous paper [U14]. We use $\Gamma$-integral structure of Iritani in [I11] for A-model. Using the corrected version, we study open mirror symmetry for quintic threefolds through log mixed Hodge theory, especially the recent result on Néron models for admissible normal functions with non-torsion extensions in the joint work [KNU14] with K. Kato and C. Nakayama. We positively use integral structures of local systems with graded polarizations over the boundary points.

In a series of joint works with Kato and Nakayama, we are constructing a fundamental diagram which consists of various kind of partial compactifications of classifying space.

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of mixed Hodge structures and their relations. We try to understand Hodge theoretic aspects of mirror symmetry in this framework of the fundamental diagram.

**Fundamental Diagram**

For a classifying space $D$ of Hodge structures of specified type, we have

\[
\begin{array}{ccc}
D_{SL(2), val} & \longrightarrow & D_{BS, val} \\
\downarrow & & \downarrow \\
\Gamma \backslash D_{\Sigma, val} & \longrightarrow & D_{\Sigma} \\
\downarrow & & \downarrow \\
\Gamma \backslash D_{\Sigma} & \longrightarrow & D_{\Sigma}^\# \\
\end{array}
\]

in pure case: [KU99], [KU02], [KU09]. For mixed case, we should extend to an amplified diagram: [KNU08], [KNU09], [KNUII], [KNU13], continuing.

**Mirror symmetry for quintic threefolds**

Mirror symmetry for the A-model of quintic threefold $V$ and the B-model of its mirror $V^\circ$ was predicted in the famous paper [CDGP91]. We recall two styles of the theorem (1) and (2) below. Every statement in the present paper is near the large radius point $q_0$ of the complexified Kähler moduli $\mathcal{K}\mathcal{M}(V)$ and the maximally unipotent monodromy point $p_0$ of the complex moduli $\mathcal{M}(V^\circ)$.

Let $t := y_1 / y_0$, $u := t / 2\pi i$ be the canonical parameters and $q := e^t = e^{2\pi iu}$ be the canonical coordinate from 2.2 below and the respective ones in 2.3 below.

The following theorem is due to Lian-Liu-Yau [LLuY97].

(1) (Potential). The potentials of the two models coincide: $\Phi_{GW}^V(t) = \Phi_{GM}^{V^\circ}(t)$.

The following theorem is formulated by Morrison [M97] and proved by Iritani [I11].

(2) (Variation of Hodge structure). The isomorphism $(q_0 \in \overline{\mathcal{K}\mathcal{M}(V)}) \sim (p_0 \in \overline{\mathcal{M}(V^\circ)})$ of neighborhoods of the compactifications, by the canonical coordinate $q = \exp(2\pi iu)$, lifts to an isomorphism, over the punctured neighborhoods $\mathcal{K}\mathcal{M}(V) \sim \mathcal{M}(V^\circ)$, of polarized $\mathbf{Z}$-variations of Hodge structure with a specified section

\[
(\mathcal{H}^V, S, \nabla^\text{even}, \mathcal{H}^V_Z, F; 1) \sim (\mathcal{H}^{V^\circ}, Q, \nabla^\text{GM}, \mathcal{H}^{V^\circ}_Z, F; \tilde{\Omega}).
\]

Our (3) below is equivalent to (1) and (2) by a log version [KU09, 2.5.14] of the nilpotent orbit theorem of Schmid [S73] (this part of [U14] is valid).

(3) (Log Hodge structure, Log period map). The isomorphism $(q_0 \in \overline{\mathcal{K}\mathcal{M}(V)}) \sim (p_0 \in \overline{\mathcal{M}(V^\circ)})$ of neighborhoods of the compactifications uniquely lifts to an isomorphism of B-model log variation of polarized Hodge structure with a specified section $\tilde{\Omega}$ for $V^\circ$ and A-model log variation of polarized Hodge structure with a specified section
1 for $V$, whose restriction over the punctured $\mathcal{KM}(V) \sim \mathcal{M}(V^\circ)$ coincides with the isomorphism of variations of polarized Hodge structure with specified sections in (2).

This rephrases as follows. Let $\sigma$ be the common monodromy cone, transformed by a level structure into End of a reference fiber of the local system, for the A-model and for the B-model. Then, we have a commutative diagram of horizontal log period maps

$$
(q_0 \in \overline{\mathcal{KM}(V)}) \sim (p_0 \in \overline{\mathcal{M}(V^\circ)})
$$

with extensions of specified sections in (2), where $(\sigma, \exp(\sigma_C)F_0)$ is the nilpotent orbit, regarded as a boundary point, and $\Gamma(\sigma)^{sp}\backslash D_\sigma$ is the fine moduli of log Hodge structures of specified type. (For fine moduli $\Gamma(\sigma)^{sp}\backslash D_\sigma$, or more generally $\Gamma\backslash D_{\Sigma}$, see [KU09].)

Open mirror symmetry for quintic threefolds

The following theorem is due to Walcher [W07] and Morrison-Walcher [MW09].

4. (Inhomogenous solutions).

Let $\mathcal{L}$ be the Picard-Fuchs differential operator for quintic mirror (cf. 2.2). Let

$$
T_A = \frac{u}{2} \pm \left(\frac{1}{4} + \frac{1}{2\pi^2} \sum_{d \text{ odd}} n_d q^{d/2}\right)
$$

be the A-model domainwall tension in [MW09], and

$$
T_B = \int_{C_-}^{C_+} \Omega
$$

be the B-model domainwall tension, where $C_\pm \subset V^\circ$ are the disjoint smooth curves coming from the two conics in $\{x_1 + x_2 = x_3 + x_4 = 0\} \cap V_\psi \subset P^4(C)$ [ibid].

Then

$$
\mathcal{L}(y_0(z)T_A(z)) = \mathcal{L}(T_B(z)) \left(\frac{15}{16\pi^2} \sqrt{z}\right) \quad \left(z = \frac{1}{(5\psi)^5}\right).
$$

Concerning this, we have the following observations.

5. (Log mixed Hodge structure, Log normal function). We describe for B-model. The same holds for A-model by (1)–(3) and the correspondence table in 2.5 below.

Put $\mathcal{H} := \mathcal{H}^V$ and $\mathcal{T} := \mathcal{T}_B$. We use $e^0 \in I^{0,0}, e^1 \in I^{1,1}$ which are a part of a basis of $\mathcal{H}_\psi^\log$ respecting the Deligne decomposition at $p_0$ (see 2.5 (3B)) and a flat sections $s^0 = e^0$, $s^1 = e^1 - ue^0$ (see 2.5 (5B)). To make the local monodromy of $T$ unipotent, we take a double cover $z^{1/2} \mapsto z$. Let $L_Q$ be the translated local system from the trivial extension $\mathbb{Q} \oplus \mathcal{H}_Q$ by $-(\mathcal{T}/y_0)s^0$ in $\mathcal{Ext}^1(\mathbb{Q}, \mathcal{H}_Q)$. Let $J_{L_Q}$ be the Néron model on a neighborhood $S$ of $p_0$ in the $z^{1/2}$-plane which lies over $L_Q$ in [KNU14]. Then,
\[ J_{L_{\mathcal{Q}}} = \mathcal{E}xt_{L_{\text{MH}/S}}^{1}(\mathcal{Z}, \mathcal{H}) \] (extension group of log mixed Hodge structures over \( S \)) in the present case ([KNU13, III, Corollary 6.1.6], cf. 1.4 below), and we have the following (5.1)–(5.3).

(5.1) The normalized tension \( T/y_{0} \) is understood as a truncated normal function by \((T/y_{0})s^{0}\). This extends as a truncated log normal function over the puncture. Then it lifts uniquely to a log normal function \( S \to J_{L_{\mathcal{Q}}} \) so that the corresponding exact sequence \( 0 \to \mathcal{H} \to H \to \mathcal{Z} \to 0 \) of log mixed Hodge structures over \( S \) is given by the liftings \( 1_{\mathcal{Z}} \) and \( 1_{\mathcal{F}} \) in \( H \) of \( 1 \in \mathcal{Z} \simeq (\text{gr}^{W})_{\mathcal{Z}} \) respecting the lattice and the Hodge filtration, respectively, which are defined as follows: \( 1_{\mathcal{Z}} := 1 - (T/y_{0})s^{0} \) with \((T/y_{0})s^{0} \in \mathcal{H}_{\mathcal{O}^{\text{log}}} = (\text{gr}^{W})_{\mathcal{O}^{\text{log}}}, \) and \( 1_{\mathcal{F}} - 1_{\mathcal{Z}} := - (\theta(T/y_{0}))e^{1} + (T/y_{0})e^{0} \).

(5.2) A splitting of the weight filtration \( W \) of the local system \( H_{\mathcal{Z}}, \) i.e., a splitting compatible with the monodromy of the local system \( H_{\mathcal{Z}}, \) is given by \( 1_{\mathcal{Z}}^{\text{spl}} = 1_{\mathcal{Z}} + s^{1}/2, \) and the log normal function over it is given by \( 1_{\mathcal{F}}^{\text{spl}} - 1_{\mathcal{Z}}^{\text{spl}} = - (\theta(T/y_{0}))e^{1} + (T/y_{0})e^{0}. \)

(5.3) (4) says that the inverse of the truncated normal function in (5.1) from its image is given by \( 16\pi^{2}/15 \) times the Picard-Fuchs differential operator \( \mathcal{L}. \)

Some geometric backgrounds of (5) are explained in Section 3.

We treat Tate twists case by case in this article.

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1. Log mixed Hodge theory

In this section, we recall some notions and results of log mixed Hodge theory from [KU09], [KNU13] and [KNU14] adapting to the present context.

1.1. Category \( \mathcal{B}(\text{log}) \)

Let \( S \) be a subset of an analytic space \( Z. \) The strong topology of \( S \) in \( Z \) is the strongest one among those topologies on \( S \) in which, for any analytic space \( A \) and any morphism \( f : A \to Z \) with \( f(A) \subset S \) as sets, \( f : A \to S \) is continuous. \( S \) is regarded as a local ringed space by the pullback sheaf of \( \mathcal{O}_{Z}. \)

Let \( \mathcal{B} \) be the category of local ringed spaces \( S \) over \( \mathcal{C} \) which have an open covering \((U_{\lambda})_{\lambda}\) satisfying the following condition: For each \( \lambda, \) there exist an analytic space \( Z_{\lambda}, \) and a subset \( S_{\lambda} \) of \( Z_{\lambda} \) such that, as local ringed space over \( \mathcal{C}, \) \( U_{\lambda} \) is isomorphic to \( S_{\lambda} \) which is endowed with the strong topology in \( Z_{\lambda} \) and the inverse image of \( \mathcal{O}_{Z_{\lambda}}. \)

A log structure on a local ringed space \( S \) is a sheaf of monoids \( M \) on \( S \) together with a homomorphism \( \alpha : M \to \mathcal{O}_{S} \) such that \( \alpha^{-1}\mathcal{O}_{S}^{\times} \cong \mathcal{O}_{S}^{\times}. \) If log structure means, locally on the underlying space, the log structure has a chart which is finitely generated, integral and saturated.
Let $B(\log)$ be the category of objects of $B$ endowed with an $fs$ log structure (more precisely, cf. [KU09]).

1.2. Ringed space $(S^{\log}, \mathcal{O}^{\log}_S)$

Let $S \in B(\log)$. As a set define
\[ S^{\log} := \{(s, h) | s \in S, h : M_s^{gp} \to S^1 \text{ homomorphism s.t. } h(u) = u/|u| (u \in \mathcal{O}^\times_{S,s})\}. \]
Endow $S^{\log}$ with the weakest topology such that the following two maps are continuous.

1. $\tau : S^{\log} \to S, (s, h) \mapsto s$.

2. For any open set $U \subset S$ and any $f \in \Gamma(U, M^{gp})$, $\tau^{-1}(U) \to S^1, (s, h) \mapsto h(f_s)$.

Then, $\tau$ is proper and surjective with fiber $\tau^{-1}(s) = (S^1)^{r(s)}$, where $r(s)$ is the rank of $(M^{gp}/\mathcal{O}^\times_s)_s$ which varies with $s \in S$.

For $s \in S$ and $t \in S^{\log}$ lying over $s$, let $q_j \in M^{gp}_s$ $(1 \leq j \leq r(s))$ be elements such that their images in $(M^{gp}/\mathcal{O}^\times_s)_s$ form a basis. Let $t_j := \log(q_j)$ and define $\mathcal{O}^{\log}_{S,t}$ to be a polynomial ring $\mathcal{O}_{S,t}[t_j | 1 \leq j \leq r(s)]$ over $\mathcal{O}_{S,s}$. Thus $\tau : (S^{\log}, \mathcal{O}^{\log}_S) \to (S, \mathcal{O}_S)$ is a morphism of ringed spaces over $C$ (more precisely, cf. [KU09]).

1.3. Graded polarized log mixed Hodge structure

Let $S \in B(\log)$. A pre-graded polarized log mixed Hodge structure on $S$ is a tuple $H = (H_S, W, (\langle, \rangle_w)_w, H_O)$ consisting of a local system of $\mathbb{Z}$-free modules $H_S$ of finite rank on $S^{\log}$, an increasing filtration $W$ of $H_Q := Q \otimes H_S$, a nondegenerate $(-1)^w$-symmetric $Q$-bilinear form $\langle, \rangle_w$ on $gr^W_w$, a locally free $\mathcal{O}_S$-module $H_O$ on $S$, a specified isomorphism $\mathcal{O}^{\log}_S \otimes Z H_S \cong \mathcal{O}^{\log}_S \otimes \mathcal{O}_S H_O$ (log Riemann-Hilbert correspondence), and a specified decreasing filtration $FH_O$ of $H_O$ such that $FH_O$ and $H_O/FH_O$ are locally free. Put $F^p := \mathcal{O}^{\log}_S \otimes \mathcal{O}_S F^p H_O$. Then $\tau_* F^p = F^p H_O$. For each integer $w$, the orthogonality condition $\langle F^p(gr^W_w), F^q(gr^W_w)\rangle_w = 0$ $(p + q > w)$ is imposed.

A pre-graded polarized log mixed Hodge structure on $S$ is a graded polarized log mixed Hodge structure on $S$ if its pullback to each $s \in S$ is a graded polarized log mixed Hodge structure on $s$ in the following sense.

Let $(H_S, W, (\langle, \rangle_w)_w, H_O)$ be a pre-graded polarized log mixed Hodge structure on a log point $s$. It is a graded polarized log mixed Hodge structure if it satisfies the following three conditions.

1. (Admissibility). For each logarithm $N$ of the local monodromy of the local system $(H_R, W, (\langle, \rangle_w)_w)$, there exists a $W$-relative $N$-filtration $M(N,W)$.

2. (Griffiths transversality). For any integer $p$, $\nabla F^p \subset H^1 \otimes \mathcal{O}^{\log}_S \otimes F^{p-1}$ is satisfied, where $H^1$ is the sheaf of $\mathcal{O}^{\log}$-module of log differential 1-forms on $(s^{\log}, \mathcal{O}^{\log}_s)$, and $\nabla = d \otimes 1_{H_Z} : O^{\log} \otimes H_Z \to \omega^{1, \log}_s \otimes H_Z$ is the log Gauss-Manin connection.

3. (Positivity). For a point $t \in s^{\log}$ and a $C$-algebra homomorphism $a : \mathcal{O}^{\log}_s \to C$, define a filtration $F(a) := C \otimes_{\mathcal{O}^{\log}_s} F_t$ on $H_{C,t}$. Then, $(H_S(t), (\langle, \rangle_w)_w, F(a))$ is a polarized Hodge structure of weight $w$ in the usual sense if $a$ is sufficiently twisted, i.e., for $(q_j)_1 \leq j \leq n \subset M_s$ inducing generators of $M_s/\mathcal{O}^\times_s$, $|\exp(a(\log q_j))| \ll 1$ for any $j$.

1.4. Néron model for admissible normal function
We review some results from [KNU14, Theorem 1.3], [KNU13, III, Section 6.1] and
[KNU10, Section 8] adapted to the situation (5) in Introduction.

For a pure case $h^{p,q} = 1 \ (p + q = 3, p, q \geq 0)$ and $h^{p,q} = 0$ otherwise, a complete fan
is constructed in [KU09, Section 12.3]. For a mixed case $h^{p,q} = 1$ (the above $(p, q)$, plus
$(p, q) = (2, 2)$) and $h^{p,q} = 0$ otherwise, over the above fan, a weak fan of Néron model
for given admissible normal function is constructed in [KNU14, Theorem 3.1], and we
have a Néron model in the following sense.

Let $S \in \mathcal{B}(\log), U := S_{\text{triv}} \subset S$ (consisting of those points with trivial log structure),
$H_{(-1)}$ be a polarized variation of Hodge structure of weight $-1$ (Tate-twisted by 2 from
$\mathcal{H}$ in Introduction (5)) on $U$ and $L_{Q}$ be a local system of $Q$-vector spaces which is an
extension of $Q$ by $H_{(-1),Q}$. An admissible normal function over $U$ for $H_{(-1)}$ underlain
by the local system $L_{Q}$ can be regarded as an admissible variation of mixed Hodge
structure which is an extension of $Z$ by $H_{(-1)}$ and lies over local system $L_{Q}$.

For any given unipotent admissible normal function over $U$ as above, $H_{(-1)}$ and
$L_{Q}$ extend to a polarized log mixed Hodge structure on $S$ and a local system on $S^{\text{log}}$,
respectively, denoted by the same symbols, and there is a relative log manifold $J_{L_{Q}}$
over $S$ (cf. [KU09]) which is strict over $S$ (i.e., endowed with the pullback log structure
from $S$) and which represents the following functor on $B/S^{o}$ ($S^{o} \in \mathcal{B}$ is the underlying
space of $S$):

$S' \mapsto \{\text{LMH } H \text{ on } S' \text{ satisfying } H(\text{gr}_{w}^W) = H_{(w)}|_{S'} \text{ (} w = -1, 0 \text{) and (*) below}\}/\text{isom.}$

(*) Locally on $S'$, there is an isomorphism $H_{Q} \simeq L_{Q}$ on $(S')^{\text{log}}$ preserving $W$.

Here $H_{(w)}|_{S'}$ is the pullback of $H_{(w)}$ by the structure morphism $S' \to S^{o}$, and $S'$ is
endowed with the pullback log structure from $S$.

Put $H' := H_{(-1)}$. In the present case, we have $J_{L_{Q}} = \mathcal{E}_{\text{LMH}}^{1}(Z, H')$ by [KNU13,
Corollary 6.1.6]. This is the subgroup of $\tau_{*}(H'_{Q_{\text{log}}}/(F^{0} + H_{Z}'))$ restricted by admissibility
condition and log-point-wise Griffiths transversality condition ([KNU10, Section 8], cf.
1.3). Define $J_{L_{Q}}$ as the image of the composite map $J_{L_{Q}} \to \tau_{*}(H'_{Q_{\text{log}}}/(F^{0} + H_{Z}')) \to
\tau_{*}(H'_{Q_{\text{log}}}/(F^{-1} + H_{Z}))$. By using the polarization, we have a commutative diagram:

\[
\begin{align*}
J_{L_{Q}} &= \mathcal{E}_{\text{LMH}}^{1}(Z, H') \subset \tau_{*}(H'_{Q_{\text{log}}}/(F^{0} + H_{Z}')) \xrightarrow{\text{pol}} \tau_{*}((F^{0})^{*}/H_{Z}') \\
J_{L_{Q}} &\subset \tau_{*}(H'_{Q_{\text{log}}}/(F^{-1} + H_{Z}) \xrightarrow{\text{pol}} \tau_{*}((F^{1})^{*}/H_{Z}').
\end{align*}
\]

2. Quintic threefolds

In this section, we give a correspondence table of $A$-model for quintic threefold and
$B$-model for its mirror. This is a correction of our previous [U14, 3] by using $\hat{\Gamma}$-integral
structure of Iritani [I11].

2.1. Quintic threefold and its mirror

Let $V$ be a general quintic threefold in $\mathbb{P}^{4}$. 
Let \( V_{\psi} := \frac{1}{5} \sum_{j=1}^{5} x_j^5 - \psi \prod_{j=1}^{5} x_j = 0 \) \((\psi \in \mathbb{P}^1)\) be a pencil of quintics in \( \mathbb{P}^4 \). Let \( \mu_5 \) be the group consisting of the fifth roots of the unity in \( \mathbb{C} \). Then the group \( G := \{(a_j) \in (\mu_5)^5 | a_1 \ldots a_5 = 1\} \) acts on \( V_{\psi} \) by \( x_j \mapsto a_j x_j \). Let \( V_{\psi}^o \) be a crepant resolution of quotient singularity of \( V_{\psi}/G \) (cf. [MW09]). Divide further by the action \((x_1, \ldots, x_5) \mapsto (a^{-1} x_1, x_2, \ldots, x_5) \) \((a \in \mu_5)\).

### 2.2. Picard-Fuchs equation on the mirror \( V^o \)

Let \( \Omega \) be a 3-form on \( V_{\psi}^o \) with a \( \log \) pole over \( \psi = \infty \) induced from

\[
\left( \frac{5}{2\pi i} \right)^3 \text{Res}_{V_{\psi}} \left( \frac{\psi}{f} \sum_{j=1}^{5} (-1)^{j-1} x_j dx_1 \wedge \cdots \wedge \hat{dx_j} \wedge \cdots \wedge dx_5 \right).
\]

Let \( z := 1/(5\psi)^5 \) and \( \theta := zd/dz \). Let

\[
\mathcal{L} := \theta^4 - 5z(5\theta + 1)(5\theta + 2)(5\theta + 3)(5\theta + 4)
\]

be the Picard-Fuchs differential operator for \( \Omega \), i.e., \( \mathcal{L} \Omega = 0 \) via the Gauss-Manin connection \( \nabla \).

At \( z = 0 \), the Picard-Fuchs differential equation \( \mathcal{L} y = 0 \) has the indicial equation \( \rho^4 = 0 \) \((\rho \text{ is indeterminate})\), i.e., maximally unipotent. By the Frobenius method, we have a basis of solutions \( y_j(z) \) \((0 \leq j \leq 3)\) as follows. Let

\[
\tilde{y}(-z; \rho) := \sum_{n=0}^{\infty} \frac{\prod_{m=1}^{5n}(5\rho + m)}{\prod_{m=1}^{n}(\rho + m)^5} (-z)^{n+\rho}
\]

be a solution of \( \mathcal{L}(\tilde{y}(-z; \rho)) = \rho^4 (-z)^{\rho} \), and let

\[
y_j(z) := \frac{1}{j!} \frac{\partial^j \tilde{y}(-z; \rho)}{\partial \rho^j} \bigg|_{\rho=0}
\]

be the Taylor expansion at \( \rho = 0 \). Then, \( y_j \) \((0 \leq j \leq 3)\) form a basis of solutions for the equation \( \mathcal{L} y = 0 \). We have

\[
y_0 = \sum_{n=0}^{\infty} \frac{(5n)!}{(n!)^5} z^n,
\]

\[
y_1 = y_0 \log z + 5 \sum_{n=1}^{\infty} \frac{(5n)!}{(n!)^5} \left( \sum_{j=n+1}^{5n} \frac{1}{j} \right) z^n.
\]

Define the canonical parameters by \( t := y_1/y_0, \ u := t/2\pi i \), and the canonical coordinate by \( q := e^t = e^{2\pi i u} \) which is a specific chart of the log structure given by the divisor \((z = 0)\) of \( \mathbb{P}^1 \) and gives a mirror map.
$y_0$ is holomorphic in $z$ and invertible at $z = 0$. Write $z = z(q)$ which is holomorphic in $q$. Then we have
\[ \log z = 2\pi i u - \frac{5}{y_0(z(q))} \sum_{n=1}^{\infty} \frac{(5n)!}{(n!)^5} \left( \sum_{j=n+1}^{5n} \frac{1}{j} \right) z(q)^n. \]

The Gauss-Manin potential of $V^\circ_z$ is
\[ \Phi_{GM}^{V^\circ} = \frac{5}{2} \left( \frac{y_1}{y_0} \frac{y_2}{y_0} - \frac{y_3}{y_0} \right). \]

Let $\tilde{\Omega} := \Omega/y_0$. Then, the Yukawa coupling at $z = 0$ is
\[ Y := -\int_{V^\circ} \tilde{\Omega} \wedge \nabla_\theta \nabla_\theta \nabla_\theta \tilde{\Omega} = \frac{5}{(1 + 5^5 z)y_0(z)^2}. \]

2.3. A-model of quintic $V$

Let $V$ be a general quintic hypersurface in $\mathbb{P}^4$. Let $T^2 = H$ be the cohomology class of a hyperplane section of $V$ in $\mathbb{P}^4$, $K(V) = R_{>0} T^2$ be the Kähler cone of $V$, and $u$ be the coordinate of $CT^2$. Put $t := 2\pi i u$. A complexified Kähler moduli is defined as
\[ \mathcal{K}\mathcal{M}(V) := \left( H^2(V, \mathbb{R}) + iK(V) \right)/H^2(V, \mathbb{Z}) \sim \Delta^*, \quad uT^2 \mapsto q := e^{2\pi i u}. \]

Let $C \in H_2(V, \mathbb{Z})$ be the homology class of a line on $V$, and $T^1 \in H^4(V, \mathbb{Z})$ be the cohomology class Poincaré duality isomorphic to $C$.

For $\beta = dC \in H_2(V, \mathbb{Z})$, define $q^\beta := q^{\int_{\beta} T^1} = q^d$. The Gromov-Witten potential of $V$ is defined as
\[ \Phi_{GW}^V := \frac{1}{6} \int_V (t T^2)^3 + \sum_{0 \neq \beta \in H_2(V, \mathbb{Z})} N_d q^\beta = \frac{5 t^3}{6} + \sum_{d > 0} N_d q^d. \]

Here the Gromov-Witten invariant $N_d$ is
\[ N_d := \int_{\overline{M}_{0,0}(\mathbb{P}^4, d)} c_{5d+1} (\pi_1^* e_1^* \mathcal{O}_{\mathbb{P}^4}(5)). \]

Note that $N_d = 0$ if $d \leq 0$. Let $N_d = \sum_{k|d} n_{d/k} k^{-3}$. Then $n_{d/k}$ is the instanton number.

2.4. Integral structure

Let $S^*$ be $\mathcal{K}\mathcal{M}(V)$ for A-model of $V$ and $\mathcal{M}(V^\circ)$ for B-model for $V^\circ$, and let $S$ be $\overline{\mathcal{K}\mathcal{M}}(V)$ for A-model and $\overline{\mathcal{M}}(V^\circ)$ for B-model (see 2.2, 2.3). Endow $S$ with the log structure associated to the divisor $S \setminus S^*$. 
The B-model variation of Hodge structure $\mathcal{H}^{V^o}$ is the usual variation of Hodge structure arising from the smooth projective family $f : X \to S^*$ of the quintic mirrors over a punctured neighborhood of the maximally unipotent monodromy point $p_0$. Its integral structure is the usual one $\mathcal{H}^{V^o}_{Z} = R^3f_*\mathcal{Z}$. This is compatible with the monodromy weight filtration $M$ around $p_0$. Define $M_{k, Z} := M_k \cap \mathcal{H}^{V^o}_Z$ for all $k$.

For the A-model $\mathcal{H}^V$ on $S^*$, the locally free sheaf on $S^*$, the Hodge filtration, and the monodromy weight filtration $M$ around the large radius point $q_0$ are given by $\mathcal{H}_O^V := \mathcal{O}_{S^*} \otimes (\bigoplus_{0 \leq p \leq 3} H^{2p}(V))$, $F^p := \mathcal{O}_{S^*} \otimes H^{2(3-p)}(V)$, and $M_{2p} := H^{2(3-p)}(V)$, respectively. Iritani defined $\hat{\Gamma}$-integral structure in more general setting in [I11, Definition 3.6]. In the present case, it is characterized as follows. Let $H$ and $C$ be a hyperplane section and a line on $V$, respectively. Then, in the present case, a basis of the $\hat{\Gamma}$-integral structure is given by $\{s(\mathcal{E}) \mid \mathcal{E} \in \mathcal{O}_V, \mathcal{O}_H, \mathcal{O}_C, \mathcal{O}_{pt}\}$ [ibid, Example 6.18], where $s(\mathcal{E})$ is a unique $\nabla^{even}$-flat section satisfying

$$s(\mathcal{E}) \sim (2\pi i)^{-3}e^{-2\pi i u H} \cdot \hat{\Gamma}(T_V) \cdot (2\pi i)^{\deg/2} \text{ch}(\mathcal{E})$$

at the large radius point $q_0$. Here, for the Chern roots $c(T_V) = \prod_{j=1}^{3}(1 + \delta_j)$, the Gamma class $\hat{\Gamma}(T_V)$ is defined by

$$\hat{\Gamma}(T_V) := \prod_{j=1}^{3}(1 + \delta_j) \exp(-\gamma c_1(V) + \sum_{k \geq 2}(-1)^k(k-1)\zeta(k) \text{ch}_k(T_V))$$

$$= \exp(\zeta(2) \text{ch}_2(T_V) - 2\zeta(3) \text{ch}_3(T_V))$$

where $\gamma$ is the Euler constant, and $\deg|_{H^{2p}(V)} := 2p$. The important point is that this class $\hat{\Gamma}(T_V)$ plays the role of a "square root" of the Todd class in Hirzebruch-Riemann-Roch ([109, 1], [I11, 1, (13)]). Denote this $\hat{\Gamma}$-integral structure by $\mathcal{H}_Z^V$. This is compatible with the monodromy weight filtration $M$ and we define $M_{k, Z} := M_k \cap \mathcal{H}^{V^o}_Z$ for all $k$. For a direct definition of $\hat{\Gamma}$-integral structure, see [I11, Definition 3.6].

In both A-model case and B-model case, the integral structures $\mathcal{H}_Z^V$ and $\mathcal{H}_Z^{V^o}$ on $S^*$ extend to the local systems of $\mathbb{Z}$-modules over $S^{log}$ ([O03], [KU09, Proposition 2.3.5]), still denoted $\mathcal{H}_Z^V$ and $\mathcal{H}_Z^{V^o}$, respectively.

Consider a diagram:

$$\begin{array}{ccc}
\tilde{S}^{log} := (R \times i(0, \infty))^r & \supset & \tilde{S}^* := (R \times i(0, \infty))^r \\
\downarrow & & \downarrow \\
S^{log} & \supset & S^* \\
\tau \downarrow & & \\
S & & 
\end{array}$$

The coordinate $u$ of $\tilde{S}^*$ extends over $\tilde{S}^{log}$. Fix base points as $u_0 = 0 + i\infty \in \tilde{S}^{log} \mapsto b := \bar{0} + i\infty \in S^{log} \mapsto q = 0 \in S$, where $q = 0$ corresponds to $q_0$ for A-model and $p_0$
for B-model. Note that fixing a base point $u = u_0$ on $\tilde{S}^{\log}$ is equivalent to fixing a base point $b$ on $S^{\log}$ and also a branch of $(2\pi i)^{-1} \log q$.

Let $B := H^V_Z(u_0) = H^V_Z(b)$ for A-model and $B := H^V_Z(u_0) = H^V_Z(b)$ for B-model.

2.5. Correspondence table

In this section, we complete the approximation in the previous paper [U14]. These results will be used in Section 3.

We use (1) and (2) in Introduction. Put $\Phi := \Phi_{GW}^V = \Phi_{GM}^V$.

(1A) Polarization of A-model of $V$.

$$S(\alpha, \beta) := (-1)^p \int_{V} \alpha \cup \beta \quad (\alpha \in H^{p,p}(V), \beta \in H^{3-p,3-p}(V)).$$

(1B) Polarization of B-model of $V^\circ$.

$$Q(\alpha, \beta) := (-1)^{(3-1)/2} \int_{V^\circ} \alpha \cup \beta = - \int_{V^\circ} \alpha \cup \beta \quad (\alpha, \beta \in H^{3}(V^\circ)).$$

(2A) $Z$-basis compatible with monodromy weight filtration.

Let $B := H^V_Z(u_0) = H^V_Z(b)$. Then we have a basis $b^0, b^1, b^2, b^3$ of $B$ compatible with the monodromy weight filtration $M$ [111, Example 6.18].

(2B) $Z$-basis compatible with monodromy weight filtration.

Let $B := H^V_Z(u_0) = H^V_Z(b)$. Then we have a basis $b^0, b^1, b^2, b^3$ of $B$ compatible with the monodromy weight filtration $M$ [ibid].

For both cases (2A) and (2B), we regard $B$ as a constant sheaf endowed with $M$ on $S^{\log}$ and also on $S$.

(3A) Specified sections inducing $Z$-basis of $gr^M$ for A-model of $V$.

$$T^3 := 1 \in H^0(V, Z), \quad T^2 := H \in H^2(V, Z),$$
$$T^1 := C \in H^4(V, Z), \quad T^0 := [pt] \in H^6(V, Z),$$

where $H$ is a hyperplane section of $V$ and $C$ is a line on $V$. Then $S(T^3, T^0) = 1$ and $S(T^2, T^1) = -1$. Hence $T^3, T^2, -T^0, T^1$ form a symplectic base for $S$ in (1A).

(3B) Specified sections inducing $Z$-basis of $gr^M$ for B-model of $V^\circ$.

We use Deligne decomposition [D97]. We consider $B$ in (2B) as a constant sheaf on $S^{\log}$. We have locally free $\mathcal{O}_S$-submodules $M_{2p} := \tau_*(\mathcal{O}_S^{\log} \otimes Z M_{2p}B)$ and $\mathcal{F}^p$ of $\tau_*(\mathcal{O}_S^{\log} \otimes Z B) = \mathcal{O}_S \otimes Z B$. The mixed Hodge structure of Hodge-Tate type $(\mathcal{M}, \mathcal{F})$ has decomposition:

$$\mathcal{O}_S \otimes Z B = \bigoplus_{p} I^{p,p}, \quad I^{p,p} := M_{2p} \cap \mathcal{F}^p \sim gr^\mathcal{M}_{2p}.$$
Transporting the basis $b^p$ ($0 \leq p \leq 3$) of $B$ in (2B), regarded as sections of the constant sheaf $B$ on $S^\log$, via isomorphism

$$I^{p,p} \sim O_S \otimes_{\mathbb{Z}} \text{gr}_{2p} B$$

we define sections $e^p \in I^{p,p}$ ($0 \leq p \leq 3$). Then $e^3, e^2, -e^0, e^1$ form a symplectic basis for $Q$ in (1B).

Note that $e^3 = \tilde{\Omega}$.

(4A) $A$-model connection $\nabla = \nabla^{even}$ of $V$.

$$\nabla_\theta T^0 := 0, \quad \nabla_\theta T^1 := T^0,$$

$$\nabla_\theta T^2 := \frac{1}{(2\pi i)^3} \frac{d^3 \Phi}{du^3} T^1 = \left( 5 + \frac{1}{(2\pi i)^3} \frac{d^3 \Phi_{\text{hol}}}{du^3} \right) T^1,$$

$$\nabla_\theta T^3 := T^2.$$

$\nabla$ is flat, i.e., $\nabla^2 = 0$.

(4B) $B$-model connection $\nabla = \nabla^{GM}$ of $V^\circ$.

$$\nabla_\theta e^0 = 0, \quad \nabla_\theta e^1 = e^0,$$

$$\nabla_\theta e^2 = \frac{1}{(2\pi i)^3} \frac{d^3 \Phi}{du^3} e^1 = Y e^1 = \frac{5}{(1 + 5^5) y_0(z)^2} \left( \frac{q}{z} \frac{dz}{dq} \right)^3 e^1,$$

$$\nabla_\theta e^3 = e^2.$$

(5A) $\nabla$-flat $\mathbb{Z}$-basis for $H^V$.

$$s^0 := T^0,$$

$$s^1 := T^1 - uT^0,$$

$$s^2 := T^2 - \left( \frac{1}{(2\pi i)^3} \frac{\partial^2 \Phi}{\partial u^2} - \frac{11}{2} \right) T^1 + \left( \frac{1}{(2\pi i)^3} \frac{\partial \Phi}{\partial u} - \frac{11}{2} u - \frac{25}{12} \right) T^0,$$

$$s^3 := T^3 - uT^2 + \left( \frac{1}{(2\pi i)^3} \left( \frac{u \partial^2 \Phi}{\partial u^2} - \frac{\partial \Phi}{\partial u} \right) - \frac{25}{12} T^1 \right.$$

$$- \left. \left( \frac{1}{(2\pi i)^3} \left( \frac{\partial \Phi}{\partial u} - 2 \Phi \right) - \frac{25i}{\pi^3} \zeta(3) \right) T^0. \right)$$

Then $s^3, s^2, -s^0, s^1$ form a symplectic basis for $S$ in (1A).

(5B) $\nabla$-flat $\mathbb{Z}$-basis for $H^{V^\circ}$.

$$s^0 := e^0,$$

$$s^1 := e^1 - ue^0,$$

$$s^2 := e^2 - \left( \frac{1}{(2\pi i)^3} \frac{\partial^2 \Phi}{\partial u^2} - \frac{11}{2} \right) e^1 + \left( \frac{1}{(2\pi i)^3} \frac{\partial \Phi}{\partial u} - \frac{11}{2} u - \frac{25}{12} \right) e^0,$$

$$s^3 := e^3 - ue^2 + \left( \frac{1}{(2\pi i)^3} \left( \frac{u \partial^2 \Phi}{\partial u^2} - \frac{\partial \Phi}{\partial u} \right) - \frac{25}{12} \right) e^1$$

$$- \left( \frac{1}{(2\pi i)^3} \left( \frac{\partial \Phi}{\partial u} - 2 \Phi \right) - \frac{25i}{\pi^3} \zeta(3) \right) e^0.$$
Then \( s^3, s^2, -s^0, s^1 \) form a symplectic basis for \( Q \) in (1B).

(6A) **Expression of the \( T^p \) by the \( s^p \).**

It is computed that \( T^p \) are written by the \( \nabla \)-flat \( \mathbb{Z} \)-basis \( s^p \) of \( \mathcal{H}_Z^V \) as follows.

\[
T^0 = s^0, \\
T^1 = s^1 + us^0, \\
T^2 := s^2 + \left( \frac{1}{(2\pi i)^3} \frac{\partial^2 \Phi}{\partial u^2} - \frac{11}{2} \right) s^1 + \left( \frac{1}{(2\pi i)^3} \left( u \frac{\partial^2 \Phi}{\partial u^2} - \frac{\partial \Phi}{\partial u} \right) + \frac{25}{12} \right) s^0, \\
T^3 = s^3 + us^2 + \left( \frac{1}{(2\pi i)^3} \frac{\partial \Phi}{\partial u} - \frac{11}{2} u + \frac{25}{12} \right) s^1 \\
+ \left( \frac{1}{(2\pi i)^3} \left( u \frac{\partial \Phi}{\partial u} - 2\Phi \right) + \frac{25}{12} u - \frac{25i}{\pi^3} \zeta(3) \right) s^0.
\]

Note that the section \( 1 = T^3 \) varies with respect to the lattice \( \mathcal{H}_Z^V \) as above while the section \( [pt] = T^0 = s^0 \) does not.

(6B) **Expression of the \( e^p \) by the \( s^p \).**

It is computed that \( e^p \) are written by the \( \nabla \)-flat \( \mathbb{Z} \)-basis \( s^p \) of \( \mathcal{H}_Z^{V^o} \) as follows.

\[
e^0 = s^0, \\
e^1 = s^1 + us^0, \\
e^2 := s^2 + \left( \frac{1}{(2\pi i)^3} \frac{\partial^2 \Phi}{\partial u^2} - \frac{11}{2} \right) s^1 + \left( \frac{1}{(2\pi i)^3} \left( u \frac{\partial^2 \Phi}{\partial u^2} - \frac{\partial \Phi}{\partial u} \right) + \frac{25}{12} \right) s^0, \\
e^3 = s^3 + us^2 + \left( \frac{1}{(2\pi i)^3} \frac{\partial \Phi}{\partial u} - \frac{11}{2} u + \frac{25}{12} \right) s^1 \\
+ \left( \frac{1}{(2\pi i)^3} \left( u \frac{\partial \Phi}{\partial u} - 2\Phi \right) + \frac{25}{12} u - \frac{25i}{\pi^3} \zeta(3) \right) s^0.
\]

Note that the normalized holomorphic 3-form \( \tilde{\Omega} = \Omega/y_0 = e^3 \) varies with respect to the lattice \( \mathcal{H}_Z^{V^o} \) as above, while the section \( e^0 = s^0 \) does not.

**Idea of proof of (4A) and (4B).** We prove (4B). (4A) follows by mirror symmetry theorems (1) and (2) in Introduction.

We improve the proof of [CoK99, Prop. 5.6.1] carefully by a log Hodge theoretic understanding of the relation among a constant sheaf and a local system on \( S^{\log} \), of the canonical extension of Deligne on \( S \), and of the Deligne decomposition.

**Idea of proofs of (5A), (5B), (6A) and (6B).** In [I11, Introduction] (cf. 2.4), the asymptotic condition in the large radius limit is stated for the flat integral section corresponding to \( \mathcal{E} = \mathcal{O}_V \in K(V) \) in the situation (5A). Up to Tate twists, this condition coincides with the one in [CDGP91, (5.5)] stated in the situation (6A). By the mirror symmetry in [I11] (cf. (2) in Introduction), this condition is interpreted in the situation (6B). Our previous results in [U14, Sections 3.5–3.6] are insufficient (see Remark...
below). In order to complete them, we compute here higher approximations in the situation (6B). The result in the situation (5B) is a linear algebraic solution of this.

Remark. The author was pointed out by Hiroshi Iritani that the definitions and the descriptions of integral structures in [U14, 3.5, 3.6] are insufficient. Actually, they were the first approximations of integral structures by means of $\text{gr}^M$, and the second proof in [ibid, 3.9] works well even in this approximation.

3. Discussions on geometries for (5) in Introduction

We discuss here the relation with geometries and local systems considered in [W07] and [MW09]. Forgetting Hodge structures, we consider only local systems corresponding to the monodromy of integral periods and tensions.

Let $V_\psi$ and $V_\psi^\circ$ be a quintic threefold and its mirror from 2.1. Let $S$ be a small neighborhood in the $z$-plane ($z$ in 2.2) of the maximal unipotent monodromy point $p_0$ endowed with the log structure associated to the divisor $p_0$.

We first consider B-model. Let the setting be as in [MW09, 4]. For $z \neq 0$ near 0, i.e., near $p_0$, let $V_\psi^\circ$ be the mirror quintic and $C_{+,z} \cup C_{-,z}$ be the disjoint union of smooth rational curves on $V_\psi^\circ$ coming from the two conics contained in $V_\psi \cap \{x_1 + x_2 = x_3 + x_4 = 0\} \subset \mathbf{P}^4(\mathbf{C})$. From the relative homology sequence for $(V_\psi^\circ, (C_{+,z} \cup C_{-,z}))$, we have

$$0 \to H_2(V_\psi^\circ; \mathbf{Z}) \to H_2(V_\psi^\circ; (C_{+,z} \cup C_{-,z}); \mathbf{Z}) \to \mathbf{Z}([C_{+,z}] - [C_{-,z}]) \to 0,$$

where $\mathbf{Z}([C_{+,z}] - [C_{-,z}])$ is $\text{Ker}(H_2(C_{+,z} \cup C_{-,z}); \mathbf{Z}) \to H_2(V_\psi^\circ; \mathbf{Z})$. The monodromy $T_\infty$ around $p_0$ interchanges $C_{+,z}$ and $C_{-,z}$.

Respecting the sequence (1), we take a family of cycles Poincaré duality isomorphic to the flat integral basis $s^p$ ($0 \leq p \leq 3$) in 2.5 (5B) and a family of chains joining from $C_{-,z}$ to $C_{+,z}$ (a choice up to integral cycles and up to half twists), and over them integrate the family of 3-forms $\Omega(z)$ with log pole over $z = 0$ ($z$ in the punctured disc in the $z$-plan) in 2.2, then we have a family of vectors $(\eta_0, \eta_1, \eta_2, \eta_3, T)$ consisting of periods and a tension. This corresponds to the data in [W07], [MW09]. Since $T_\infty(T) = -(T + \eta_1 + \eta_0)$ by [W07, (3.14)], we find $T + \frac{1}{2} \eta_1 + \frac{1}{4} \eta_0 = \frac{15}{2 \pi^2} \tau$ is an eigenvector of the monodromy $T_\infty$ with eigenvalue $-1$.

The family of sequences (1) ($z \neq 0$) forms an exact sequence of local systems of $\mathbf{Z}$-modules. To make the monodromy of this system unipotent, we take a double cover $z^{1/2} \mapsto z$. Let $S$ be a neighborhood disc of $p_0$ in the $z^{1/2}$-plane endowed with log structure associated to the divisor $p_0$ in $S$, and let $S^{\log}$ be as in 1.2. Let $S^*$ be the punctured disc $S \setminus \{p_0\}$. Pull back the above local system to $S^*$ and then extend it over $S^{\log}$.

Applying Tate twist $(-3)$ and Poincaré duality isomorphism to the left and the right ends of this exact sequence, we have a local system $L'$ over $S^{\log}$ which is an extension of $\mathbf{Z}(-2)$ by $\mathcal{H}_{\mathbf{Z}}$:

$$0 \to \mathcal{H}_{\mathbf{Z}} \to L' \to \mathbf{Z}(-2) \to 0.$$
Let $1 \in \mathbb{Z} \cong gr_{4}^{W} \mathbb{Z}(-2)$, take a lifting $1_{Z} := 1 - (T/\eta_{0})s^{0}$ in $L'$ of 1, and extend $\nabla$ on $\mathcal{H}_{Z}$ over $L'$ by $\nabla(1_{Z}) = 0$. We look for a $T_{\infty}^{-1}$-invariant $\nabla$-flat element associated to $1_{Z}$. This is computed as $1_{Z}^{sp1} := 1_{Z} - (s^{1}/2)$, and we know that $L'$ coincides with $H_{Z}$ in (5) in Introduction.

For the relative monodromy weight filtration $M = M(N,W)$, we see that $1_{Z} \in M_{4}$ and $s^{1} \in M_{2}$ are the smallest filters containing the elements in question. Taking the graded quotients by $M$ of the sequence (2), we have

\[\operatorname{gr}_{6}^{M} \mathcal{H}_{Z} \twoheadrightarrow \operatorname{gr}_{6}^{M} L',\]
\[0 \to \operatorname{gr}_{4}^{M} \mathcal{H}_{Z} \to \operatorname{gr}_{4}^{M} L' \to \mathbb{Z}(-2) \to 0,\]
\[0 \to \operatorname{gr}_{2}^{M} \mathcal{H}_{Z} \to \operatorname{gr}_{2}^{M} L' \to (2\text{-torsion}) \to 0,\]
\[\operatorname{gr}_{0}^{M} \mathcal{H}_{Z} \twoheadrightarrow \operatorname{gr}_{0}^{M} L'.\]

The 2-torsion in the third sequence of (3) corresponds to a half twist of chains from $C_-$ to $C_+$. Standing on a half integral point and looking at the integral points nearby, we have two orientations. These correspond to the two orientations of a half twist of the chains, and also correspond to $T_{\pm} := \pm\left(\frac{5}{2}T - \frac{9}{4}\right) - \frac{2}{3}$ in [W07]. $T_{-}$ is different from $-T_{+}$ by the complementary half twist, i.e., $T_{+} + T_{-} = -\eta_{1}$.

For $A$-model, we consider the setting in [W07, 2.1]. Let $V = V_{\psi}$ with $\psi = 0$ from 2.1 be a Fermat quintic threefold in $\mathbb{P}^{4}(\mathbb{C})$ and $L_{g} := V \cap \mathbb{P}^{4}(\mathbb{R})$ be a Lagrangian submanifold of its real locus. From the exact sequence of relative homology for $(V, L_{g})$,

\[H_{6}(V;\mathbb{Z}) \twoheadrightarrow H_{6}(V, L_{g};\mathbb{Z}),\]
\[0 \to H_{4}(V;\mathbb{Z}) \to H_{4}(V, L_{g};\mathbb{Z}) \to H_{3}(L_{g};\mathbb{Z}) \to 0,\]
\[0 \to H_{2}(V;\mathbb{Z}) \to H_{2}(V, L_{g};\mathbb{Z}) \to H_{1}(L_{g};\mathbb{Z}) \to 0,\]
\[H_{0}(V;\mathbb{Z}) \twoheadrightarrow H_{0}(V, L_{g};\mathbb{Z}).\]

Let $H' = H_{*}(V)$, $H = H_{*}(V, L_{g})$ and $H'' = H_{*}(L_{g})$, and let

\[H_{\text{even}}(V) := \bigoplus_{0 \leq p \leq 3} (H')_{2p}, \quad H_{\text{even}}(V, L_{g}) := \bigoplus_{0 \leq p \leq 3} H_{2p}, \quad H_{\text{odd}}(L_{g}) := \bigoplus_{0 \leq p \leq 1} (H'')_{2p+1}.\]

Then we have an exact sequence

\[0 \to H_{\text{even}}(V) \to H_{\text{even}}(V, L_{g}) \to H_{\text{odd}}(L_{g}) \to 0.\]

The weight filtration $W$ is given by $W_{3}H_{\text{even}}(V, L_{g}) := H_{\text{even}}(V)$, $W_{4}H_{\text{even}}(V, L_{g}) := H_{\text{even}}(V, L_{g})$, and the relative monodromy weight filtration $M = M(N,W)$ is given by $M_{2p}H_{\text{even}}(V, L_{g}) = H_{\leq 2p}(V, L_{g})$ ($0 \leq p \leq 3$).

In the above setting, the projection from $\mathbb{P}^{4}(\mathbb{R})$ to the real hyperplane $\{x_{5} = 0\} = \mathbb{P}^{3}(\mathbb{R})$ with center $(0,0,0,0,1)$ induces a homeomorphism $L_{g} \simeq \mathbb{P}^{3}(\mathbb{R})$. Therefore there are two choices of flat $U(1)$ connections on $L_{g}$. Denote $L_{g}$ endowed with these...
structures by $Lg_{\pm}$. Morrison-Walcher [MW09, 3] explain the relation between $Lg_{\pm}$ for A-model of $V$ and $C_{\pm}$ for B-model of $V^\circ$.

After pulling back to the double cover $z^{1/2} \to z$ ($z \neq 0$) and extending over $S^{\log}$, the sequence for A-model (5) and the sequence for B-model (2), and the set of sequences for A-model (4) and the set of sequences for B-model (3), respectively, seem to correspond in mirror symmetry. By Poincaré duality isomorphisms, $H^{even}(V) = H^{even}(V)(-3)$ and $H^{even}(Lg) \simeq H_{odd}(Lg)$.

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