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京都大学
Semi-classical Asymptotics for the Partition Function of an Abstract Bose Field Model

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Semi-classical asymptotics for the partition function of an abstract Bose field model is considered.

Keywords: semi-classical asymptotics, Bose field, partition function, second quantization, Fock space.

I. INTRODUCTION

In quantum mechanics, in which a physical constant $\hbar := h/2\pi$ (h : the Planck constant) plays an important role, the limit $\hbar \to 0$ for various quantities (if it exists) is called the classical limit. Trace formulas in the abstract boson Fock space and the classical limit for the trace $Z(\beta\hbar)$ (the partition function) of the heat semigroup of a perturbed second quantization operator were derived by Arai [2], where $\beta > 0$ denotes the inverse temperature. Generally speaking, the classical limit is regarded as the zero-th order approximation in $\hbar$. From this point of view, it is interesting to derive higher order asymptotics of various quantities in $\hbar$. Such asymptotics are called semi-classical asymptotics. In this paper the asymptotic formula for $Z(\beta\hbar)$ is stated, which is derived in [1].

II. A CLASSICAL LIMIT IN THE ABSTRACT BOSON FOCK SPACE

In this section we review a classical limit for the trace of a perturbed second quantization operator and some fundamental facts related to it.

Let $\mathcal{H}$ be a real separable Hilbert space, and $A$ be a strictly positive self-adjoint operator acting in $\mathcal{H}$. We denote by $\{\mathcal{H}_s(A)\}_{s \in \mathbb{R}}$ the Hilbert scale associated with $A$ [3]. For all $s \in \mathbb{R}$, the dual space of $\mathcal{H}_s(A)$ can be naturally identified with $\mathcal{H}_{-s}(A)$. 
We denote by $\mathcal{J}_{1}(\mathcal{H})$ the ideal of the trace class operators on $\mathcal{H}$. Let $\gamma > 0$ be fixed. Throughout this paper, we assume the following.

Assumption I. $A^{\gamma-1} \in \mathcal{J}_{1}(\mathcal{H})$.

Under Assumption I, the embedding mapping of $\mathcal{H}$ into

$$E := \mathcal{H}_{\gamma}(A)$$

is Hilbert-Schmidt. Hence, by Minlos' theorem, there exists a unique probability measure $\mu$ on $(E, \mathcal{B})$ such that the Borel field $\mathcal{B}$ is generated by $\{\phi(f) | f \in \mathcal{H}_{\gamma}(A)\}$ and

$$\int_{E} e^{i\phi(f)} d\mu(\phi) = e^{-\|f\|_{\mathcal{H}}^{2}/2}, \quad f \in \mathcal{H}_{\gamma}(A),$$

where $\|\cdot\|_{\mathcal{H}}$ denotes the norm of $\mathcal{H}$.

The complex Hilbert space $L^{2}(E, d\mu)$ is canonically isomorphic to the boson Fock space over $\mathcal{H}$, which is called the $Q$-space representation of it [3]. We denote by $d\Gamma(A)$ the second quantization of $A$ and set

$$H_{0} = d\Gamma(A).$$

Then for all $\beta > 0$, $e^{-\beta H_{0}} \in \mathcal{J}_{1}(L^{2}(E, d\mu))$.

Definition 2.1. A mapping $V$ of a Banach space $X$ into a Banach space $Y$ is said to be polynomially continuous if there exists a polynomial $P$ of two real variables with positive coefficients such that

$$\|V(\phi) - V(\psi)\| \leq P(\|\phi\|, \|\psi\|)\|\phi - \psi\|, \quad \phi, \psi \in X.$$  

Let $V$ be a real valued function on $E$. Throughout this paper, we assume the following.

Assumption II. The function $V$ is bounded from below, 3-times Fréchet differentiable, and $V, V', V'', V'''$ are polynomially continuous.

For $\hbar > 0$, we define $V_{\hbar}$ by

$$V_{\hbar}(\phi) := V(\sqrt{\hbar} \phi), \quad \phi \in E.$$  

and set

$$H_{\hbar} := H_{0} + \frac{1}{\hbar}V_{\hbar},$$

where $+ \in$ denotes the quadratic form sum.

Under Assumption I, II, for all $\beta > 0$, $e^{-\beta H_{\hbar}} \in \mathcal{J}_{1}(L^{2}(E, d\mu))$ [2].

Theorem 2.2. [2]. Let $\beta > 0$. Then

$$\lim_{\hbar \to 0} \frac{\text{Tr} e^{-\beta H_{\hbar}}}{\text{Tr} e^{-\beta H_{0}}} = \int_{E} \exp \left( -\beta V \left( \sqrt{\frac{\beta}{2}} A^{-1/2} \phi \right) \right) d\mu(\phi).$$
III. A CLASS OF LOCALLY CONVEX SPACES

In this section we introduce a class of locally convex spaces, which gives a general framework for the asymptotic analysis discussed in this paper.

We denote by $\mathbb{R}_+$ the set of the nonnegative real numbers.

**Definition 3.1.** A mapping $f$ from $\mathbb{R}_+$ to a locally convex space $X$ is said to be locally bounded if for all $\delta > 0$ and every continuous seminorm $p$ on $X$,

$$p_\delta(f) := \sup_{0 \leq \epsilon \leq \delta} p(f(\epsilon)) < \infty.$$ 

We denote by $(X_{\mathbb{R}_+})_{1b}$ the linear space of the locally bounded mappings from $\mathbb{R}_+$ to $X$. The topology defined by the seminorms $\{p_\delta\}$ turns $(X_{\mathbb{R}_+})_{1b}$ into a locally convex space. If $X$ is a Fréchet space, $(X_{\mathbb{R}_+})_{1b}$ is a Fréchet space.

Let $\{E_n\}_{n \in \mathbb{N}}$ be a family of Banach spaces with the property that $E_{n+1} \subset E_n$, $\|\phi\|_n \leq \|\phi\|_{n+1}$, $\phi \in E_{n+1}$, for all $n \in \mathbb{N}$, where $\|\cdot\|_n$ denotes the norm of $E_n$. Then, the topology defined by the norms $\{\|\cdot\|_n\}_{n \in \mathbb{N}}$ turns $\bigcap_{n \in \mathbb{N}} E_n$ into a Fréchet space.

**Definition 3.2.** Let $f$ be a mapping from $\mathbb{R}_+$ to $\bigcap_{p \in \mathbb{N}} L^p(X, dP; Y)$. We say that $f$ is in $\bigcap_{p \in \mathbb{N}} L^p(X, dP; Y)_{\text{u.i.}}$ if and only if for each $\delta > 0$, there exists a nonnegative function $g \in \bigcap_{p \in \mathbb{N}} L^p(X, dP)$ such that

$$\sup_{0 \leq \epsilon \leq \delta} \|f(\epsilon)(x)\|_Y \leq g(x),$$ 

$P$-a.e.x.

The set $\bigcap_{p \in \mathbb{N}} L^p(X, dP; Y)_{\text{u.i.}}$ is a linear subspace of $\bigcap_{p \in \mathbb{N}} L^p(X, dP; Y)_{\text{1.b.}}$. In what follows, we omit $x$ in $f(\epsilon)(x)$.

Let $X_1, \cdots, X_n$ and $Z$ be non-empty sets and $G$ be a real-valued function on $X_1 \times \cdots \times X_n$ and $F_j$ be a mapping from $Z$ to $X_j$, $j = 1, \cdots, n$. We define $G(F_1, \cdots, F_n)$, the real-valued function on $Z$, by

$$G(F_1, \cdots, F_n)(z) = G(F_1(z), \cdots, F_n(z)), \quad z \in Z.$$ 

Then we can prove the following propositions.
PROPOSITION 3.3. Let $Q$ be a polynomial of $n$ real valuables. Then the mapping 
$$(F_1, \ldots, F_n) \mapsto Q(\|F_1\|, \ldots, \|F_n\|)$$
from $\left((\bigcap_{p \in \mathbb{N}} L^p(X, dP; Y))_{u.i}^{\mathbb{R}_{+}}\right)^n$ to $\left((\bigcap_{p \in \mathbb{N}} L^p(X, dP))_{u.i}^{\pi_{+}}\right)$ is continuous.

PROPOSITION 3.4. Let $Z_j$ be a Banach space $(j = 1, \cdots, n)$, $L$ be a continuous multilinear form on $Z_1 \times \cdots \times Z_n$, and $V_j$ be a polynomially continuous mapping from $Y$ to $Z_j (j = 1, \cdots, n)$. Then the mapping $(F_1, \cdots, F_n) \mapsto L(V_1 \circ F_1, \cdots, V_n \circ F_n)$ from $\left((\bigcap_{p \in \mathbb{N}} L^p(X, dP; Y))_{u.i}^{\mathbb{R}_{+}}\right)^n$ to $\left((\bigcap_{p \in \mathbb{N}} L^p(X, dP))_{u.i}^{\pi_{+}}\right)$ is continuous.

IV. AN ASYMPTOTIC FORMULA

Let $\{\lambda_n\}_{n=1}^{\infty}$ be the eigenvalues of $A$, and $\{e_n\}_{n=1}^{\infty}$ be the complete orthonormal system (CONS) of $\mathcal{H}$ with $Ae_n = \lambda_n e_n$, and

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n^{\gamma-9}} < \infty$$

(4.1)

Let $\varphi$ be a bijection from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$. For all $n, m \in \mathbb{N}$, we set $f_{n,m} = e_{\varphi(n,m)}$. Then $\{f_{n,m}\}_{n,m=1}^{\infty}$ is a CONS of $\mathcal{H}$. For all $\phi \in E$, we define

$$\phi_n := \phi(e_n), \quad \phi_{n,m} := \phi(f_{n,m}).$$

Then $\{\phi_n\}_n$ and $\{\phi_{n,m}\}_{n,m}$ are families of independent Gaussian random variables such that for all $n, m, n', m' \in \mathbb{N}$,

$$\int_E \phi_n d\mu(\phi) = 0, \quad \int_E \phi_n \phi_m d\mu(\phi) = \delta_{nm}$$

(4.2)

$$\int_E \phi_{n,m} \phi_{n',m'} d\mu(\phi) = \delta_{nn'} \delta_{mm'}.$$  

(4.3)

For all $m_1, \cdots, m_p \in \mathbb{N}$, we have

$$\sup_{n_1, \cdots, n_p \in \mathbb{N}} \int_E |\phi_{n_1}|^{m_1} \cdots |\phi_{n_p}|^{m_p} d\mu(\phi) < \infty.$$  

(4.4)

For all $N, M \in \mathbb{N}$, we set

$$F_{N,M}(\epsilon, \omega, s) = \sqrt{\frac{2}{\beta}} \sum_{n=1}^{N} \frac{\phi_n}{\sqrt{\lambda_n}} e_n + \sum_{n=1}^{N} \sum_{m=1}^{M} \sqrt{\frac{4\epsilon^2 \lambda_n}{\beta(\epsilon^2 \lambda_n^2 + (2\pi m)^2)}} (\psi_{n,m} \cos(2\pi ms) + \theta_{n,m} \sin(2\pi ms)) e_n, \quad \epsilon \geq 0, \omega = (\phi, \psi, \theta) \in \Omega, \quad 0 \leq s \leq 1.$$  

(4.5)

Then we have

$$\frac{\text{Tr} e^{-\beta H_0}}{\text{Tr} e^{-\beta H}} = \lim_{N,M \to \infty} \int_{\Omega} \exp \left(-\beta \int_{0}^{1} V(F_{N,M}(\epsilon, \omega, s)) \, ds \right) d\nu(\omega),$$

(4.6)
where $\epsilon = \beta \hbar$ (See [2], Lemma 5.2, Lemma 5.3.).

We set

$$Z(\epsilon) = \lim_{N,M \to \infty} \int_{\Omega} \exp \left( -\beta \int_{0}^{1} F_{N,M}(\epsilon, \omega, s) ds \right) d\nu(\omega), \quad \epsilon \geq 0, \quad (4.7)$$

For all $n, m \in \mathbb{N}$, we set

$$\alpha_{n,m}(\epsilon) = \sqrt{\frac{4\epsilon^2 \lambda_n}{\beta (\epsilon^2 \lambda_n^2 + (2\pi m)^2)}}, \quad \epsilon \geq 0.$$

Then, for all $\delta > 0$, there exists a constant $C > 0$ such that

$$|\alpha_{n,m}(\epsilon)| \leq \frac{C \sqrt{\lambda_n}}{m}, \quad n, m \in \mathbb{N}, \quad 0 \leq \epsilon \leq \delta. \quad (4.8)$$

$$|\alpha'_{n,m}(\epsilon)| \leq \frac{C \sqrt{\lambda_n}}{m}, \quad n, m \in \mathbb{N}, \quad 0 \leq \epsilon \leq \delta. \quad (4.9)$$

$$|\alpha''_{n,m}(\epsilon)| \leq \frac{C \lambda_n^{5/2}}{m}, \quad n, m \in \mathbb{N}, \quad 0 \leq \epsilon \leq \delta. \quad (4.10)$$

$$|\alpha''_{n,m}(\epsilon)| \leq \frac{C (\lambda_n^{5/2} + \lambda_n^{9/2})}{m}, \quad n, m \in \mathbb{N}, \quad 0 \leq \epsilon \leq \delta. \quad (4.11)$$

We denote by $\mu_{[0,1]}^{(L)}$ the Lebesgue measure on $[0,1]$. Then by (4.8),(4.9),(4.10) and (4.11), we can prove the following lemma.

**Lemma 4.1.** \{F_{N,M}\}_{N,M \in \mathbb{N}}, \{F'_{N,M}\}_{N,M \in \mathbb{N}}, \{F''_{N,M}\}_{N,M \in \mathbb{N}}, \{F'''_{N,M}\}_{N,M \in \mathbb{N}} are Cauchy nets in $(\cap_{p \in \mathbb{N}} L^p(\Omega \times [0,1], d(\nu \otimes \mu_{[0,1]}^{(L)}))_{u.i}^{\mathbb{R}_+}.\cdot$

For all $N, M \in \mathbb{N}$, we set

$$G_{N,M}(\epsilon, \omega) = \exp \left( -\beta \int_{0}^{1} V(F_{N,M}(\epsilon, \omega, s)) ds \right), \quad \epsilon \geq 0, \quad \omega \in \Omega.$$

Then by Proposition 3.4 and Lemma 4.1, we can prove the following lemma.

**Lemma 4.2.** \{G_{N,M}\}_{N,M \in \mathbb{N}}, \{G'_{N,M}\}_{N,M \in \mathbb{N}}, \{G''_{N,M}\}_{N,M \in \mathbb{N}}, \{G'''_{N,M}\}_{N,M \in \mathbb{N}} are Cauchy nets in $(\cap_{p \in \mathbb{N}} L^p(\Omega, d\nu))^{\mathbb{R}_+}.$

By Lemma 4.2 and the fact that $\alpha_{n,m}$ is infinitely differentiable for all $n, m \in \mathbb{N}$, \int_{\Omega} H_{N,M}(\epsilon, \omega) d\nu(\omega) with $H_{N,M} = G_{N,M}, G'_{N,M}, G''_{N,M}, G'''_{N,M}$ uniformly converges in $\epsilon$. Hence one can interchange the limit $\lim_{N,M \to \infty}$ with differentiations in $\epsilon$ and see that $Z$ is 3-times continuously differentiable in $\mathbb{R}_+$. We can prove the following theorem.
THEOREM 4.3. For all $\beta > 0$,

\[
\frac{\text{Tr} e^{-\beta H_{\hbar}}}{\text{Tr} e^{-\beta H_{0}}}
= \frac{1}{E} \exp \left( -\beta V\left( \sqrt{\frac{2}{\beta}} A^{-1/2} \phi \right) \right) d\mu(\phi)
- \frac{\beta^{3} \hbar^{2}}{2} \sum_{m=1}^{\infty} \int_{E^{2}} d\mu(\phi) d\mu(\psi) \exp \left( -\beta V\left( \sqrt{\frac{2}{\beta}} A^{-1/2} \phi \right) \right)
\times V'' \left( \sqrt{\frac{2}{\beta}} A^{-1/2} \phi \right) A^{1/2} \left( \frac{1}{\sqrt{\beta \pi m}} \sum_{n=1}^{\infty} \psi_{n,m} e_{n} \right), A^{1/2} \left( \frac{1}{\sqrt{\beta \pi m}} \sum_{n=1}^{\infty} \psi_{n,m} e_{n} \right) + o(\hbar^{2})
\]

as $\hbar \to 0$.

REFERENCES

