<table>
<thead>
<tr>
<th>項目</th>
<th>内容</th>
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</thead>
<tbody>
<tr>
<td>Title</td>
<td>Semi-classical Asymptotics for the Partition Function of an Abstract Bose Field Model (Mathematical aspects of quantum fields and related topics)</td>
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<tr>
<td>Author(s)</td>
<td>相原祐太</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2014), 1921: 102-107</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2014-10</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/223404">http://hdl.handle.net/2433/223404</a></td>
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<td>Right</td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
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Kyoto University
Semi-classical Asymptotics for the Partition Function of an Abstract Bose Field Model

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Semi-classical asymptotics for the partition function of an abstract Bose field model is considered.

Keywords: semi-classical asymptotics, Bose field, partition function, second quantization, Fock space.

I. INTRODUCTION

In quantum mechanics, in which a physical constant $\hbar := h/2\pi$ (h: the Planck constant) plays an important role, the limit $\hbar \to 0$ for various quantities (if it exists) is called the classical limit. Trace formulas in the abstract boson Fock space and the classical limit for the trace $Z(\beta \hbar)$ (the partition function) of the heat semigroup of a perturbed second quantization operator were derived by Arai [2], where $\beta > 0$ denotes the inverse temperature. Generally speaking, the classical limit is regarded as the zero-th order approximation in $\hbar$. From this point of view, it is interesting to derive higher order asymptotics of various quantities in $\hbar$. Such asymptotics are called semi-classical asymptotics. In this paper the asymptotic formula for $Z(\beta \hbar)$ is stated, which is derived in [1].

II. A CLASSICAL LIMIT IN THE ABSTRACT BOSON FOCK SPACE

In this section we review a classical limit for the trace of a perturbed second quantization operator and some fundamental facts related to it.

Let $\mathcal{H}$ be a real separable Hilbert space, and $A$ be a strictly positive self-adjoint operator acting in $\mathcal{H}$. We denote by $\{\mathcal{H}_s(A)\}_{s \in \mathbb{R}}$ the Hilbert scale associated with $A$ [3]. For all $s \in \mathbb{R}$, the dual space of $\mathcal{H}_s(A)$ can be naturally identified with $\mathcal{H}_{-s}(A)$. 
We denote by $\mathcal{J}_1(\mathcal{H})$ the ideal of the trace class operators on $\mathcal{H}$. Let $\gamma > 0$ be fixed. Throughout this paper, we assume the following.

**Assumption I.** $A^{\gamma} \in \mathcal{J}_1(\mathcal{H})$.

Under Assumption I, the embedding mapping of $\mathcal{H}$ into

$$E := \mathcal{H}_{-\gamma}(A)$$

is Hilbert-Schmidt. Hence, by Minlos' theorem, there exists a unique probability measure $\mu$ on $(E, \mathcal{B})$ such that the Borel field $\mathcal{B}$ is generated by $\{\phi(f) | f \in \mathcal{H}_{\gamma}(A)\}$ and

$$\int_{E} e^{i\phi(f)} d\mu(\phi) = e^{-\|f\|_{\mathcal{H}}^{2}/2}, \quad f \in \mathcal{H}_{\gamma}(A),$$

where $\|\cdot\|_{\mathcal{H}}$ denotes the norm of $\mathcal{H}$.

The complex Hilbert space $L^2(E, d\mu)$ is canonically isomorphic to the boson Fock space over $\mathcal{H}$, which is called the $Q$-space representation of it [3]. We denote by $d\Gamma(A)$ the second quantization of $A$ and set

$$H_0 = d\Gamma(A).$$

Then for all $\beta > 0$, $e^{-\beta H_0} \in \mathcal{J}_1(L^2(E, d\mu))$.

**Definition 2.1.** A mapping $V$ of a Banach space $X$ into a Banach space $Y$ is said to be polynomially continuous if there exists a polynomial $P$ of two real variables with positive coefficients such that

$$\|V(\phi) - V(\psi)\| \leq P(\|\phi\|, \|\psi\|)\|\phi - \psi\|, \quad \phi, \psi \in X.$$

Let $V$ be a real valued function on $E$. Throughout this paper, we assume the following.

**Assumption II.** The function $V$ is bounded from below, 3-times Fréchet differentiable, and $V, V', V'', V'''$ are polynomially continuous.

For $\hbar > 0$, we define $V_\hbar$ by

$$V_\hbar(\phi) := V(\sqrt{\hbar} \phi), \quad \phi \in E.$$ 

and set

$$H_\hbar := H_0 + \frac{1}{\hbar} V_\hbar,$$

where $+$ denotes the quadratic form sum.

Under Assumption I, II, for all $\beta > 0$, $e^{-\beta H_\hbar} \in \mathcal{J}_1(L^2(E, d\mu))$ [2].

**Theorem 2.2.** [2]. Let $\beta > 0$. Then

$$\lim_{\hbar \to 0} \frac{\text{Tr} e^{-\beta H_\hbar}}{\text{Tr} e^{-\beta H_0}} = \int_{E} \exp \left( -\beta V \left( \sqrt{\frac{2}{\beta}} A^{-1/2} \phi \right) \right) d\mu(\phi).$$
III. A CLASS OF LOCALLY CONVEX SPACES

In this section we introduce a class of locally convex spaces, which gives a general framework for the asymptotic analysis discussed in this paper.

We denote by $\mathbb{R}_+$ the set of the nonnegative real numbers.

**Definition 3.1.** A mapping $f$ from $\mathbb{R}_+$ to a locally convex space $X$ is said to be locally bounded if for all $\delta > 0$ and every continuous seminorm $p$ on $X$,

$$p_\delta(f) := \sup_{0 \leq \epsilon \leq \delta} p(f(\epsilon)) < \infty.$$  

We denote by $(X^{\mathbb{R}_+})_{lb}$ the linear space of the locally bounded mappings from $\mathbb{R}_+$ to $X$. The topology defined by the seminorms $\{p_\delta\}_{\delta > 0}$ turns $(X^{\mathbb{R}_+})_{lb}$ into a locally convex space. If $X$ is a Fréchet space, $(X^{\mathbb{R}_+})_{lb}$ is a Fréchet space.

Let $(E_n)_{n \in \mathbb{N}}$ be a family of Banach spaces with the property that $E_{n+1} \subset E_n$, $\|\phi\|_n \leq \|\phi\|_{n+1}$, $\phi \in E_{n+1}$, for all $n \in \mathbb{N}$, where $\| \cdot \|_n$ denotes the norm of $E_n$. Then, the topology defined by the norms $\{\| \cdot \|_n\}_{n \in \mathbb{N}}$ turns $\bigcap_{n \in \mathbb{N}} E_n$ into a Fréchet space.

**Definition 3.2.** Let $f$ be a mapping from $\mathbb{R}_+$ to $\bigcap_{p \in \mathbb{N}} L^p(X, dP; Y)$. We say that $f$ is in $(\bigcap_{p \in \mathbb{N}} L^p(X, dP; Y))_{ui}^{\mathbb{R}_+}$ if and only if for each $\delta > 0$, there exists a nonnegative function $g \in \bigcap_{p \in \mathbb{N}} U(X, dP)$ such that

$$\sup_{0 \leq \epsilon \leq \delta} \| f(\epsilon)(x) \|_Y \leq g(x), \quad P-a.e.x.$$  

The set $(\bigcap_{p \in \mathbb{N}} L^p(X, dP; Y))_{ui}^{\mathbb{R}_+}$ is a linear subspace of $(\bigcap_{p \in \mathbb{N}} L^p(X, dP; Y))_{lb}^{\mathbb{R}_+}$. In what follows, we omit $x$ in $f(\epsilon)(x)$.

Let $X_1, \cdots, X_n$ and $Z$ be non-empty sets and $G$ be a real-valued function on $X_1 \times \cdots \times X_n$ and $F_j$ be a mapping from $Z$ to $X_j$, $j = 1, \cdots, n$. We define $G(F_1, \cdots, F_n)$, the real-valued function on $Z$, by

$$G(F_1, \cdots, F_n)(z) = G(F_1(z), \cdots, F_n(z)), \quad z \in Z.$$  

Then we can prove the following propositions.
PROPOSITION 3.3. Let $Q$ be a polynomial of $n$ real valuables. Then the mapping $(F_1, \cdots, F_n) \mapsto Q(\|F_1\|, \cdots, \|F_n\|)$ from $\left(\bigcap_{p \in \mathbb{N}} L^p(X, dP; Y)\right)^{R_+}_{\text{u.i.}}$ to $\left(\bigcap_{p \in \mathbb{N}} L^p(X, dP)\right)^{R_+}_{\text{u.i.}}$ is continuous.

PROPOSITION 3.4. Let $Z_j$ be a Banach space ($j = 1, \cdots, n$), $L$ be a continuous multilinear form on $Z_1 \times \cdots \times Z_n$, and $V_j$ be a polynomially continuous mapping from $Y$ to $Z_j (j = 1, \cdots, n)$. Then the mapping $(F_1, \cdots, F_n) \mapsto L(V_1 \circ F_1, \cdots, V_n \circ F_n)$ from $\left(\bigcap_{p \in \mathbb{N}} L^p(X, dP; Y)\right)^{R_+}_{\text{u.i.}}$ to $\left(\bigcap_{p \in \mathbb{N}} L^p(X, dP)\right)^{R_+}_{\text{u.i.}}$ is continuous.

IV. AN ASYMPTOTIC FORMULA

Let $\{\lambda_n\}_{n=1}^\infty$ be the eigenvalues of $A$, and $\{e_n\}_{n=1}^\infty$ be the complete orthonormal system (CONS) of $\mathcal{H}$ with $Ae_n = \lambda_ne_n$, and
\[
\sum_{n=1}^\infty \frac{1}{\lambda_n^{-\gamma}} < \infty \tag{4.1}
\]
Let $\varphi$ be a bijection from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$. For all $n, m \in \mathbb{N}$, we set $f_{n,m} = e_{\varphi(n,m)}$. Then $\{f_{n,m}\}_{n,m=1}^\infty$ is a CONS of $\mathcal{H}$. For all $\phi \in E$, we define
\[
\phi_n := \phi(e_n), \quad \phi_{n,m} := \phi(f_{n,m}).
\]
Then $\{\phi_n\}$ and $\{\phi_{n,m}\}_{n,m}$ are families of independent Gaussian random variables such that for all $n, m, n', m' \in \mathbb{N},$
\[
\int_E \phi_n d\mu(\phi) = 0, \quad \int_E \phi_m d\mu(\phi) = 0, \quad \int_E \phi_n \phi_m d\mu(\phi) = \delta_{nm} \tag{4.2}
\]
\[
\int_E \phi_{n,m} \phi_{n',m'} d\mu(\phi) = \delta_{nn'} \delta_{mm'} \tag{4.3}
\]
For all $m_1, \cdots, m_p \in \mathbb{N}$, we have
\[
\sup_{n_1, \cdots, n_p \in \mathbb{N}} \int_E |\phi_{n_1}|^{m_1} \cdots |\phi_{n_p}|^{m_p} d\mu(\phi) < \infty. \tag{4.4}
\]
For all $N, M \in \mathbb{N}$, we set
\[
F_{N,M}(\epsilon, \omega, s) = \sqrt{\frac{2}{\beta}} \sum_{n=1}^N \frac{\phi_n}{\sqrt{\lambda_n}} e_n + \sum_{n=1}^N \sum_{m=1}^M \sqrt{\frac{4 \epsilon^2 \lambda_n}{\beta (\epsilon^2 \lambda_n^2 + (2\pi m)^2)}} (\psi_{n,m} \cos(2\pi ms) + \theta_{n,m} \sin(2\pi ms)) e_n, \quad s \in [0, 1], \quad \epsilon \geq 0, \quad \omega = (\phi, \psi, \theta) \in \Omega, \quad 0 \leq s \leq 1. \tag{4.5}
\]
Then we have
\[
\frac{\text{Tr} e^{-\beta H_0}}{\text{Tr} e^{-\beta H}} = \lim_{N,M \to \infty} \int_{\Omega} \exp \left( -\beta \int_0^1 V(F_{N,M}(\epsilon, \omega, s)) \, ds \right) d\nu(\omega), \tag{4.6}
\]
where \( \epsilon = \beta \hbar \) (See [2], Lemma 5.2, Lemma 5.3.).

We set

\[
Z(\epsilon) = \lim_{N,M \to \infty} \int_{\Omega} \exp \left( -\beta \int_{0}^{1} F_{N,M}(\epsilon, \omega, s) \, ds \right) \, d\nu(\omega), \quad \epsilon \geq 0, \quad (4.7)
\]

For all \( n, m \in \mathbb{N} \), we set

\[
\alpha_{n,m}(\epsilon) = \sqrt{\frac{4\epsilon^{2} \lambda_{n}}{\beta(\epsilon^{2} \lambda_{n}^{2} + (2\pi m)^{2})}}, \quad \epsilon \geq 0.
\]

Then, for all \( \delta > 0 \), there exists a constant \( C > 0 \) such that

\[
|\alpha_{n,m}(\epsilon)| \leq \frac{C\sqrt{\lambda} n}{m}, \quad n, m \in \mathbb{N}, \quad 0 \leq \epsilon \leq \delta. \quad (4.8)
\]

\[
|\alpha'_{n,m}(\epsilon)| \leq \frac{C\sqrt{\lambda} n}{m}, \quad n, m \in \mathbb{N}, \quad 0 \leq \epsilon \leq \delta. \quad (4.9)
\]

\[
|\alpha''_{n,m}(\epsilon)| \leq \frac{C\lambda_{n}^{5/2}}{m}, \quad n, m \in \mathbb{N}, \quad 0 \leq \epsilon \leq \delta. \quad (4.10)
\]

\[
|\alpha'''_{n,m}(\epsilon)| \leq \frac{C(\lambda_{n}^{5/2} + \lambda_{n}^{9/2})}{m}, \quad n, m \in \mathbb{N}, \quad 0 \leq \epsilon \leq \delta. \quad (4.11)
\]

We denote by \( \mu_{[0,1]}^{(L)} \) the Lebesgue measure on \([0,1]\). Then by (4.8),(4.9),(4.10) and (4.11), we can prove the following lemma.

**LEMMA 4.1.** \( \{F_{N,M}\}_{N,M \in \mathbb{N}}, \{F_{N,M}'\}_{N,M \in \mathbb{N}}, \{F_{N,M}''\}_{N,M \in \mathbb{N}}, \{F_{N,M}'''\}_{N,M \in \mathbb{N}} \) are Cauchy nets in \( (\bigcap_{p \in \mathbb{N}} U(\Omega \times [0,1], d(\nu \otimes \mu_{[0,1]}^{(L)}))_{u.i}^{\mathbb{R}_{+}}) \).

For all \( N, M \in \mathbb{N} \), we set

\[
G_{N,M}(\epsilon, \omega) = \exp \left( -\beta \int_{0}^{1} V(F_{N,M}(\epsilon, \omega, s)) \, ds \right), \quad \epsilon \geq 0, \quad \omega \in \Omega.
\]

Then by Proposition 3.4 and Lemma 4.1, we can prove the following lemma.

**LEMMA 4.2.** \( \{G_{N,M}\}_{N,M \in \mathbb{N}}, \{G_{N,M}'\}_{N,M \in \mathbb{N}}, \{G_{N,M}''\}_{N,M \in \mathbb{N}}, \{G_{N,M}'''\}_{N,M \in \mathbb{N}} \) are Cauchy nets in \( (\bigcap_{p \in \mathbb{N}} L^{p}(\Omega, d\nu))_{u.i}^{\mathbb{R}_{+}} \).

By Lemma 4.2 and the fact that \( \alpha_{n,m} \) is infinitely differentiable for all \( n, m \in \mathbb{N} \),

\[
\int_{\Omega} H_{N,M}(\epsilon, \omega) \, d\nu(\omega) \quad \text{with} \quad H_{N,M} = G_{N,M}, G_{N,M}', G_{N,M}'' \text{ uniformly converges in} \ \epsilon.
\]

Hence one can interchange the limit \( \lim_{N,M \to \infty} \) with differentiations in \( \epsilon \) and see that \( Z \) is 3-times continuously differentiable in \( \mathbb{R}_{+} \).

We can prove the following theorem.
THEOREM 4.3. For all $\beta > 0$,

\[
\frac{\text{Tr}e^{-\beta H_{\hbar}}}{\text{Tr}e^{-\beta H_{0}}} = \int_{E} \exp \left(-\beta V \left(\sqrt{\frac{2}{\beta}} A^{-1/2} \phi \right) \right) d\mu(\phi)
\]
\[
- \frac{\beta^{3} \hbar^{2}}{2} \sum_{m=1}^{\infty} \int_{E^{2}} d\mu(\phi) d\mu(\psi) \exp \left(-\beta V \left(\sqrt{\frac{2}{\beta}} A^{-1/2} \phi \right) \right)
\]
\[
\times \left( V'' \left(\sqrt{\frac{2}{\beta}} A^{-1/2} \phi \right) \right) \left(A^{1/2} \left(\frac{1}{\sqrt{\beta \pi m}} \sum_{n=1}^{\infty} \psi_{n,m} e_{n} \right), A^{1/2} \left(\frac{1}{\sqrt{\beta \pi m}} \sum_{n=1}^{\infty} \psi_{n,m} e_{n} \right) \right)
\]
\[
+ o(\hbar^{2})
\]

as $\hbar \to 0$.

REFERENCES

