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京都大学
Semi-classical Asymptotics for the Partition Function of an Abstract Bose Field Model

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Semi-classical asymptotics for the partition function of an abstract Bose field model is considered.

Keywords: semi-classical asymptotics, Bose field, partition function, second quantization, Fock space.

I. INTRODUCTION

In quantum mechanics, in which a physical constant $\hbar := h/2\pi$ ($h$ : the Planck constant) plays an important role, the limit $\hbar \to 0$ for various quantities (if it exists) is called the classical limit. Trace formulas in the abstract boson Fock space and the classical limit for the trace $Z(\beta \hbar)$ (the partition function) of the heat semigroup of a perturbed second quantization operator were derived by Arai [2], where $\beta > 0$ denotes the inverse temperature. Generally speaking, the classical limit is regarded as the zero-th order approximation in $\hbar$. From this point of view, it is interesting to derive higher order asymptotics of various quantities in $\hbar$. Such asymptotics are called semi-classical asymptotics. In this paper the asymptotic formula for $Z(\beta \hbar)$ is stated, which is derived in [1].

II. A CLASSICAL LIMIT IN THE ABSTRACT BOSE FOCK SPACE

In this section we review a classical limit for the trace of a perturbed second quantization operator and some fundamental facts related to it.

Let $\mathcal{H}$ be a real separable Hilbert space, and $A$ be a strictly positive self-adjoint operator acting in $\mathcal{H}$. We denote by $\{\mathcal{H}_s(A)\}_{s \in \mathbb{R}}$ the Hilbert scale associated with $A$ [3]. For all $s \in \mathbb{R}$, the dual space of $\mathcal{H}_s(A)$ can be naturally identified with $\mathcal{H}_{-s}(A)$.
We denote by $\mathcal{J}_1(\mathcal{H})$ the ideal of the trace class operators on $\mathcal{H}$. Let $\gamma > 0$ be fixed. Throughout this paper, we assume the following.

**Assumption I.** $A^{3-\gamma} \in \mathcal{J}_1(\mathcal{H})$.

Under Assumption I, the embedding mapping of $\mathcal{H}$ into 

$$E := \mathcal{H}_{-\gamma}(A)$$

is Hilbert-Schmidt. Hence, by Minlos' theorem, there exists a unique probability measure $\mu$ on $(E, \mathcal{B})$ such that the Borel field $\mathcal{B}$ is generated by $\{\phi(f) \mid f \in \mathcal{H}_{\gamma}(A)\}$ and

$$\int_{E} e^{i\phi(f)} d\mu(\phi) = e^{-\Vert f \Vert_{\mathcal{H}}^2/2}, \quad f \in \mathcal{H}_{\gamma}(A),$$

where $\Vert \cdot \Vert_{\mathcal{H}}$ denotes the norm of $\mathcal{H}$.

The complex Hilbert space $L^2(E, d\mu)$ is canonically isomorphic to the boson Fock space over $\mathcal{H}$, which is called the $Q$-space representation of it [3]. We denote by $d\Gamma(A)$ the second quantization of $A$ and set

$$H_0 = d\Gamma(A).$$

Then for all $\beta > 0$, $e^{-\beta H_0} \in \mathcal{J}_1(L^2(E, d\mu))$.

**Definition 2.1.** A mapping $V$ of a Banach space $X$ into a Banach space $Y$ is said to be polynomially continuous if there exists a polynomial $P$ of two real variables with positive coefficients such that

$$\Vert V(\phi) - V(\psi) \Vert \leq P(\Vert \phi \Vert, \Vert \psi \Vert) \Vert \phi - \psi \Vert, \quad \phi, \psi \in X.$$

Let $V$ be a real valued function on $E$. Throughout this paper, we assume the following.

**Assumption II.** The function $V$ is bounded from below, 3-times Fréchet differentiable, and $V, V', V'', V'''$ are polynomially continuous.

For $\hbar > 0$, we define $V_{\hbar}$ by

$$V_{\hbar}(\phi) := V(\sqrt{\hbar} \phi), \quad \phi \in E.$$ 

and set

$$H_{\hbar} := H_0 + \frac{1}{\hbar} V_{\hbar},$$

where $+$ denotes the quadratic form sum.

Under Assumption I, II, for all $\beta > 0$, $e^{-\beta H_{\hbar}} \in \mathcal{J}_1(L^2(E, d\mu))$ [2].

**Theorem 2.2.** [2]. Let $\beta > 0$. Then

$$\lim_{\hbar \to 0} \frac{\text{Tr} e^{-\beta H_{\hbar}}}{\text{Tr} e^{-\beta H_0}} = \int_{E} \exp \left( -\beta V \left( \sqrt{\frac{2}{\beta}} A^{-1/2} \phi \right) \right) d\mu(\phi).$$
III. A CLASS OF LOCALLY CONVEX SPACES

In this section we introduce a class of locally convex spaces, which gives a general framework for the asymptotic analysis discussed in this paper.

We denote by $\mathbb{R}_+$ the set of the nonnegative real numbers.

**DEFINITION 3.1.** A mapping $f$ from $\mathbb{R}_+$ to a locally convex space $X$ is said to be locally bounded if for all $\delta > 0$ and every continuous seminorm $p$ on $X$,

$$p_\delta(f) := \sup_{0 \leq \epsilon \leq \delta} p(f(\epsilon)) < \infty.$$  

We denote by $(X^{\mathbb{R}_+})_{1b}$ the linear space of the locally bounded mappings from $\mathbb{R}_+$ to $X$. The topology defined by the seminorms $\{p_\delta\}_{p,\delta}$ turns $(X^{\mathbb{R}_+})_{1b}$ into a locally convex space. If $X$ is a Fréchet space, $(X^{\mathbb{R}_+})_{1b}$ is a Fréchet space.

Let $\{E_n\}_{n \in \mathbb{N}}$ be a family of Banach spaces with the property that $E_{n+1} \subset E_n$, $\|\phi\|_n \leq \|\phi\|_{n+1}$, $\phi \in E_{n+1}$, for all $n \in \mathbb{N}$, where $\|\cdot\|_n$ denotes the norm of $E_n$. Then, the topology defined by the norms $\{\|\cdot\|_n\}_{n \in \mathbb{N}}$ turns $\bigcap_{n \in \mathbb{N}}E_n$ into a Fréchet space.

**DEFINITION 3.2.** Let $f$ be a mapping from $\mathbb{R}_+$ to $\bigcap_{p \in \mathbb{N}}L^p(X, dP;Y)$. We say that $f$ is in $((\bigcap_{p \in \mathbb{N}}L^p(X, dP;Y))_{1b})^{\mathbb{R}_+}$ if and only if for each $\delta > 0$, there exists a nonnegative function $g \in (\bigcap_{p \in \mathbb{N}}L^p(X, dP))_{u.i.}$ such that

$$\sup_{0 \leq \epsilon \leq \delta} \|f(\epsilon)(x)\|_Y \leq g(x),$$

$P$-a.e.x.

The set $(\bigcap_{p \in \mathbb{N}}L^p(X, dP;Y))_{u.i.}$ is a linear subspace of $(\bigcap_{p \in \mathbb{N}}L^p(X, dP;Y))_{1b}^{\mathbb{R}_+}$. In what follows, we omit $x$ in $f(\epsilon)(x)$.

Let $X_1, \cdots, X_n$ and $Z$ be non-empty sets and $G$ be a real-valued function on $X_1 \times \cdots \times X_n$ and $F_j$ be a mapping from $Z$ to $X_j$, $j = 1, \cdots, n$. We define $G(F_1, \cdots, F_n)$, the real-valued function on $Z$, by

$$G(F_1, \cdots, F_n)(z) = G(F_1(z), \cdots, F_n(z)), \quad z \in Z.$$  

Then we can prove the following propositions.
Proposition 3.3. Let $Q$ be a polynomial of $n$ real valuables. Then the mapping

$$(F_1, \cdots, F_n) \mapsto Q(\|F_1\|, \cdots, \|F_n\|)$$

from \(\bigcap_{p \in \mathbb{N}} L^p(X, dP; Y)_{u.i}^{\mathbb{R}_+} \) to \(\bigcap_{p \in \mathbb{N}} L^p(X, dP)_{u.i}^{\mathbb{R}_+} \) is continuous.

Proposition 3.4. Let $Z_j$ be a Banach space \((j = 1, \cdots, n)\), $L$ be a continuous multilinear form on $Z_1 \times \cdots \times Z_n$, and $V_j$ be a polynomially continuous mapping from $Y$ to $Z_j$ \((j = 1, \cdots, n)\). Then the mapping

$$(F_1, \cdots, F_n) \mapsto L(V_1 \circ F_1, \cdots, V_n \circ F_n)$$

from \(\bigcap_{p \in \mathbb{N}} L^p(X, dP; Y)_{u.i}^{\mathbb{R}_+} \) to \(\bigcap_{p \in \mathbb{N}} L^p(X, dP)_{u.i}^{\mathbb{R}_+} \) is continuous.

IV. An Asymptotic Formula

Let \(\{\lambda_n\}_{n=1}^{\infty}\) be the eigenvalues of $A$, and \(\{e_n\}_{n=1}^{\infty}\) be the complete orthonormal system (CONS) of $\mathcal{H}$ with $Ae_n = \lambda_n e_n$, and

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n^{-\theta}} < \infty \quad (4.1)$$

Let $\varphi$ be a bijection from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$. For all $n, m \in \mathbb{N}$, we set $f_{n,m} = e_{\varphi(n,m)}$. Then \(\{f_{n,m}\}_{n,m=1}^{\infty}\) is a CONS of $\mathcal{H}$. For all $\phi \in E$, we define

$$\phi_n := \phi(e_n), \quad \phi_{n,m} := \phi(f_{n,m}).$$

Then \(\{\phi_n\}_n\) and \(\{\phi_{n,m}\}_{n,m}\) are families of independent Gaussian random variables such that for all $n, m, n', m' \in \mathbb{N}$,

$$\int_E \phi_n d\mu(\phi) = 0, \quad \int_E \phi_n \phi_m d\mu(\phi) = \delta_{nm} \quad (4.2)$$

$$\int_E \phi_{n,m} \phi_{n',m'} d\mu(\phi) = \delta_{nn'} \delta_{mm'} \quad (4.3)$$

For all $m_1, \cdots, m_p \in \mathbb{N}$, we have

$$\sup_{n_1, \cdots, n_p \in \mathbb{N}} \int_E |\phi_{n_1}|^{m_1} \cdots |\phi_{n_p}|^{m_p} d\mu(\phi) < \infty \quad (4.4)$$

For all $N, M \in \mathbb{N}$, we set

$$F_{N,M}(\epsilon, \omega, s) = \sqrt{\frac{2}{\beta}} \sum_{n=1}^{N} \frac{\phi_n}{\sqrt{\lambda_n}} e_n + \sum_{n=1}^{N} \sum_{m=1}^{M} \sqrt{\frac{4\epsilon^2 \lambda_n}{\beta (\epsilon^2 \lambda_n^2 + (2\pi m)^2)}} (\psi_{n,m} \cos(2\pi ms) + \theta_{n,m} \sin(2\pi ms)) e_n, \quad \epsilon \geq 0, \quad \omega = (\phi, \psi, \theta) \in \Omega, \quad 0 \leq s \leq 1 \quad (4.5)$$

Then we have

$$\frac{\text{Tr} e^{-\beta H_0}}{\text{Tr} e^{-\beta H_N}} = \lim_{N, M \to \infty} \int_{\Omega} \exp \left( -\beta \int_0^1 V(F_{N,M}(\epsilon, \omega, s)) ds \right) d\nu(\omega), \quad (4.6)$$
where $\varepsilon = \beta \hbar$ (See [2], Lemma 5.2, Lemma 5.3.).

We set

$$Z(\varepsilon) = \lim_{N,M \to \infty} \int_{\Omega} \exp \left( -\beta \int_{0}^{1} F_{N,M}(\varepsilon, \omega, s) ds \right) d\nu(\omega), \quad \varepsilon \geq 0,$$

(4.7)

For all $n, m \in \mathbb{N}$, we set

$$\alpha_{n,m}(\varepsilon) = \sqrt{\frac{4\varepsilon^{2}\lambda_{n}}{\beta(\varepsilon^{2}\lambda_{n}^{2} + (2\pi m)^{2})}} \varepsilon \geq 0.$$

Then, for all $\delta > 0$, there exists a constant $C > 0$ such that

$$|\alpha_{n,m}(\varepsilon)| \leq \frac{c\sqrt{\lambda}n}{m}, \quad n, m \in \mathbb{N}, \quad 0 \leq \varepsilon \leq \delta.$$  

(4.8)

$$|\alpha_{n,m}'(\varepsilon)| \leq \frac{c\sqrt{\lambda}n}{m}, \quad n, m \in \mathbb{N}, \quad 0 \leq \varepsilon \leq \delta.$$  

(4.9)

$$|\alpha_{n,m}''(\varepsilon)| \leq \frac{C\lambda_{n}^{5/2}}{m}, \quad n, m \in \mathbb{N}, \quad 0 \leq \varepsilon \leq \delta.$$  

(4.10)

$$|\alpha_{n,m}'''(\varepsilon)| \leq \frac{C(\lambda_{n}^{5/2} + \lambda_{n}^{9/2})}{m}, \quad n, m \in \mathbb{N}, \quad 0 \leq \varepsilon \leq \delta.$$  

(4.11)

We denote by $\mu_{[0,1]}^{(L)}$ the Lebesgue measure on $[0,1]$. Then by (4.8),(4.9),(4.10) and (4.11), we can prove the following lemma.

LEMA 4.1. \{F_{N,M}\}_{N,M \in \mathbb{N}}, \{F_{N,M}'\}_{N,M \in \mathbb{N}}, \{F_{N,M}''\}_{N,M \in \mathbb{N}}, \{F_{N,M}'''\}_{N,M \in \mathbb{N}} are Cauchy nets in $$(\bigcap_{p \in \mathbb{N}} L^{p}(\Omega \times [0,1], d(\nu \otimes \mu_{[0,1]}^{(L)} E))_{u.i}^{\mathbb{R}_{+}} \cdot$$

For all $N, M \in \mathbb{N}$, we set

$$G_{N,M}(\varepsilon, \omega) = \exp \left( -\beta \int_{0}^{1} V(F_{N,M}(\varepsilon, \omega, s)) ds \right) \quad \varepsilon \geq 0, \quad \omega \in \Omega.$$

Then by Proposition 3.4 and Lemma 4.1, we can prove the following lemma.

LEMA 4.2. \{G_{N,M}\}_{N,M \in \mathbb{N}}, \{G_{N,M}'\}_{N,M \in \mathbb{N}}, \{G_{N,M}''\}_{N,M \in \mathbb{N}}, \{G_{N,M}'''\}_{N,M \in \mathbb{N}} are Cauchy nets in $$(\bigcap_{p \in \mathbb{N}} L^{p}(\Omega, d\nu))_{u.i}^{\mathbb{R}_{+}} \cdot$$

By Lemma 4.2 and the fact that $\alpha_{n,m}$ is infinitely differentiable for all $n, m \in \mathbb{N}$, $$\int_{\Omega} H_{N,M}(\varepsilon, \omega) d\nu(\omega)$$ converges in $\varepsilon$. Hence one can interchange the limit $\lim_{N,M \to \infty}$ with differentiations in $\varepsilon$ and see that $Z$ is 3-times continuously differentiable in $\mathbb{R}_{+}$.

We can prove the following theorem.
**Theorem 4.3.** For all $\beta > 0$,

$$\frac{\text{Tr} e^{-\beta H_A}}{\text{Tr} e^{-\beta H_0}} = \int_E \exp \left( -\beta V \left( \sqrt{\frac{2}{\beta^2}} A^{-1/2} \phi \right) \right) d\mu(\phi)$$

$$- \frac{\beta^3 \hbar^2}{2} \sum_{m=1}^\infty \int_{E^2} d\mu(\phi) d\mu(\psi) \exp \left( -\beta V \left( \sqrt{\frac{2}{\beta^2}} A^{-1/2} \phi \right) \right)$$

$$\times V'' \left( \sqrt{\frac{2}{\beta^2}} A^{-1/2} \phi \right) \left( A^{1/2} \left( \frac{1}{\sqrt{\beta^2 \pi^2 m}} \sum_{n=1}^\infty \psi_{n,m} e_n \right), A^{1/2} \left( \frac{1}{\sqrt{\beta^2 \pi^2 m}} \sum_{n=1}^\infty \psi_{n,m} e_n \right) \right)$$

$$+ o(\hbar^2)$$

as $\hbar \to 0$.

**References**

