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ON EVALUATION FIBER SEQUENCES

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ABSTRACT. We denote the n-th projective space of a topological monoid G by B_nG and the classifying space by BG. Our main result states that, if G is a topological group, then the evaluation fiber sequence $\mathrm{Map}_0(B_nG, BG) \to \mathrm{Map}(B_nG, BG) \to BG$ extends to the right. This theorem is proved by the technique of A_n-maps.

1. INTRODUCTION

The aim of this note is to present the result of the author's paper [Tsu].

First of all, let us review the history of A_n-theory. H-space was introduced by J.-P. Serre, which is named after H. Hopf. A pointed space G is said to be an *H-space* if a continuous binary operation $m : G \times G \to G$ of which the identity element is the basepoint is given. Every topological monoid is of courese an H-space. But the converse does not hold.

J. F. Adams [Ada60] proved that an n-dimensional sphere S^n admits a structure of an H-space if and only if $n = 0, 1, 3$ or 7. But I. M. James [Jam57] proved that S^7 never admit a homotopy associative H-structure. An H-space (G, m) is said to be *homotopy associative* if the maps $m \circ (\mathrm{id}_G \times m)$ and $m \circ (m \times \mathrm{id}_G)$ are homotopic. So, there is some difference between H-spaces and topological monoids. Then, how about the difference between homotopy associative H-spaces and topological monoids?

J. D. Stasheff [Sta63a] introduced the notion of *A_n-spaces*. H-space and homotopy associative H-space are nothing but A_2-spaces and A_3-spaces, respectively. Every topological monoid is an A_∞-space. For general n, A_n-space is an H-space with some *higher homotopy associativity of order n* in some sense. This homotopy associativity data is called *A_n-form*. His work revealed that being a topological monoid is much far from being an H-space.

The same thing can be said for maps between topological monoids. A map $f : G \to G'$ is said to be an *H-map* if the maps $(g_1, g_2) \mapsto f(g_1g_2)$ and $(g_1, g_2) \mapsto f(g_1)f(g_2)$ are homotopic. Every continuous homomorphism between topological monoids is an H-map but the converse is false.

M. Sugawara [Sug60] given a condition for that a map between topological monoid induces a map between their classifying spaces. As is well-known, a homomorphism $G \to G'$ induces a map $Bf : BG \to BG'$ between the classifying spaces. He introduced the notion *strongly multiplicative map* and proved that a strongly multiplicative map induces a map between the classifying spaces. After that, Stasheff [Sta63b] proved the converse for path-connected topological monoids. Being a strongly multiplicative map requires a map to preserve the infinitely higher homotopy associativity of topological monoids. Stasheff weakened the condition and defined *A_n-map* between topological monoids. A_n-map is also defined by the data of preservation of higher homotopy associativity, called *A_n-form*.

By definition of A_n-map, one may expect that A_n-map is generalized for morphism between A_n-maps. Stasheff [Sta70] described the condition only for $n \leq 4$ because it needs combinatorially complicated cell complex, called *multiplihedra*. Abstractly, this was done by Boardman–Vogt [BV73]

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using their "W-construction". Independently, N. Iwase also constructed those complexes in his master thesis [Iwa83]. He described their combinatorial structure as well.

2. Space of A_n-maps

Now, to state our result, we get back to the work of Stasheff about A_n-maps between topological monoids. Consider a path-connected topological monoid G. Let us denote the n-th projective space and the classifying space by B_nG and $BG = B_\infty G$, respectively. We remark that there is the canonical inclusion $B_{n_1}G \subset B_{n_2}G$ for $n_1 < n_2$ and the homeomorphism $B_1G \cong \Sigma G$ with the reduced suspension. There is a universal principal G-fibration $EG \to BG$ with EG contractible. The restriction $E_nG \to B_nG$ over B_nG is also a principal G-fibration. Using the canonical homotopy cofiber sequence

$$E_nG \longrightarrow B_nG \longrightarrow B_{n+1}G$$

and the homotopy fiber sequence

$$E_nG \longrightarrow B_nG \longrightarrow BG,$$

Stasheff's result [Sta63b, Theorem 4.5] is rephrased as follows.

Theorem 2.1 (Stasheff, 1963). *Let G and G' be connected topological monoids which are CW complex. Then, a pointed map $f : G \to G'$ is an A_n-map if and only if the reduced suspension $\Sigma f : \Sigma G \to \Sigma G' \subset BG'$ can be extended to a map $B_nG \to BG'$.*

Theorem 2.1 states nothing about the correspondence of A_n-forms. Our result refines this point.

Let us denote the space of continuous maps between X and Y by $\mathrm{Map}(X, Y)$ and that of pointed ones by $\mathrm{Map}_0(X, Y)$. To guarantee the exponential law, we always work in the category of compactly generated spaces and the mapping spaces are considered in the manner of compactly generated spaces. The space of A_n-maps with A_n-forms between G and G' is denoted by $\mathcal{A}_n(G, G')$.

Theorem 2.2 (T). *Let G be a well-pointed topological monoid of homotopy type of a CW complex and G' a well-pointed grouplike topological monoid. Then the following composite is a weak equivalence.*

$$\mathcal{A}_n(G, G') \xrightarrow{B_n} \mathrm{Map}_0(B_nG, B_nG') \xrightarrow{(\iota_n)_\#} \mathrm{Map}_0(B_nG, BG').$$

A pointed space X is said to be *well-pointed* if the basepoint of X has the homotopy extension property. A topological monoid G is said to be *grouplike* if $\pi_0(G)$ is a group with respect to the multiplication induced from that of G. The map B_n is given by Sugawara's construction [Sug60] and the map $(\iota_n)_\#$ is the composition with the inclusion $\iota_n : B_nG \to BG$. In [Tsu], the author constructs a topological category $\mathcal{A}_n$ of topological monoids and A_n-maps between them and realizes B_n as a continuous functor from $\mathcal{A}_n$ to the category of (compactly generated) pointed spaces.

3. Evaluation fiber sequence

Next, we explain the main result on evaluation fiber sequences. If X is well-pointed, then the evaluation

$$\mathrm{Map}(X, Y) \longrightarrow Y$$

at the basepoint is a Hurewicz fibration of which the fiber over the basepoint is $\mathrm{Map}_0(X, Y)$. This fiber sequence is called the *evaluation fiber sequence.* Roughly, our second result states that this fiber sequence extends to the right if $X = B_nG$ and $Y = BG$. One may think that this is strange. Because it implies not only that the mapping space $\mathrm{Map}_0(B_nG, BG)$ is equivalent to a topological monoid, but also the connecting map $G \to \mathrm{Map}_0(B_nG, BG)$ is equivalent to a homomorphism between topological monoids. But Theorem 2.2 claims that $\mathrm{Map}_0(B_nG, BG)$ is equivalent to the topological monoid $\mathcal{A}_n(G,G)$.

There is a well-known adjunction

$$\mathrm{Map}_0(\Sigma X, Y) \cong \mathrm{Map}_0(X, \Omega Y)$$

for pointed spaces X and Y, where ΩY is the space of based loops in Y.

For a topological monoid G, define the subspace $\overline{\mathrm{Map}}_0(B_nG, BG) \subset \mathrm{Map}_0(B_nG, BG)$ consisting of maps $B_nG \to BG$ which restricts to a map $\Sigma G \to BG$ with the adjoint $G \to \Omega BG \simeq G$ is a homotopy equivalence. We also define the subspace $\overline{\mathrm{Map}}(B_nG, BG) \subset \mathrm{Map}(B_nG, BG)$ as the union of the path-components that intersect with $\overline{\mathrm{Map}}_0(B_nG, BG)$.

For a topological group G, the composition of the conjugation $G \to G$ by an element of G to an A_n-map from the left defines the left action of G on $\mathcal{A}_n(G,G)$. In particular, the composition to the identity map defines a map $\delta : G \to \mathcal{A}_n(G,G)$.

Note that there is a natural well-pointed replacement $\mathcal{W}G \to G$ of a topological monoid G, which is a homomorphism between topological monoids and is a weak equivalence.

Theorem 3.1 (T). *Let G be a well-pointed topological group of homotopy type of a CW complex. Consider a map $BG \to B\mathcal{W}\mathcal{A}_n(G,G;\mathrm{eq})$ defined by the composite*

$$BG \xleftarrow{\simeq} B\mathcal{W}G \xrightarrow{B\mathcal{W}\delta} B\mathcal{W}\mathcal{A}_n(G,G;\mathrm{eq}).$$

Then the sequence

$$\overline{\mathrm{Map}}_0(B_nG, BG) \to \overline{\mathrm{Map}}(B_nG, BG) \to BG \to B\mathcal{W}\mathcal{A}_n(G,G;\mathrm{eq}),$$

is a homotopy fiber sequence.

When $n = \infty$, the above sequence can be extended as

$$\overline{\mathrm{Map}}_0(BG, BG) \to \overline{\mathrm{Map}}(BG, BG) \xrightarrow{e} BG \to B\mathcal{W}\overline{\mathrm{Map}}_0(BG, BG) \to B\mathcal{W}\overline{\mathrm{Map}}(BG, BG),$$

where the subspaces $\overline{\mathrm{Map}}_0(BG, BG) \subset \mathrm{Map}_0(BG, BG)$ and $\overline{\mathrm{Map}}(BG, BG) \subset \mathrm{Map}(BG, BG)$ are exactly those of homotopy equivalences $BG \to BG$. For details, see Gottlieb's paper [Got73].

4. A_n-TYPES OF GAUGE GROUPS

Theorem 2.2 and 3.1 may be applied in many situations. As an application of them, we obtain the result for A_n-types of gauge groups. For a principal G-bundle $P \to B$, the topological group $\mathcal{G}(P)$ consisting of G-equivariant maps $P \to P$ that induces the identity on B is called the *gauge group* of P. The associated bundle $\mathrm{ad}\,P = P \times_G G$ with respect to the adjoint action of G is called the *adjoint bundle* and becomes a bundle of topological groups. The space of sections $\Gamma(\mathrm{ad}\,P)$ is naturally isomorphic to the gauge group $\mathcal{G}(P)$. For example, the classification of the A_n-types of the gauge groups of principal SU(2)-bundles over S^4 is investigated by Kono [Kon91], by Crabb–Sutherland [CS00], by Tsukuda [Tsu01] and by the author [Tsu12]. For homotopy types, there are many related works.

Theorem 4.1 (Kishimoto–Kono, 2010 and T). *Let G be a well-pointed topological group and B be a pointed space, both of which have the pointed homotopy types of CW complexes. For a principal G-bundle P over B classified by $\epsilon : B \to BG$, the following conditions are equivalent:*

(i) $\mathrm{ad}\,P$ *is A_n-trivial,*
(ii) the map $(\epsilon, \iota_n) : B \vee B_nG \to BG$ *extends over the product* $B \times B_nG$,
(iii) the composite $B \xrightarrow{\epsilon} BG \to B\mathcal{W}\mathcal{A}_n(G,G;\mathrm{eq})$ *is null-homotopic.*

A bundle of topological monoids $E \to B$ is said to be *A_n-trivial* if there exist a topological monoid G and a "fiberwise A_n-equivalence" $B \times G \to \operatorname{ad} P$. The A_n-triviality of $\operatorname{ad} P$ implies the A_n-equivalence of $\mathcal{G}(P)$ and $\operatorname{Map}(B, G)$. The equivalence of (i) and (ii) have already been known by Kishimoto–Kono [KK10].

In [Tsu], the author applied Theorem 2.2 and 3.1 to higher homotopy commutativity and to cyclic maps as well.

5. Future work

It seems that Theorem 3.1 has several applications to homotopy theory of gauge groups and higher homotopy commutativity. Another direction is generalizations of Theorem 2.2. The author is trying to generalize Theorem 2.2 for the A_∞-functors between small topological categories.

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