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Local and end deformation properties of uniform embeddings in κ-cones

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1. Introduction

This article is a continuation of systematic study of topological properties of spaces of uniform embeddings and groups of uniform homeomorphisms in metric manifolds [2, 4, 5, 8, 9]. Since the notion of uniform continuity and uniform topology depends on the choice of metrics, it is essential to select reasonable classes of metric manifolds (M, d).

J. M. Kister [4] treated the case of the standard Euclidean space $\mathbb{R}^n$ using Alexander's trick. This argument implies that the group $\mathcal{H}_b^u(\mathbb{R}^n)$ of bounded uniform homeomorphisms of $\mathbb{R}^n$ endowed with the uniform topology is contractible in a weak sense of [7, Section 5.6]. A.V. Černavskiĭ [2] considered the case where M is the interior of a compact manifold N and the metric d is a restriction of some metric on N. Recently, in [8] we considered the class of metric covering spaces over compact manifolds and obtained a local deformation theorem for uniform embeddings in those spaces. From this local deformation theorem we also deduced a global deformation result on groups of uniform homeomorphisms of metric manifolds with finitely many Euclidean ends. As a corollary, it follows that $\mathcal{H}_b^u(\mathbb{R}^n)$ is contractible as a topological space.

This expository article is mainly based on [9], where we study deformation property of uniform embeddings in the κ-cone ends ($\kappa \leq 0$) over compact Lipschitz metric manifolds. First we interpret the (local) contractibility of the group $\mathcal{H}_b^u(X, d)$ of bounded uniform homeomorphisms of a metric space (X, d) endowed with the uniform topology in term of continuous selection of uniform isotopies (Section 3). This clarifies the meaning of the weaker notion of (local) contractibility of $\mathcal{H}_b^u(X, d)$ due to [7, Section 5.6], which is defined by continuous selection of level-wise bounded uniform isotopies. We also list up basic properties of Alexander isotopies on κ-cones as level-wise bounded isotopies. Local deformation property (LD) of uniform embeddings was introduced in the earlier version of [9]. We extend the results to the κ-cone ends over compact Lipschitz metric manifolds (Section 4). We also introduce the notion of end deformation property (ED) of uniform embeddings in proper product ends and extend the result on Euclidean ends in [8] to the case of 0-cone ends of any compact Lipschitz metric manifolds (Section 5). The preliminary section 2 is devoted to the basics on metric manifolds and spaces of uniform embeddings.

2. Metric manifolds and spaces of uniform embeddings

This section includes basics on metric spaces and uniform embeddings needed in this paper.

2.1. **Metric manifolds.**

In this article, maps between topological spaces are always assumed to be continuous. As usual, any metric space (X, d) is given the topology τ_d induced by the metric d. A metric space is said to be proper if any bounded closed subset is compact.

Suppose (X, d) is a metric space. For $A \subset X$ and $\varepsilon \in [0, \infty]$ let $O_\varepsilon(A)$ denote the open ε-neighborhood of A in (X, d). For $A, B \subset X$, we write $A \subset_u B$ in X and call B a uniform neighborhood of A in X if $O_\varepsilon(A) \subset B$ for some $\varepsilon > 0$.

An n-manifold means a topological n-manifold M possibly with boundary. The symbols ∂M and $\mathrm{Int}\, M$ denote the boundary and interior of M as a manifold.

A metric n-manifold means an n-manifold with a fixed compatible metric. A basic model is the space-form M^n_κ (the simply connected complete Riemannian n-manifold with constant sectional curvature κ). For example, $M^n_\kappa = \mathbb{S}^n, \mathbb{R}^n, \mathbb{H}^n$ for $\kappa = 1, 0, -1$ respectively. Here, $\mathbb{R}^n$ is the Euclidean n-space with the standard Euclidean metric d_0, $\mathbb{H}^n$ is the hyperbolic n-space and $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$ is the standard unit n-sphere in $\mathbb{R}^{n+1}$ with the standard spherical metric d_1 defined by $\cos d_1(x, y) = \langle x, y \rangle$, where $\langle\ ,\ \rangle$ is the standard Euclidean inner product. Note that $M^n_{\pm\kappa}$ is homothetic to $M^n_{\pm 1}$ for any $\kappa > 0$. Let $\mathbb{R}^n_{\geq 0} = \{x \in \mathbb{R}^n \mid x_n \geq 0\}$.

Definition 2.1. A Lipschitz metric n-manifold is a metric n-manifold (M, d) which satisfies the following condition:

- $(\natural)$ For each point $x \in M$ there exists an open neighborhood U of x in M and an open subset V of $\mathbb{R}^n_{\geq 0}$ such that $(U, d|_U)$ is bi-Lipschitz homeomorphic to $(V, d_0|_V)$.

Any Riemannian manifold with the induced path-length metric is a Lipschitz metric manifold.

2.2. **Spaces of uniform embeddings.**

Suppose (X, d) and (Y, ϱ) are metirc spaces. A map $f : (X, d) \to (Y, \varrho)$ is said to be (i) uniformly continuous if for each $\varepsilon > 0$ there is a $\delta > 0$ such that if $x, x' \in X$ and $d(x, x') < \delta$ then $\varrho(f(x), f(x')) < \varepsilon$, (ii) coarsely uniform if for each $R > 0$ there is a $S > 0$ such that if $x, x' \in X$ and $d(x, x') < R$ then $\varrho(f(x), f(x')) < S$, (iii) Lipschitz if there exists $K > 0$ such that $\varrho(f(x), f(x')) \leq K d(x, x')$ for any $x, x' \in X$.

A map $f : (X, d) \to (Y, \varrho)$ is called a uniform (coarsely uniform, Lipschitz) homeomorphism if f is bijective and both f and f^{-1} are uniformly continuous (coarsely uniform, Lipschitz, respectively). A uniform embedding is a uniform homeomorphism onto its image.

Let $\mathcal{C}((X,d),(Y,\varrho))$ denote the space of continuous maps $f:(X,d)\to(Y,\varrho)$. The metric ϱ on Y induces the sup-metric $\tilde{\varrho}$ on $\mathcal{C}(X,Y)$ defined by

$$\tilde{\varrho}(f,g)=\sup\{\varrho(f(x),g(x))\mid x\in X\}\in[0,\infty].$$

The topology on $\mathcal{C}((X,d),(Y,\varrho))$ induced by the sup-metric $\tilde{\varrho}$ is called the uniform topology. Below the space $\mathcal{C}((X,d),(Y,\varrho))$ and its subspaces are endowed with the sup-metric $\tilde{\varrho}$ and the uniform topology, unless otherwise specified. For notational simplicity, the metric $\tilde{\varrho}$ is denoted by the same symbol ϱ. Let $\mathcal{C}^u((X,d),(Y,\varrho))$ denote the subspace consisting of uniformly continuous maps $f:(X,d)\to(Y,\varrho)$.

Let $\mathcal{H}(X,d)$ denote the group of homeomorphisms h of (X,d) onto itself. This group and its subgroups are endowed with the sup-metric and the uniform topology. Let $\mathcal{H}_b(X,d):=\{h\in\mathcal{H}(X,d)\mid d(h,\mathrm{id}_X)<\infty\}$ (the subgroup of bounded homeomorphisms).

Let $\mathcal{H}^u(X,d)$ denote the subgroup of uniform homeomorphisms of (X,d) onto itself. It is a topological group, while the whole group $\mathcal{H}(X,d)$ is not necessarily a topological group under the uniform topology. We also use the following notations:

$$\mathcal{H}^u_b(X,d):=\mathcal{H}^u(X,d)\cap\mathcal{H}_b(X,d),\quad \mathcal{H}^u(X,d;C):=\{h\in\mathcal{H}^u(X,d)\mid h=\mathrm{id}\text{ on }C\}$$

for $C\subset X$, and for $\alpha>0$ let $\mathcal{H}^u(\mathrm{id}_X,\alpha;X,d)$ denote the open α-neighborhood of id_X in $\mathcal{H}^u(X,d)$. The group $\mathcal{H}^u_b(X,d)$ is an open (and closed) subgroup of $\mathcal{H}^u(X,d)$, so it includes the connected component of id_X in $\mathcal{H}^u(X,d)$. The group $\mathcal{H}^u_b(X,d)$ also includes the subgroups $\mathcal{H}_c(X,d)\subset\mathcal{H}_0(X,d)$;

(i) $\mathcal{H}_c(X,d)=\{h\in\mathcal{H}(X,d)\mid \operatorname{supp}h:\text{compact}\}$,
(ii) $\mathcal{H}_0(X,d)=\{h\in\mathcal{H}(X,d)\mid h:\text{asymptotic to id}\}$.

Here (i) $\operatorname{supp}h=cl_X\{x\in X\mid h(x)\neq x\}$ and (ii) h is asymptotic to id if for any $\varepsilon>0$ there exists a compact subset K of (X,d) such that $d(h(x),x)<\varepsilon$ $(x\in X-K)$.

For subsets $F,C\subset X$, let $\mathcal{E}^u(F,(X,d);C)$ denote the space of uniform embeddings

$$f:(F,d|_F)\to(X,d)\quad\text{with}\quad f=\mathrm{id}\ \text{ on } F\cap C$$

with the sup-metric and the uniform topology.

Suppose $M=(M,d)$ is a metric n-manifold and $F,C\subset M$. An embedding $f:F\to M$ is said to be proper if $f^{-1}(\partial M)=F\cap\partial M$. Let $\mathcal{E}^u_*(F,M;C)$ denote the subspace of $\mathcal{E}^u(F,M;C)$ consisting of proper uniform embeddings and for $\alpha>0$ let $\mathcal{E}^u_*(i_F,\alpha;F,M;C)$ denote the open α-neighborhood of the inclusion map $i_F:F\subset M$ in the space $\mathcal{E}^u_*(F,M;C)$. Let $\mathcal{E}^u_*(F,M;C)_b=\{f\in\mathcal{E}^u_*(F,M;C)\mid d(f,i_F)<\infty\}$.

By deformations of uniform embeddings we mean the following type of homotopies in spaces of uniform embeddings:

Definition 2.2. Suppose $\mathcal{W}$ is a neighborhood of i_F in $\mathcal{E}^u_*(F, M; C)$ and $H \subset F$, $D \subset C$. An admissible deformation of $\mathcal{W}$ over H in $\mathcal{E}^u_*(F, M; D)$ means a homotopy

$$\varphi : \mathcal{W} \times [0,1] \longrightarrow \mathcal{E}^u_*(F, M; D)$$

which satisfies the following conditions:

(1) For each $f \in \mathcal{W}$ (i) $\varphi_0(f) = f$, (ii) $\varphi_1(f) = \mathrm{id}$ on H,
(iii) if $f = \mathrm{id}$ on $F \cap \partial M$, then $\varphi_t(f) = \mathrm{id}$ on $F \cap \partial M$ $(t \in [0,1])$.
(2) $\varphi_t(i_F) = i_F$ $(t \in [0,1])$.

2.3. **κ-cones.**

First we recall the definition of κ-cones over metric spaces (cf. [1, Chapter I.5]). The cone $C(Y)$ over a set Y is obtained from the set $Y \times [0,\infty)$ by collapsing the subset $Y \times \{0\}$ to one point. The equivalence class of the point $(x, s) \in Y \times [0,\infty)$ is denoted by sx. For $sx \in C(Y)$ and $t \in [0,\infty)$ the scalor multiplication is defined by $t(sx) = (ts)x$. Let

$$C(Y)_r := \{ty \in C(Y) \mid t \geq r\} \quad (r \geq 0).$$

Any function $f : Y \to Y'$ induces the associated function

$$C(f) : C(Y) \to C(Y') : C(f)(ty) = tf(y).$$

Suppose (Y, d) is a metric space. The metric d_π on Y is define by

$$d_\pi(x, y) := \min\{d(x, y), \pi\} \quad (x, y \in Y).$$

Definition 2.3. For $\kappa \leq 0$, the κ-cone $C_\kappa(Y, d)$ over (Y, d) is the metric space $(C(Y), \tilde{d}_\kappa)$, where the metric $\tilde{d}_\kappa$ on $C(Y)$ is defined by the following formula:

(i) for $\kappa = 0$: $\tilde{d}_\kappa(sx, ty)^2 = s^2 + t^2 - 2st\cos d_\pi(x, y)$,
(ii) for $\kappa < 0$: $\cosh(\sqrt{-\kappa}\,\tilde{d}_\kappa(sx, ty))$
$$= \cosh(\sqrt{-\kappa}\,s)\cosh(\sqrt{-\kappa}\,t) - \sinh(\sqrt{-\kappa}\,s)\sinh(\sqrt{-\kappa}\,t)\cos d_\pi(x, y).$$

The κ-cone ends over (Y, d) mean the subspaces $C_\kappa(Y, d)_r = (C(Y)_r, \tilde{d}_\kappa)$ $(r > 0)$ of the κ-cone $C_\kappa(Y, d)$.

The scalor multiplication $C_\kappa(Y, d) \times [0,\infty) \ni (u, t) \mapsto tu \in C_\kappa(Y, d)$ is continuous.

Example 2.1. (cf. [1, Proposition 5.8]) Consider the standard unit $(n-1)$-sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$ with the standard spherical metric d_1. Then, the κ-cone $C_\kappa(\mathbb{S}^{n-1})$ is isometric to M^n_κ for $\kappa \leq 0$. An isometry $\varphi : C_\kappa(\mathbb{S}^{n-1}) \approx M^n_\kappa$ is obtained by

$$\varphi(ty) = \exp_o(ty) \qquad (ty \in C_\kappa(\mathbb{S}^{n-1})),$$

where o is any fixed point of M^n_κ, $\exp_o : T_oM^n_\kappa \to M^n_\kappa$ is the exponential map and $\mathbb{S}^{n-1}$ is identified to the unit sphere $S(T_oM^n_\kappa)$ in the tangent space $T_oM^n_\kappa$.

The next remark is useful for the unified treatment of the cases $\kappa = 0$ and $\kappa < 0$.

Remark 2.1. (1) Consider the functions $\lambda_\kappa : \mathbb{R} \to \mathbb{R}$ $(\kappa \leq 0)$ defined by

$$\lambda_\kappa(u) = \begin{cases} \frac{u}{2} & (\kappa = 0) \\ \sinh\left(\frac{\sqrt{-\kappa}}{2}u\right) & (\kappa < 0). \end{cases}$$

It is seen that λ_κ is a diffeomorphism which is a monotonically increasing odd function. We use the abbreviation: $\lambda^2_\kappa(u) = (\lambda_\kappa(u))^2$.

(2) We have the following equalities for the metric $\tilde{d}_\kappa$ $(\kappa \leq 0)$: $\qquad (sx, ty \in C(Y))$

(i) $\lambda^2_\kappa(\tilde{d}_\kappa(sx, ty)) = \lambda^2_\kappa(s - t) + \lambda_\kappa(2s)\lambda_\kappa(2t)\sin^2 \frac{1}{2}d_\pi(x, y)$,

(ii) (a) $\tilde{d}_\kappa(sy, ty) = |s-t| \leq \tilde{d}_\kappa(sx, ty)$, (b) $\lambda_\kappa(\tilde{d}_\kappa(tx, ty)) = \lambda_\kappa(2t)\sin\frac{1}{2}d_\pi(x, y)$.

Suppose $f : (Y, d) \to (Y', d')$ is a map between metric spaces. Then, for any $\kappa \leq 0$ the κ-cone extension $C_\kappa(f) \equiv C(f) : C_\kappa(Y, d) \to C_\kappa(Y', d')$ is continuous. If $K \geq 1$ and $f : (Y, d) \to (Y', d')$ is K-Lipschitz, then so is the map $f_\pi \equiv f : (Y, d_\pi) \to (Y', d'_\pi)$.

Lemma 2.1. The following conditions are equivalent.

(1) f_π is Lipschitz. (2) $C_\kappa(f)$ is Lipschitz. (3) $C_\kappa(f)$ is uniformly continuous.

3. Uniform isotopies and level-wise uniform isotopies

3.1. Uniform isotopies.

Suppose (X, d) is a metric space. An isotopy on X is a homeomorphism $H \in \mathcal{H}(X \times [0, 1])$ which preserves $[0, 1]$-factor (i.e., $H(x, t) = (H_t(x), t)$ $((x, t) \in X \times [0, 1])$). By $\mathcal{H}(X \times [0, 1])^I$ we denote the subgroup of $\mathcal{H}(X \times [0, 1])$ consisting of all isotopies on X. If H is an isotopy on X, then $H_t \in \mathcal{H}(X)$ for each $t \in [0, 1]$ and we obtain the associated function $\widehat{H} : [0, 1] \to \mathcal{H}(X) : \widehat{H}(t) = H_t$.

The product space $X \times [0, 1]$ is given the metric $\tilde{d}$ defined by

$$\tilde{d}((x, t), (y, s)) = d(x, y) + |t - s| \quad ((x, t), (y, s) \in X \times [0, 1]).$$

Definition 3.1. An isotopy H on X is said to be

(i) a (bounded) uniform isotopy on (X, d) if $H \in \mathcal{H}^u_{(b)}(X \times [0, 1], \tilde{d})$,

(ii) a level-wise bounded (uniform) isotopy on (X, d) if $H_t \in \mathcal{H}^{(u)}_b(X, d)$ for each $t \in [0, 1]$.

Let $\mathcal{H}^u_{(b)}(X \times [0, 1], \tilde{d})^I$ and $\mathcal{H}^{lb(u)}(X \times [0, 1], \tilde{d})^I$ denote the subgroups of $\mathcal{H}(X \times [0, 1])^I$ consisting of (bounded) uniform isotopies and level-wise bounded (uniform) isotopies on (X, d), endowed with the sup-metric and the uniform topology defined by $\tilde{d}$.

Uniform isotopies are reinterpreted correctly as continuous paths in $\mathcal{H}^u(X,d)$.

Lemma 3.1.

(1) If H is a (bounded) uniform isotopy on (X,d), then H is a level-wise (bounded) uniform isotopy and the function $\widehat{H} : [0,1] \to \mathcal{H}^u_{(b)}(X,d)$, $\widehat{H}(t) = H_t$ is continuous.

(2) Any map $h : [0,1] \to \mathcal{H}^u_{(b)}(X,d)$ induces a (bounded) uniform isotopy $\tilde{h}$ on (X,d) by

$$\tilde{h}(x,t) = (h(t)(x),t) \quad ((x,t) \in X \times [0,1]).$$

(3) There exists a natural isometry

$$\eta : \big(\mathcal{H}^u_{(b)}(X \times [0,1],\tilde{d})^I,\tilde{d}\big) \longrightarrow \big(\mathcal{C}([0,1],(\mathcal{H}^u_{(b)}(X,d),d))_u,d\big) : \ \eta(H) = \widehat{H}, \ \ \eta^{-1}(h) = \tilde{h}.$$

From this lemma we obtain the following exponential law.

Proposition 3.1. For any topological space Z there exist natural isometries

$$\begin{aligned}(\mathcal{C}(Z \times [0,1],(\mathcal{H}^u_{(b)}(X,d),d)),d) \ &\underset{\cong}{\longrightarrow} \ \big(\mathcal{C}\big(Z,(\mathcal{C}([0,1],(\mathcal{H}^u_{(b)}(X,d),d))_u,d)\big),d\big) \\ &\underset{\cong}{\xleftarrow{\eta_\#}} \ \big(\mathcal{C}\big(Z,(\mathcal{H}^u_{(b)}(X \times [0,1],\tilde{d})^I,\tilde{d})\big),\tilde{d}\big).\end{aligned}$$

3.2. **Local contractibility of $\mathcal{H}^u(X,d)$.**

In the literature some authors introduced weaker notions of local contractibility of homeomorphism groups of noncompact manifolds (cf. [7, Section 5.6]). (One should note that these weaker notions are also denoted by the same terminology "local contractibility" and this ambiguity might lead to some misunderstanding.) In this subsection, for the group $\mathcal{H}^u_b(X,d)$ we clarify the relationship between the weaker notion and the standard notion of local contractibility.

First note that a topological group G is locally contractible if and only if a neighborhood of the unit element of G is contractible in G. Thus, Proposition 3.1 induces the following interpretation in term of continuous selection of uniform isotopies.

Corollary 3.1. (1) $\mathcal{H}^u_b(X,d)$ is locally contractible if and only if there exists a neighborhood $\mathcal{U}$ of id_X in $\mathcal{H}^u_b(X,d)$ and a map $\Phi : \mathcal{U} \to \mathcal{H}^u_b(X \times [0,1],\tilde{d})^I$ such that for each $h \in \mathcal{U}$ the image $\Phi(h)$ is a uniform isotopy from h to id_X.

(2) $\mathcal{H}^u_b(X,d)$ is contractible if and only if we can take $\mathcal{U} = \mathcal{H}^u_b(X,d)$ in (1).

The general formulation in [7, Section 5.6] leads to the following definition for $\mathcal{H}^u_b(X,d)$:

Definition 3.2. We say that $\mathcal{H}^{(u)}_b(X,d)$ is locally contractible* if there exist a neighborhood $\mathcal{U}$ of id_X in $\mathcal{H}^{(u)}_b(X,d)$ and a map

$$\Phi : \mathcal{U} \longrightarrow \mathcal{H}^{lb(u)}(X \times [0,1],\tilde{d})^I$$

such that for each $h \in \mathcal{U}$ the image $\Phi(h)$ is a level-wise bounded (uniform) isotopy from h to id_X. The map Φ is called a local contraction* of $\mathcal{H}_b^{(u)}(X,d)$. If we can take $\mathcal{U} = \mathcal{H}_b^{(u)}(X,d)$, then we say that $\mathcal{H}_b^{(u)}(X,d)$ is contractible* and call the map Φ a contraction* of $\mathcal{H}_b^{(u)}(X,d)$.

Obviously, if $\mathcal{H}_b^u(X,d)$ is (locally) contractible, then $\mathcal{H}_b^u(X,d)$ is (locally) contractible*. In the next subsection we see that the Alexander isotopies in a κ-cone yield a contraction*, but they do not induce a continuous contraction in general.

3.3. Alexander isotopies in κ-cones.

In this section we discuss basic properties of Alexander isotopies in the κ-cone $C_\kappa(X,d) = (C(X), \tilde{d}_\kappa)$ ($\kappa \le 0$) over any compact metric space (X,d) (cf. [4]). The metric $\tilde{d}_\kappa$ on $C_\kappa(X,d)$ induces the sup-metric $\tilde{d}_\kappa$ on $\mathcal{H}(C_\kappa(X,d))$ and the metric $\tilde{\tilde{d}}_\kappa$ on $C_\kappa(X,d) \times [0,1]$.

The κ-cone $C_\kappa(X,d)$ admits the radial transformations

$$\theta_t \in \mathcal{H}(C_\kappa(X,d)): \ \ \theta_t(u) = tu \qquad (t \in (0,\infty)).$$

For each $h \in \mathcal{H}_b(C_\kappa(X,d))$ we obtain

$$h_t \in \mathcal{H}_b(C_\kappa(X,d)) \ \ (t \in [0,1]): \quad h_t = \begin{cases} \theta_t h(\theta_{1/t}) & (t \in (0,1]), \\ \mathrm{id} & (t = 0), \end{cases}$$

and the Alexander isotopy for h;

$$\Phi(h) : C_\kappa(X,d) \times [0,1] \to C_\kappa(X,d) \times [0,1] :$$
$$\Phi(h)(u,t) = (h_t(u), t) = \begin{cases} (th(\tfrac{1}{t}u), t) & (t \in (0,1]), \\ (u, 0) & (t = 0), \end{cases}$$

which is a level-wise bounded isotopy on $C_\kappa(X,d)$ from id to h.

The function $\eta_h : [0,1] \longrightarrow \mathcal{H}_b(C_\kappa(X,d)) : \eta_h(t) = h_t$ is always continuous at $t = 0$.

Lemma 3.2. The isotopy $\Phi(h)$ is a uniform isotopy on $C_\kappa(X,d)$ if and only if η_h is continuous and $h_t \in \mathcal{H}_b^u(C_\kappa(X,d))$ $(t \in [0,1])$. In this case, $\Phi(h)$ is bounded.

We obtain an isometric embedding

$$\Phi : \mathcal{H}_b(C_\kappa(X,d)) \ni h \longmapsto \Phi(h) \in \mathcal{H}^{lb}(C_\kappa(X,d) \times [0,1], \tilde{\tilde{d}}_\kappa)^I,$$

which forms a contraction* of $\mathcal{H}_b(C_\kappa(X,d))$. (For $\kappa = 0$, it restricts to a contraction* $\Phi : \mathcal{H}_b^u(C_0(X,d)) \to \mathcal{H}^{lbu}(C_0(X,d) \times [0,1], \tilde{\tilde{d}}_0)^I$.) However, the associate functions

$$\varphi : \mathcal{H}_b(C_\kappa(X,d)) \times [0,1] \longrightarrow \mathcal{H}_b(C_\kappa(X,d)): \ \ \varphi(h,t) = h_t$$

and $\varphi : \mathcal{H}_b^u(C_0(X,d)) \times [0,1] \to \mathcal{H}_b^u(C_0(X,d))$ are not continuous for any compact Lipschitz metric manifold (X,d) as shown in the examples below.

Example 3.1. For any non-zero vector $v \in \mathbb{R}^n$ we can find $h \in \mathcal{H}_b^u(\mathbb{R}^n)$ such that

(i) $h((2k+1)v) = (2k+1)v$ $(k \in \mathbb{N})$ and

(ii) there exists $c > 0$ such that $d(h(2kv), 2kv) > c \quad (k \in \mathbb{N})$.

For any such h, the function η_h is not continuous at $t = 1$.

Proof. Assume the contrary, that η_h is continuous at $t = 1$. Then, there exists $t_0 \in [0, 1)$ such that $d(h_t, h) < c$ for any $t \in (t_0, 1]$. Since $\frac{2k}{2k+1} \to 1$ $(k \to \infty)$, there exists $k \in \mathbb{N}$ such that $t := \frac{2k}{2k+1} > t_0$. It follows that

$$h_t(2kv) = th\left(\tfrac{1}{t}2kv\right) = th((2k+1)v) = t(2k+1)v = 2kv \quad \text{so that}$$

$$d(h_t, h) \geq d(h_t(2kv), h(2kv)) = d(2kv, h(2kv)) > c.$$

This contradicts $d(h_t, h) < c$. □

This argument extends to the κ-cone case.

Example 3.2. For any compact Lipschitz metric manifold (X, d) there exists $h \in \mathcal{H}_b^u(C_\kappa(X, d))$ for which the function η_h is not continuous at $t = 1$.

One can expect that the function φ restricts to *continuous* contractions of some subgroups of $\mathcal{H}_b^u(C_\kappa(X, d))$. Consider the following subgroup of $\mathcal{H}_b^u(C_\kappa(X, d))$:

$$\mathcal{G} = \mathcal{G}(C_\kappa(X, d)) := \{h \in \mathcal{H}_b^u(C_\kappa(X, d)) \mid \Phi(h) \text{ is a uniform isotopy.}\}.$$

Proposition 3.2.

(1) (i) The function φ restricts to a contraction $\varphi : \mathcal{G} \times [0, 1] \to \mathcal{G}$.

(ii) The subset $\mathcal{G}$ is maximal in $\mathcal{H}_b^u(C_\kappa(X, d))$ in the following sense:

($\natural$) If $\mathcal{S} \subset \mathcal{H}_b^u(C_\kappa(X, d))$ and $\varphi : \mathcal{S} \times [0, 1] \to \mathcal{H}^u(C_\kappa(X, d))$ is continuous, then $\mathcal{S} \subset \mathcal{G}$.

(2) Both $\mathcal{H}_0(C_\kappa(X, d))$ and $\mathcal{H}_c(C_\kappa(X, d))$ are normal subgroups of $\mathcal{G}$ and the contraction φ of $\mathcal{G}$ restricts to the contractions of these subgroups.

In particular, in the case $\mathbb{R}^n = C_0(\mathbb{S}^{n-1})$, the function φ restricts to a continuous contraction of $\mathcal{H}_c(\mathbb{R}^n)$.

Remark 3.1. Let $\mathcal{H}_c(\mathbb{R}^n)_{co}$ denote the group $\mathcal{H}_c(\mathbb{R}^n)$ endowed with the compact-open topology. For $n \geq 2$, the function $\varphi : \mathcal{H}_c(\mathbb{R}^n)_{co} \times [0, 1] \to \mathcal{H}_c(\mathbb{R}^n)_{co}$ is not continuous at $(\mathrm{id}_{\mathbb{R}^n}, 0)$.

In [8] we showed that $\mathcal{H}_b^u(\mathbb{R}^n)$ is contractible in the usual sense. There we adopted the following strategy: (1) Since $\mathbb{R}^n$ admits a metric covering projection onto the flat torus, $\mathbb{R}^n$ has the property (LD) and there exists a contraction χ_t of a small open ball $\mathcal{H}^u(\mathrm{id}, \varepsilon; \mathbb{R}^n)$ in $\mathcal{H}_b^u(\mathbb{R}^n)$ (see Section 4). (2) Given $\alpha > 0$, using χ_t and a similarity transformation θ_γ, we can find $\beta > 0$ and a contraction ψ_t of the open ball $\mathcal{H}^u(\mathrm{id}, \alpha; \mathbb{R}^n)$ in $\mathcal{H}^u(\mathrm{id}, \beta; \mathbb{R}^n)$. (3) The iteration of (2) yields a sequence of contractions ψ_t^i of $\mathcal{H}^u(\mathrm{id}, \alpha_i; \mathbb{R}^n)$ in $\mathcal{H}^u(\mathrm{id}, \alpha_{i+1}; \mathbb{R}^n)$

for some increasing sequence $\alpha_i \in \mathbb{R}$ $(i \in \mathbb{N})$. A contraction of $\mathcal{H}_b^u(\mathbb{R}^n)$ is obtained by composing these contractions ψ_t^i. In this argument, we need not change the scale factor γ of the similarity transformation θ_γ continuously.

4. Local deformation property of uniform embeddings

In this section we summarize basic facts on local deformation property of uniform embeddings (LD) and discuss the case of κ-cone ends ($\kappa \le 0$) over compact Lipschitz metric manifolds. Throughout this section $M = (M, d)$ denotes a metric manifold.

4.1. Definition and basic facts.

Definition 4.1.

(1) An admissible tuple in M is a tuple (X, W', W, Z, Y) such that $X \subset_u W' \subset W \subset M$ and $Z \subset_u Y \subset M$. Let $\mathcal{S}(M)$ denote the collection of all admissible tuples in M.

(2) An admissible deformation for $\mathcal{X} = (X, W', W, Z, Y) \in \mathcal{S}(M)$ is an admissible deformation

$$\varphi : \mathcal{W} \times [0,1] \longrightarrow \mathcal{E}_*^u(W, M; Z)$$

of a neighborhood $\mathcal{W}$ of i_W in $\mathcal{E}_*^u(W, M; Y)$ over X (in the sense of Definition 2.2) such that for each $(f, t) \in \mathcal{W} \times [0,1]$

(i) $\varphi_t(f) = f$ on $W - W'$ and (ii) $\varphi_t(f)(W) = f(W)$.

Definition 4.2. (LD)

(1) For $\mathcal{X} \in \mathcal{S}(M)$, $(\mathrm{LD})_M(\mathcal{X}) \iff \exists$ an admissible deformation for $\mathcal{X}$.

(2) For $A \subset M$ $\quad A : (\mathrm{LD})_M$

$$\iff (\mathrm{LD})_M(\mathcal{X}) \text{ for any } \mathcal{X} = (X, W', W, Z, Y) \in \mathcal{S}(M) \text{ with } X \subset A$$

(3) $M : (\mathrm{LD}) \iff M : (\mathrm{LD})_M \iff (\mathrm{LD})_M(\mathcal{X})$ for any $\mathcal{X} \in \mathcal{S}(M)$

Basic Example 4.1. The local deformation theorem by A.V. Černavskiĭ ([2]) and R. D. Edwards - R. C. Kirby ([3, Theorem 5.1]) is restated as follows:

(∗) Any relatively compact subset K of M has the property $(\mathrm{LD})_M$.

The condition (LD) has the following basic properties.

Proposition 4.1.

(1) Suppose $h : (M, d) \approx (M', d')$ is a uniform homeomorphism. Then

(i) for any $A \subset M$, $\quad A : (\mathrm{LD})_M \iff h(A) : (\mathrm{LD})_{M'}$ and

(ii) $M : (\mathrm{LD}) \iff M' : (\mathrm{LD})$

(2) (i) Suppose $A \subset B \subset M$. Then $B : (\mathrm{LD})_M \implies A : (\mathrm{LD})_M$.

(ii) Suppose $A \subset_u N \subset M$ and N is an n-manifold. Then,

$$A : (\mathrm{LD})_{(N,d|_N)} \iff A : (\mathrm{LD})_M.$$

(3) Suppose $A \subset_u U \subset M$ and $B \subset M$. Then $U,\ B : (\mathrm{LD})_M \implies A \cup B : (\mathrm{LD})_M$.

(4) Suppose K is a relatively compact subset of M. Then, for any $A \subset M$

$$A : (\mathrm{LD})_M \iff A \cup K : (\mathrm{LD})_M.$$

(5) Suppose $M = K \cup \cup_{i=1}^m L_i$, K is relatively compact, each L_i is an n-manifold and closed in M, and $d(L_i, L_j) > 0$ for any $i \neq j$. Then,

$$M : (\mathrm{LD}) \iff (L_i, d|_{L_i}) : (\mathrm{LD})\ (i = 1, \cdots, m).$$

(6) If M : (LD), then $\mathcal{H}^u(M)$ and $\mathcal{H}^u(M, \partial M)$ are locally contractible.

4.2. **Examples.**

In this subsection we list some basic examples of metric manifolds with the property (LD). This enables us to deduce deformation results in more complicated metric manifolds from those in simpler pieces.

[1] Metric covering spaces and geometric group actions

The notion of metric covering spaces was introduced in [8] as a natural metric version of Riemannian coverings in the smooth category.

Definition 4.3. A map $\pi : (X, d) \to (Y, \varrho)$ is called a metric covering projection if it satisfies the following conditions:

$(*)_1$ There exists an open cover $\mathcal{U}$ of Y such that for each $U \in \mathcal{U}$ the inverse $\pi^{-1}(U)$ is the disjoint union of open subsets of X each of which is mapped isometrically onto U by π.

$(*)_2$ For each $y \in Y$ the fiber $\pi^{-1}(y)$ is uniformly discrete in X.

$(*)_3$ $\varrho(\pi(x), \pi(x')) \leq d(x, x')$ for any $x, x' \in X$.

The next theorem follows from Basic Example 4.1 and the Arzela-Ascoli theorem.

Theorem 4.1. If $\pi : (M, d) \to (N, \varrho)$ is a metric covering projection and (N, ϱ) is a compact metric manifold, then (M, d) has the property (LD).

This result extends to the case of metric manifolds with geometric group actions. An action Φ of a (discrete) group G on a locally compact metric space (X, d) is called geometric if it is proper, cocompact and isometric (cf. [1, Chapter I.8]). We work in a slightly more general setting.

Definition 4.4. The action Φ of G on X is said to be

(1) locally isometric if for every $x \in X$ there exists $\varepsilon \in (0, \infty]$ such that

$(\natural)_x$ each $g \in G$ maps $O_\varepsilon(x)$ isometrically onto $O_\varepsilon(gx)$, and

(2) locally geometric if it is proper, cocompact and locally isometric.

Theorem 4.2. A metric manifold has the property (LD) if it admits a locally geometric group action.

[2] The κ-cone ends ($\kappa \leq 0$) over compact Lipschitz metric manifolds

The property (LD) for a metric manifold depends only on the property (LD) for its ends (Proposition 4.1(5)). This observation leads us to study of the property (LD) for typical metric ends.

The Euclidean space $\mathbb{R}^n$, the hyperbolic space $\mathbb{H}^n$ and the cylinder over any compact metric manifold have the property (LD), since they admit free geometric group actions (Theorem 4.2). Hence their ends also have the property (LD). This implies that the κ-cone ends $C_\kappa(\mathbb{S}^{n-1})_1$ ($\kappa \leq 0$) have the property (LD). By Proposition 4.1 (1), (3) this generalizes to the following form.

Theorem 4.3. Suppose (N, d) is a compact Lipschitz metric manifold. Then for any $\kappa \leq 0$ the κ-cone end $C_\kappa(N, d)_1$ has the property (LD).

5. End deformation property of uniform embeddings

In this section we introduce the notion of end deformation property of uniform embeddings (ED) in proper product ends and study its basic feature.

5.1. Definition and basic facts.

Definition 5.1. An n-dimensional proper product end is a metric n-manifold (L, d) such that (i) the metric d is proper and (ii) there exists a homeomorphism $\theta : S \times [1, \infty) \approx L$ for some compact $(n-1)$-manifold S.

Suppose (L, d) is a proper product end. A subset F of L is said to be cofinal if $L - F$ is relatively compact in L. By $\mathcal{CF}(L)$ we denote the collection of cofinal closed subsets of L. We fix a homeomorphism $\theta : S \times [1, \infty) \approx L$ and let $L_I := \theta(S \times I)$ for $I \subset [1, \infty)$ and $L_r := L_{[r,\infty)}$ for $r \geq 1$. Note that (i) $F \subset L$ is cofinal if and only if $L_r \subset F$ for some $r \in [1, \infty)$ and (ii) $d(L_{\{1\}}, L_r) \to \infty$ $(r \to \infty)$ since the metric d is proper.

Definition 5.2. Suppose (L, d) is a proper product end. We say that (L, d) has the property (ED) and write (L, d) : (ED) if the following condition is satisfied:

(∗) For any $F \in \mathcal{CF}(L)$ and any $\alpha > 0$ there exist $\beta > 0$, $H \in \mathcal{CF}(L)$ with $H \subset F$ and an admissible deformation over H (in the sense of Definition 2.2)

$$\varphi : \mathcal{E}^u_*(i_F, \alpha; F, (L, d)) \times [0, 1] \longrightarrow \mathcal{E}^u_*(i_F, \beta; F, (L, d))$$

such that

$$(*)_1 \ \varphi_t(f) = f \ \text{ on } \ \mathrm{Fr}_L F \quad \text{for each } (f,t) \in \mathcal{E}^u_*(i_F, \alpha; F, (L,d)) \times [0,1].$$

Remark 5.1. The deformation φ in Definition 5.2 also satisfies the following condition:

$$(*)_2 \ \ \varphi_t(f)(F) = f(F) \quad \text{for each } (f,t) \in \mathcal{E}^u_*(i_F, \alpha; F, (L,d)) \times [0,1].$$

Proposition 5.1. Suppose (L,d), (L',d') are proper product ends and $h : (L,d) \approx (L',d')$ is a uniform, coarsely uniform homeomorphism. If (L,d) : (ED), then so is (L',d').

Example 5.1. Suppose (N,d) is a compact metric manifold. If $C_0(N,d)_1$: (LD), then $C_0(N,d)_1$: (ED). In particular, the 0-cone end $C_0(N,d)_1$ over any compact Lipschitz metric manifold (N,d) has the property (ED).

5.2. **End deformation theorem for uniform embeddings.**

Suppose (M,d) is a metric n-manifold. A proper product end of M is a closed subset L of M such that $\mathrm{Fr}_M L$ is compact and $(L, d|_L)$ is a proper product end. Suppose L is a proper product end of M. We say that the end L is isolated if $d(M - L, L_r) \to \infty$ as $r \to \infty$. Let $\mathcal{CF}(M,L)$ denote the collection of closed subsets F of M such that $F \cap L \in \mathcal{CF}(L)$.

If a metric manifold M has finitely many isolated proper product ends with the property (ED), then the spaces of uniform embeddings in M admit strong deformation retractions of the following form:

Theorem 5.1. Suppose (M,d) is a metric manifold and $L(1), \cdots, L(m) \subset M$ are pairwise disjoint isolated proper product ends of M. If $(L(i), d|_{L(i)})$: (ED) for each $i = 1, \cdots, m$, then for any $F \subset M$ with $F \in \mathcal{CF}(M, L(i))$ $(i = 1, \cdots, m)$ and any $s_i > r_i > 1$ with $L(i)_{r_i} \subset \mathrm{Int}_M(F \cap L(i))$ $(i = 1, \cdots, m)$ there exists a strong deformation retraction

$$\varphi \ \text{ of } \ \mathcal{E}^u_*(F,M)_b \ \text{ onto } \ \mathcal{E}^u_*(F, M; \cup_{i=1}^m L(i)_{s_i})_b$$

such that

(1) for each $(f,t) \in \mathcal{E}^u_*(F,M)_b \times [0,1]$
  (i) $\varphi_t(f) = f$ on $f^{-1}(M - \mathrm{Int}_M(\cup_{i=1}^m L(i)_{r_i})) - \mathrm{Int}_M(\cup_{i=1}^m L(i)_{r_i})$,
  (ii) if $\cup_{i=1}^m L(i)_{r_i} \subset f(F)$, then $\varphi_t(f)(F) = f(F)$,
  (iii) if $f = \mathrm{id}$ on $F \cap \partial M$, then $\varphi_t(f) = \mathrm{id}$ on $F \cap \partial M$.

(2) for each $\alpha > 0$ there exists $\beta > 0$ such that

$$\varphi(\mathcal{E}^u_*(i_F, \alpha; F, M) \times [0,1]) \subset \mathcal{E}^u_*(i_F, \beta; F, M).$$

As a corollary we have the following deformation theorem for uniform homeomorphisms.

Corollary 5.1. Suppose (M,d) is a metric manifold and $L(1), \cdots, L(m) \subset M$ are pairwise disjoint isolated proper product ends of M such that $(L(i), d|_{L(i)})$ has the property (ED) and $L(i)_2 \subset \mathrm{Int}_M L(i)$ for $i = 1, \cdots, m$. Let $L_r = L(1)_r \cup \cdots \cup L(m)_r$ $(r \geq 1)$. Then there exists a strong deformation retraction φ of $\mathcal{H}^u_b(M)$ onto $\mathcal{H}^u_b(M; L_3)$ such that

(1) for each $(h,t) \in \mathcal{H}^u_b(M)\times[0,1]$ (i) $\varphi_t(h) = h$ on $h^{-1}(M - \mathrm{Int}_M L_2) - \mathrm{Int}_M L_2$,
(ii) if $h = \mathrm{id}$ on ∂M, then $\varphi_t(h) = \mathrm{id}$ on ∂M,

(2) for each $\alpha > 0$ there exists $\beta > 0$ such that

$$\varphi(\mathcal{H}^u(\mathrm{id}_M, \alpha; M) \times [0,1]) \subset \mathcal{H}^u(\mathrm{id}_M, \beta; M).$$

We close this article with a problem which requires further investigation.

Problem 5.1. Examine the property (ED) for the κ-cone ends $C_\kappa(M)_1$ ($\kappa < 0$) over typical compact Riemannian manifolds M.

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