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COHOMOLOGY OF TORIC ORIGAMI MANIFOLDS

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ABSTRACT. Toric origami manifolds, introduced in [2], are generalizations of symplectic toric manifolds. In this note, we study the topology of orientable toric origami manifolds with acyclic proper faces. This note is based on the joint work with Anton Ayzenberg, Mikiya Masuda and Seonjeong Park, and more details can be found in our paper [1].

1. TORIC ORIGAMI MANIFOLDS

In this section we recall the definitions and properties of toric origami manifolds and origami templates. Details can be found in [2], [8] or [5].

A folded symplectic form on a $2n$-dimensional manifold M is a closed 2-form ω whose top power ω^n vanishes transversally on a subset W and whose restriction to points in W has maximal rank. Then W is a codimension-one submanifold of M and is called the fold. If W is empty, ω is a genuine symplectic form. The pair (M,ω) is called a folded symplectic manifold. Since the restriction of ω to W has maximal rank, it has a one-dimensional kernel at each point of W. This determines a line field on W called the null foliation. If the null foliation is the vertical bundle of some principal S^1-fibration $W\to X$ over a compact base X, then the folded symplectic form ω is called an origami form and the pair (M,ω) is called an origami manifold. The action of a torus T on an origami manifold (M,ω) is Hamiltonian if it admits a moment map $\mu\colon M\to \mathfrak{t}^*$ to the dual Lie algebra of the torus, which satisfies the conditions: (1) μ is equivariant with respect to the given action of T on M and the coadjoint action of T on the vector space $\mathfrak{t}^*$ (this action is trivial for the torus); (2) μ collects Hamiltonian functions, that is, $d\langle\mu,V\rangle=\imath_{V^\#}\omega$ for any $V\in\mathfrak{t}$, where $V^\#$ is the vector field on M generated by V.

Definition. A toric origami manifold (M,ω,T,μ), abbreviated as M, is a compact connected origami manifold (M,ω) equipped with an effective Hamiltonian action of a torus T with $\dim T=\frac{1}{2}\dim M$ and with a choice of a corresponding moment map μ.

When the fold W is empty, a toric origami manifold is a symplectic toric manifold. A theorem of Delzant [3] says that symplectic toric manifolds are classified by their moment images called Delzant polytopes. Recall that a Delzant polytope in $\mathbb{R}^n$ is a simple convex polytope, whose normal fan is smooth (with respect to some given lattice $\mathbb{Z}^n\subset\mathbb{R}^n$). In other words, all normal vectors to facets of P have rational coordinates, and, whenever facets $F_1,\dots,F_n$ meet in a vertex of P, the primitive normal vectors $\nu(F_1),\dots,\nu(F_n)$ form a basis of the lattice $\mathbb{Z}^n$. Let $\mathcal{D}_n$ denote the set of all Delzant polytopes in $\mathbb{R}^n$ (w.r.t. a given lattice) and $\mathcal{F}_n$ be the set of all their facets.

The moment data of a toric origami manifold can be encoded into an origami template (G,Ψ_V,Ψ_E), where

- G is a connected graph (loops and multiple edges are allowed) with the vertex set V and edge set E;

- $\Psi_V : V \to \mathcal{D}_n$;
- $\Psi_E : E \to \mathcal{F}_n$;

subject to the following conditions:

- If $e \in E$ is an edge of G with endpoints $v_1, v_2 \in V$, then $\Psi_E(e)$ is a facet of both polytopes $\Psi_V(v_1)$ and $\Psi_V(v_2)$, and these polytopes coincide near $\Psi_E(e)$ (this means there exists an open neighborhood U of $\Psi_E(e)$ in $\mathbb{R}^n$ such that $U \cap \Psi_V(v_1) = U \cap \Psi_V(v_2)$).
- If $e_1, e_2 \in E$ are two edges of G adjacent to $v \in V$, then $\Psi_E(e_1)$ and $\Psi_E(e_2)$ are disjoint facets of $\Psi(v)$.

The facets of the form $\Psi_E(e)$ for $e \in E$ are called the fold facets of the origami template.

Example. The following picture is an example of origami templates.

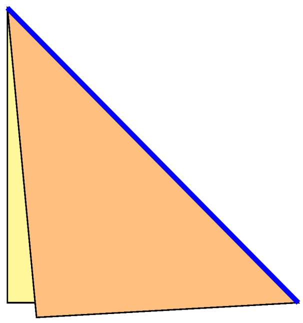

FIGURE 1. The origami template with two copies of isosceles right triangles

The following is a generalization of the theorem by Delzant to toric origami manifolds.

Theorem 1.1 ([2]). *Assigning the moment data of a toric origami manifold induces a one-to-one correspondence*

$$\{\textit{toric origami manifolds}\} \leftrightsquigarrow \{\textit{origami templates}\}$$

up to equivariant origami symplectomorphism on the left-hand side, and affine equivalence on the right-hand side.

Denote by $|(G, \Psi_V, \Psi_E)|$ the topological space constructed from the disjoint union $\bigsqcup_{v \in V} \Psi_V(v)$ by identifying facets $\Psi_E(e) \subset \Psi_V(v_1)$ and $\Psi_E(e) \subset \Psi_V(v_2)$ for any edge $e \in E$ with endpoints v_1, v_2.

An origami template (G, Ψ_V, Ψ_E) is called coörientable if the graph G has no loops (this means all edges have different endpoints). Then the corresponding toric origami manifold is also called coörientable. If M is orientable, then M is coörientable [5]. If M is coörientable, then the action of T^n on M is locally standard [5, lemma 5.1].

Let (G, Ψ_V, Ψ_E) be an origami template and M the associated toric origami manifold which is supposed to be orientable in the following. The topological space $|(G, \Psi_V, \Psi_E)|$ is a manifold with corners with the face structure induced from the face structures on polytopes $\Psi_V(v)$, and $|(G, \Psi_V, \Psi_E)|$ is homeomorphic to M/T as a manifold with corners. The space $|(G, \Psi_V, \Psi_E)|$ has the same homotopy type as the graph G, thus $M/T \cong |(G, \Psi_V, \Psi_E)|$ is either contractible or homotopy equivalent to a wedge of circles.

Under the correspondence of Theorem 1.1 the fold facets of the origami template correspond to the connected components of the fold W of M. If $F = \Psi_E(e)$ is a fold facet of the template (G, Ψ_V, Ψ_E), then the corresponding component $Z = \mu^{-1}(F)$ of the fold $W \subset M$ is a principal S^1-bundle over a compact space B. The space B is a $(2n-2)$-dimensional symplectic toric manifold corresponding to the Delzant polytope F.

2. Betti numbers of toric origami manifolds

From the classifying Theorem 1.1, a natural question is how to describe the cohomology ring and T-equivariant cohomology ring of a toric origami manifold M in terms of its corresponding origami template. If M is simply connected, i.e. the associated graph G is a tree, this question is answered by Masuda and Panov in [7] and Holm and Pires in [5]. However, if M is non-simply connected, this question is still open in general, even for the betti numbers unless the case where $\dim M = 4$ is solved by Holm and Pires in [6]. In this section, we will give an explicit formula for the betti numbers of M when M is orientable and every proper face of M/T is acyclic. Our first main result is the following.

Theorem 2.1. *Let M be an orientable toric origami manifold of dimension $2n$ $(n \geq 2)$ such that every proper face of M/T is acyclic. Then*

$$\begin{aligned} b_{2i+1}(M) &= 0 \quad \textit{for } 1 \leq i \leq n-2, \\ b_1(M) &= b_{2n-1}(M) = b_1(M/T). \end{aligned}$$

Moreover, $H^*(M)$ is torsion free.

We can describe $b_{2i}(M)$ in terms of the face numbers of M/T and $b_1(M)$. Let $\mathcal{P}$ be the simplicial poset dual to $\partial(M/T)$. As usual, we define

$$\begin{aligned} f_i = & \text{ the number of } (n-1-i)\text{-faces of } M/T \\ = & \text{ the number of } i\text{-simplices in } \mathcal{P} \quad \text{for } i = 0, 1, \ldots, n-1 \end{aligned}$$

and the h-vector $(h_0, h_1, \ldots, h_n)$ by

$$\sum_{i=0}^{n} h_i t^{n-i} = (t-1)^n + \sum_{i=0}^{n-1} f_i (t-1)^{n-1-i}. \tag{2.1}$$

Theorem 2.2. *Let M be an orientable toric origami manifold of dimension $2n$ such that every proper face of M/T is acyclic. Let b_j be the j-th Betti number of M and $(h_0, h_1, \ldots, h_n)$ be the h-vector of M/T. Then*

$$\sum_{i=0}^{n} b_{2i} t^i = \sum_{i=0}^{n} h_i t^i + b_1(1 + t^n - (1-t)^n),$$

in other words, $b_0 = h_0 = 1$ and

$$\begin{aligned} b_{2i} &= h_i - (-1)^i \binom{n}{i} b_1 \quad \textit{for } 1 \leq i \leq n-1, \\ b_{2n} &= h_n + (1 - (-1)^n) b_1. \end{aligned}$$

Example. Let M be the 4-dimensional toric origami manifold corresponding to the origami template shown on fig.2 (Example 3.15 of [2]). It is easy to check that M satisfies the condition of our theorems from fig.2. The f-vector $(f_0, f_1) = (8, 8)$, so the h-vector $(h_0, h_1, h_2) = (1, 6, 1)$. Then applying Theorem 2.1 and Theorem 2.2, we have $b_0 = b_4 = 1$, $b_1 = b_3 = 1$ and $b_2 = 8$.

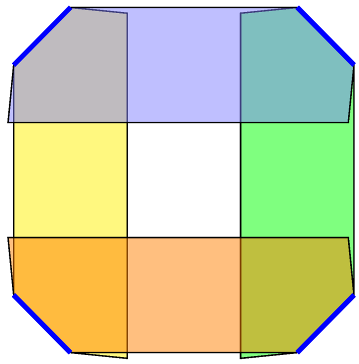

FIGURE 2. The origami template with four polygons

3. Towards the ring structure

A torus manifold M of dimension $2n$ is an orientable connected closed smooth manifold with an effective smooth action of an n-dimensional torus T having a fixed point ([4]). An orientable toric origami manifold with acyclic proper faces in the orbit space has a fixed point, so it is a torus manifold. The action of T on M is called *locally standard* if every point of M has a T-invariant open neighborhood equivariantly diffeomorphic to a T-invariant open set of a faithful representation space of T. Then the orbit space M/T is a nice manifold with corners. The torus action on an orientable toric origami manifold is locally standard. In this section, we study the cohomology ring structure of an orientable toric origami manifold with acyclic proper faces of the orbit space.

Let $\mathcal{P}$ be the poset dual to the face poset of M/T as before.

Proposition 3.1. *Let M be a locally standard torus manifold such that every proper face of M/T is acyclic, and the free part of the action gives a trivial principal bundle $M^\circ \to M^\circ/T$. Then $H_T^*(M) \cong \mathbb{Z}[\mathcal{P}] \oplus \tilde{H}^*(M/T)$ as graded rings.*

Let $\pi\colon ET \times_T M \to BT$ be the projection. Since $\pi^*(H^2(BT))$ maps to zero by the restriction homomorphism $\iota^*\colon H_T^*(M) \to H^*(M)$, ι^* induces a graded ring homomorphism

$$\bar{\iota}^*\colon H_T^*(M)/(\pi^*(H^2(BT))) \to H^*(M). \tag{3.1}$$

Proposition 3.2. *Let M be an orientable toric origami manifold of dimension $2n$ such that every proper face of M/T is acyclic, then $\bar{\iota}^*$ in (3.1) is an isomorphism except in degrees 2, 4 and $2n-1$. Moreover, the rank of the cokernel of $\bar{\iota}^*$ in degree 2 is $nb_1(M)$ and the rank of the kernel of $\bar{\iota}^*$ in degree 4 is $\binom{n}{2}b_1(M)$.*

Example. Let M be the toric origami manifold corresponding to the origami template shown on fig.2. Topologically M/T is homeomorphic to $S^1 \times [0,1]$ and the boundary of M/T as a manifold with corners consists of two boundaries of 4-gons. The multi-fan of M is the union of two copies of the fan of $\mathbb{C}P^1 \times \mathbb{C}P^1$ with the product torus action. Indeed, if v_1, v_2 are primitive edge vectors in the fan of $\mathbb{C}P^1 \times \mathbb{C}P^1$ which spans a 2-dimensional cone, then the other primitive edge vectors $v_3, \dots, v_8$ in the multi-fan of M are

$$v_3 = -v_1, \quad v_4 = -v_2, \quad \text{and} \quad v_i = v_{i-4} \quad \text{for } i = 5, \dots, 8$$

and the 2-dimensional cones in the multi-fan are

$$\angle v_1v_2, \quad \angle v_2v_3, \quad \angle v_3v_4, \quad \angle v_4v_1,$$
$$\angle v_5v_6, \quad \angle v_6v_7, \quad \angle v_7v_8, \quad \angle v_8v_5,$$

where $\angle vv'$ denotes the 2-dimensional cone spanned by vectors v, v'. Note that

$$\tau_i\tau_j = 0 \text{ if } v_i, v_j \text{ do not span a 2-dimensional cone.} \tag{3.2}$$

We have

$$\pi^*(u) = \sum_{i=1}^{8} \langle u, v_i\rangle \tau_i \quad \text{for any } u \in H^2(BT). \tag{3.3}$$

Let v_1^*, v_2^* be the dual basis of v_1, v_2. Taking $u = v_1^*$ or v_2^* , we see that

$$\tau_1 + \tau_5 = \tau_3 + \tau_7, \quad \tau_2 + \tau_6 = \tau_4 + \tau_8 \quad \text{in } H_T^*(M)/(\pi^*(H^2(BT))). \tag{3.4}$$

Since we applied (3.3) for the basis v_1^*, v_2^* of $H^2(BT)$, there is no other essentially new linear relation among τ_i's.

Now, multiply the equations (3.4) by τ_i and use (3.2). Then we obtain

$$\begin{aligned} &\tau_i^2 = 0 \quad \text{for any } i,\\ &(\mu_1 :=)\tau_1\tau_2 = \tau_2\tau_3 = \tau_3\tau_4 = \tau_4\tau_1,\\ &(\mu_2 :=)\tau_5\tau_6 = \tau_6\tau_7 = \tau_7\tau_8 = \tau_8\tau_5 \quad \text{in } H_T^*(M)/(\pi^*(H^2(BT))). \end{aligned}$$

Our argument shows that these together with (3.2) are the only degree two relations among τ_i's in $H_T^*(M)/(\pi^*(H^2(BT)))$. The kernel of

$$\iota^*\colon H_T^{even}(M;\mathbb{Q})/(\pi^*(H^2(BT;\mathbb{Q}))) \to H^{even}(M;\mathbb{Q})$$

in degree 4 is spanned by $\mu_1 - \mu_2$.

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