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Spaces of equivariant maps to toric varieties

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Abstract

The main purpose of this note is to announce the recent result in [13] concerning the homotopy type of spaces of algebraic maps from a real projective space to a compact smooth real toric variety. This note is also based on the joint work with Andrzej Kozlowski and Masahiro Ohno [8].

Toric varieties. An irreducible normal algebraic variety X (over $\mathbb{C}$) is called *a toric variety* if it has an algebraic action of the complex algebraic torus $\mathbb{T}_{\mathbb{C}}^r=(\mathbb{C}^*)^r$, such that the orbit $\mathbb{T}_{\mathbb{C}}^r\cdot *$ of some point $*\in X$ is dense in X and isomorphic to $\mathbb{T}_{\mathbb{C}}^r$.

A strong convex rational polyhedral cone σ in $\mathbb{R}^n$ is a subset of $\mathbb{R}^n$ of the form $\sigma=\{\sum_{k=1}^s a_k\mathbf{n}_k | a_k\geq 0\}$, such that the set $\{\mathbf{n}_k\}_{k=1}^s\subset \mathbb{Z}^n$ does not contain any line.

A finite collection Σ of strongly convex rational polyhedral cones in $\mathbb{R}^n$ is called *a fan* if every face of element of Σ is belongs to Σ and the intersection of any two elements of Σ is a face of each other. It is well-known that a toric variety X is completely characterized up to isomorphism by the fan Σ. We denote by X_Σ the corresponding toric variety associated to Σ. A cone σ in $\mathbb{R}^n$ is called *smooth* (reps. *simplicial*) if it is generated by a subset of a basis of $\mathbb{Z}^n$ (resp. a subset of a basis of $\mathbb{R}^n$). A fan Σ is called *complete* if $\bigcup_{\sigma\in\Sigma}\sigma=\mathbb{R}^n$. It is known that X_Σ is compact iff Σ is complete, and that X_Σ is smooth iff every $\sigma\in\Sigma$ is smooth [4, Theorem 1.3.12]. It is also known that $\pi_1(X_\Sigma)$ is isomorphic to the quotient of $\mathbb{Z}^n$ by the subgroup generated by $\bigcup_{\sigma\in\Sigma}\sigma\cap\mathbb{Z}^n$. [4, Theorem 12.1.10]. In particular, X_Σ is simply connected if it is compact.

Real toric varieties. For a fan Σ, let $X_{\Sigma,\mathbb{R}}$ denote the subspace of X_Σ consisting of all real points of X_Σ. Alternatively the space $X_{\Sigma,\mathbb{R}}$ is given by replacing the complex numbers $\mathbb{C}$ by the real numbers $\mathbb{R}$ everywhere in the given definitions of a toric variety X_Σ [18, Def. 6.1], and it is called *a real toric variety.* Note that $X_{\Sigma,\mathbb{R}}$ with the intersection $X_{\Sigma,\mathbb{R}}=X_\Sigma\cap\mathbb{R}\mathrm{P}^N$ when X_Σ is a toric variety embedded in $\mathbb{C}\mathrm{P}^N$.

Homogenous coordinates of $X_{\Sigma,\mathbb{K}}$. We shall use the symbols $\{z_k\}_{k=1}^r$ to denote variables of polynomials, and we assume that $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$. For polynomials $f_1,\cdots,f_s\in$

$\mathbb{K}[z_1, \cdots, z_r]$, let $V_{\mathbb{K}}(f_1, \cdots, f_s)$ denote the affine variety

$$V_{\mathbb{K}}(f_1, \cdots, f_s) = \{\boldsymbol{x} \in \mathbb{K}^r \mid f_k(\boldsymbol{x}) = 0 \text{ for each } 1 \leq k \leq s\}. \tag{1.1}$$

Let $\Sigma(1) = \{\rho_1, \cdots, \rho_r\}$ denote the set of all one dimensional cones in a fan Σ, and let $\mathbf{n}_k \in \mathbb{Z}^n$ denote the generator of $\rho_k \cap \mathbb{Z}^n$ such that $\rho_k \cap \mathbb{Z}^n = \mathbb{Z}_{\geq 0} \cdot \mathbf{n}_k$ (called *the primitive element* of ρ_k) for each $1 \leq k \leq r$. Define the affine variety $Z_{\Sigma,\mathbb{K}} \subset \mathbb{K}^r$ by

$$Z_{\Sigma,\mathbb{K}} = V_{\mathbb{K}}(z^{\hat{\sigma}} | \sigma \in \Sigma), \tag{1.2}$$

where $z^{\hat{\sigma}}$ denotes the monomial $z^{\hat{\sigma}} = \prod_{1 \leq k \leq r, \mathbf{n}_k \notin \sigma} z_k \in \mathbb{Z}[z_1, \cdots, z_r]$. Let $\mathbb{T}^r_{\mathbb{K}} = (\mathbb{K}^*)^r$ and define the subgroup $G_{\Sigma,\mathbb{K}} \subset \mathbb{T}^r_{\mathbb{K}}$ by

$$G_{\Sigma,\mathbb{K}} = \{(\mu_1, \cdots, \mu_r) \in \mathbb{T}^r_{\mathbb{K}} \mid \prod_{k=1}^{r} \mu_k^{\langle \mathbf{m}, \mathbf{n}_k \rangle} = 1 \text{ for all } \mathbf{m} \in \mathbb{Z}^n\}. \tag{1.3}$$

It is known that that there is an isomorphism $X_{\Sigma,\mathbb{K}} \cong (\mathbb{K}^r \setminus Z_{\Sigma,\mathbb{K}})//G_{\Sigma,\mathbb{K}}$ for $\mathbb{K} = \mathbb{C}$ if the set $\{\mathbf{n}_1, \cdots, \mathbf{n}_r\}$ spans $\mathbb{R}^n$, where the group $G_{\Sigma,\mathbb{K}}$ acts on the complement $\mathbb{K}^r \setminus Z_{\Sigma,\mathbb{K}}$ by the coordinate-wise multiplications and the space $(\mathbb{K}^r \setminus Z_{\Sigma,\mathbb{K}})//G_{\Sigma,\mathbb{K}}$ denotes its orbit space.

It is known that $G_{\Sigma,\mathbb{K}}$ acts freely on the complement $\mathbb{K}^r \setminus Z_{\Sigma,\mathbb{K}}$ if Σ is smooth and $\mathbb{K} = \mathbb{C}$. In this case, for $\mathbb{K} = \mathbb{C}$ there are isomorphisms

$$X_{\Sigma,\mathbb{K}} \cong (\mathbb{K}^r \setminus Z_{\Sigma,\mathbb{K}})/G_{\Sigma,\mathbb{K}} \quad \text{and} \quad G_{\Sigma,\mathbb{K}} \cong \mathbb{T}^{r-n}_{\mathbb{K}}. \tag{1.4}$$

Note that (1.4) also holds for $\mathbb{K} = \mathbb{R}$ if Σ is smooth and complete [19, Lemma 7.3].

We say that a set of primitive elements $\{\mathbf{n}_{i_1}, \cdots, \mathbf{n}_{i_k}\}$ is *primitive* if they do not lie in any cone in Σ but every proper subset does. It is known that

$$Z_{\Sigma,\mathbb{K}} = \bigcup_{\{\mathbf{n}_{i_1}, \cdots, \mathbf{n}_{i_k}\}:\ \text{primitive}} V_{\mathbb{K}}(z_{i_1}, \cdots, z_{i_k}). \tag{1.5}$$

So $Z_{\Sigma,\mathbb{K}}$ is a closed variety with real dimension $(r - r_{\text{min}}) \dim_{\mathbb{R}} \mathbb{K}$, where we set

$$r_{\text{min}} = \min \big\{ k \in \mathbb{Z}_{\geq 1} \mid \{\mathbf{n}_{i_1}, \cdots, \mathbf{n}_{i_k}\} \text{ is primitive} \big\}. \tag{1.6}$$

Spaces of continuous maps. For connected spaces X and Y, let $\text{Map}(X, Y)$ be the space of all continuous maps $f : X \to Y$ and $\text{Map}^*(X, Y)$ the corresponding subspace of all based continuous maps. If $m \geq 2$ and $g \in \text{Map}^*(\mathbb{R}\text{P}^{m-1}, X)$, let $F(\mathbb{R}\text{P}^m, X; g)$ denote the subspace of $\text{Map}^*(\mathbb{R}\text{P}^m, X)$ given by

$$F(\mathbb{R}\text{P}^m, X; g) = \{f \in \text{Map}^*(\mathbb{R}\text{P}^m, X) : f | \mathbb{R}\text{P}^{m-1} = g\}, \tag{1.7}$$

where we identify $\mathbb{R}\text{P}^{m-1} \subset \mathbb{R}\text{P}^m$ by putting $x_m = 0$. It is known that there is a homotopy equivalence $F(\mathbb{R}\text{P}^m, X; g) \simeq \Omega^m X$ if it is not an empty set.

Assumptions. From now on, we assume that the following two conditions are satisfied:

(1.7.1) Let Σ be a complete smooth fan in $\mathbb{R}^n$, $\Sigma(1)=\{\rho_1,\cdots,\rho_r\}$ be the set of all one-dimension cones in Σ, and all primitive elements $\{\mathbf{n}_1,\cdots,\mathbf{n}_r\}$ of the fan Σ spans $\mathbb{R}^n$, where $\mathbf{n}_k\in\mathbb{Z}^n$ denotes the primitive element of ρ_k for $1\leq k\leq r$.

(1.7.2) Let $D=(d_1,\cdots,d_r)$ be an r-tuple of positive integers such that $\sum_{k=1}^r d_k\mathbf{n}_k=\mathbf{0}$.

Then we can identify $X_{\Sigma,\mathbb{K}}=(\mathbb{K}^r\setminus Z_{\Sigma,\mathbb{K}})/G_{\Sigma,\mathbb{K}}$ and we denote by $[a_1,\cdots,a_r]$ the corresponding element of $X_{\Sigma,\mathbb{K}}$ for each $(a_1,\cdots,a_r)\in\mathbb{K}^r\setminus Z_{\Sigma,\mathbb{K}}$.

Spaces of polynomials representing algebraic maps. Let $\mathcal{H}^{\mathbb{K}}_{d,m}\subset\mathbb{K}[z_0,\dots,z_m]$ denote the $\mathbb{K}$-vector subspace consisting of all homogeneous polynomials of degree d. Let $A_D(m)$ denote the space $A^{\mathbb{K}}_D(m)=\mathcal{H}^{\mathbb{K}}_{d_1,m}\times\mathcal{H}^{\mathbb{K}}_{d_2,m}\times\cdots\times\mathcal{H}^{\mathbb{K}}_{d_r,m}$ and let $A^{\mathbb{K}}_{D,\Sigma}(m)\subset A^{\mathbb{K}}_D(m)$ denote the subspace

$$(1.8)\qquad A^{\mathbb{K}}_{D,\Sigma}=\big\{(f_1,\cdots,f_r)\in A^{\mathbb{K}}_D(m)\mid F(\boldsymbol{x})\notin Z_{\Sigma,\mathbb{K}}\text{ for any }\boldsymbol{x}\in\mathbb{R}^{m+1}\setminus\{\mathbf{0}\}\big\},$$

where we set $F(\boldsymbol{x})=(f_1(\boldsymbol{x}),\cdots,f_r(\boldsymbol{x}))$.

Because $(1,1,\cdots,1)\in\mathbb{K}^r\setminus Z_{\Sigma,\mathbb{K}}$, we choose $x_0=[1,\cdots,1]\in X_{\Sigma,\mathbb{K}}$ as the base-point of $X_{\Sigma,\mathbb{K}}$. Define the subspace $A_D(m,X_{\Sigma,\mathbb{K}})\subset A^{\mathbb{K}}_{D,\Sigma}(m)$ by

$$(1.9)\qquad A_D(m,X_{\Sigma,\mathbb{K}})=\{(f_1,\cdots,f_r)\in A^{\mathbb{K}}_{D,\Sigma}(m)|\ (f_1(\boldsymbol{e}_1),\cdots,f_r(\boldsymbol{e}_1))=(1,1,\cdots,1)\},$$

where $\boldsymbol{e}_1=(1,0,\cdots,0)\in\mathbb{R}^{m+1}$, and let us choose $[\boldsymbol{e}_1]=[1:0:\cdots:0]$ as the base-point of $\mathbb{RP}^m$. Define the natural map $j'_{D,\mathbb{K}}:A^{\mathbb{K}}_{D,\Sigma}(m)\to\mathrm{Map}(\mathbb{RP}^m,X_{\Sigma,\mathbb{K}})$ by

$$(1.10)\qquad j'_{D,\mathbb{K}}(f_1,\cdots,f_r)([\boldsymbol{x}])=[f_1(\boldsymbol{x}),\cdots,f_r(\boldsymbol{x})]\quad\text{for }\boldsymbol{x}=(x_0,\cdots,x_m)\in\mathbb{R}^{m+1}\setminus\{\mathbf{0}\}.$$

Since the space $A^{\mathbb{K}}_{D,\Sigma}(m)$ is connected, the image of $j'_{D,\mathbb{K}}$ lies in a connected component of $\mathrm{Map}(\mathbb{RP}^m,X_{\Sigma,\mathbb{K}})$, which is denoted by $\mathrm{Map}_D(\mathbb{RP}^m,X_{\Sigma,\mathbb{K}})$. This gives the natural map

$$(1.11)\qquad j'_{D,\mathbb{K}}:A^{\mathbb{K}}_{D,\Sigma}(m)\to\mathrm{Map}_D(\mathbb{RP}^m,X_{\Sigma,\mathbb{K}}).$$

Note that $j'_{D,\mathbb{K}}(f_1,\cdots,f_r)\in\mathrm{Map}^*(\mathbb{RP}^m,X_{\Sigma,\mathbb{K}})$ if $(f_1,\cdots,f_r)\in A^{\mathbb{K}}_D(m,X_\Sigma)$. Hence, if we set $\mathrm{Map}^*_D(\mathbb{RP}^m,X_{\Sigma,\mathbb{K}})=\mathrm{Map}^*(\mathbb{RP}^m,X_{\Sigma,\mathbb{K}})\cap\mathrm{Map}_D(\mathbb{RP}^m,X_{\Sigma,\mathbb{K}})$, we have the natural map

$$(1.12)\qquad i_{D,\mathbb{K}}=j'_{D,\mathbb{K}}|A_D(m,X_{\Sigma,\mathbb{K}}):A_D(m,X_{\Sigma,\mathbb{K}})\to\mathrm{Map}^*_D(\mathbb{RP}^m,X_{\Sigma,\mathbb{K}}).$$

Suppose that $m\geq 2$ and let us choose a fixed element $(g_1,\cdots,g_r)\in A_D(m-1,X_{\Sigma,\mathbb{K}})$. For each $1\leq k\leq r$, let $B^{\mathbb{K}}_k=\{g_k+z_mh:h\in\mathcal{H}^{\mathbb{K}}_{d_k-1,m}\}$. Then define the subspace $A_D(m,X_{\Sigma,\mathbb{K}};g)\subset A_D(m,X_{\Sigma,\mathbb{K}})$ by

$$(1.13)\qquad A_D(m,X_{\Sigma,\mathbb{K}};g)=A_D(m,X_{\Sigma,\mathbb{K}})\cap(B^{\mathbb{K}}_1\times B^{\mathbb{K}}_2\times\cdots\times B^{\mathbb{K}}_r).$$

It is easy to see that $i_{D,\mathbb{K}}(f_1,\cdots,f_r)|\mathbb{R}\mathrm{P}^{m-1}=g$ if $(f_1,\cdots,f_r)\in A_D(m,X_{\Sigma,\mathbb{K}};g)$, where g denotes the map in $\mathrm{Map}_D^*(\mathbb{R}\mathrm{P}^{m-1},X_{\Sigma,\mathbb{K}})$ given by

$$(1.14)\qquad g([x_0:\cdots:x_{m-1}])=[g_1(\boldsymbol{x}),\cdots,g_r(\boldsymbol{x})]\quad\text{for } \boldsymbol{x}=(x_0,\cdots,x_{m-1})\in\mathbb{R}^m\setminus\{\mathbf{0}\}.$$

Then, one can define the map $i'_{D,\mathbb{K}}:A_D(m,X_{\Sigma,\mathbb{K}};g)\to F(\mathbb{R}\mathrm{P}^m,X_{\Sigma,\mathbb{K}};g)\simeq\Omega^m X_{\Sigma,\mathbb{K}}$ by

$$(1.15)\qquad i'_{D,\mathbb{K}}=i_{D,\mathbb{K}}|A_D(m,X_{\Sigma,\mathbb{K}};g):A_D(m,X_{\Sigma,\mathbb{K}};g)\to F(\mathbb{R}\mathrm{P}^m,X_{\Sigma,\mathbb{K}};g)\simeq\Omega^m X_{\Sigma,\mathbb{K}}.$$

Now consider the action of $G_{\Sigma,\mathbb{K}}$ on the space $A^{\mathbb{K}}_{D,\Sigma}(m)$ given by the coordinate-wise multiplication and define the space $\widetilde{A_D}(m,X_{\Sigma,\mathbb{K}})$ by the quotient space

$$(1.16)\qquad \widetilde{A_D}(m,X_{\Sigma,\mathbb{K}})=A^{\mathbb{K}}_{D,\Sigma}(m)/G_{\Sigma,\mathbb{K}}.$$

It is easy to see that one can define the map $j_{D,\mathbb{K}}:\widetilde{A_D}(m,X_{\Sigma,\mathbb{K}})\to\mathrm{Map}_D(\mathbb{R}\mathrm{P}^m,X_{\Sigma,\mathbb{K}})$ by

$$(1.17)\qquad j_{D,\mathbb{K}}([f_1,\cdots,f_r])([x_0,\cdots,x_r])=[f_1(\boldsymbol{x}),\cdots,f_r(\boldsymbol{x})]\quad\text{for } \boldsymbol{x}\in\mathbb{R}^{m+1}\setminus\{\mathbf{0}\}.$$

Let $d_{\min}$ and $D_{\mathbb{R}}(d_1,\cdots,d_r;m,r)$ be the positive integer defined by

$$(1.18)\qquad d_{\min}=\min\{d_1,d_2,\cdots,d_r\},\qquad D(d_1,\cdots,d_r;m)=(r_{\min}-m-1)d_{\min}-2.$$

From now on we write $(X_{\Sigma,\mathbb{K}},Z_{\Sigma,\mathbb{K}},G_{\Sigma,\mathbb{K}})=(X_\Sigma,Z_\Sigma,G_\Sigma)$ if $\mathbb{K}=\mathbb{C}$.

The main results. The main results of this note are stated as follows.

Theorem 1.1 ([13]). *Let Σ be a complete smooth fan in $\mathbb{R}^n$, let $\{d_k:1\leq k\leq r\}$ be the set of positive integers satisfying the conditions (1.7.1), (1.7.2), and let $X_{\Sigma,\mathbb{R}}$ be a smooth compact real toric variety associated to the fan Σ. Then if $1\leq m\leq r_{\min}-2$ and $D=(d_1,\cdots,d_r)\in(\mathbb{Z}_{\geq 1})^r$, the map*

$$i'_{D,\mathbb{R}}:A_D(m,X_{\Sigma,\mathbb{R}};g)\to F(\mathbb{R}\mathrm{P}^m,X_{\Sigma,\mathbb{R}};g)\simeq\Omega^m X_{\Sigma,\mathbb{R}}$$

is a homology equivalence through dimension $D(d_1,\cdots,d_r;m)$. □

Theorem 1.2 ([13]). *Under the same assumptions as Theorem 1.1, if $1\leq m\leq r_{\min}-2$ and $D=(d_1,\cdots,d_r)\in(\mathbb{Z}_{\geq 1})^r$, the maps*

$$\begin{cases} j_{D,\mathbb{R}}:\widetilde{A_D}(m,X_{\Sigma,\mathbb{R}})\to\mathrm{Map}_D(\mathbb{R}\mathrm{P}^m,X_{\Sigma,\mathbb{R}})\\ i_{D,\mathbb{R}}:A_D(m,X_{\Sigma,\mathbb{R}})\to\mathrm{Map}^*_D(\mathbb{R}\mathrm{P}^m,X_{\Sigma,\mathbb{R}})\end{cases}$$

are homology equivalences through dimension $D(d_1,\cdots,d_r;m)$. □

Remark 1.3. (i) A map $f : X \to Y$ is called *a homology equivalence through dimension* D if $f_* : H_k(X,\mathbb{Z}) \xrightarrow{\cong} H_k(Y,\mathbb{Z})$ is an isomorphism for any $k \leq D$.

(ii) Let G be a finite group and let $f : X \to Y$ be a G-equivariant map between G-spaces X and Y. Then the map $f : X \to Y$ is called *a G-equivariant homology equivalence through dimension D* if $f_*^H : H_k(X^H,\mathbb{Z}) \xrightarrow{\cong} H_k(Y^H,\mathbb{Z})$ is an isomorphism for any $k \leq D$ and any subgroup $H \subset G$, where $W^H = \{x \in W \mid g \cdot x = x \text{ for any } g \in H\}$ for a G-space W and f^H denotes the restriction map $f^H = f|X^H$.

(iii) Note that the complex conjugation on $\mathbb{C}$ naturally induces the $\mathbb{Z}/2$-action on the space X_Σ, and it is easy to see that $(X_\Sigma)^{\mathbb{Z}/2} = X_{\Sigma,\mathbb{R}}$. Similarly, it also induces the $\mathbb{Z}/2$-actions on the space $A_{D,\mathbb{C}}(m,X_\Sigma)$, $\widetilde{A}_{D,\mathbb{C}}(m,X_\Sigma)$, $A_{D,\mathbb{C}}(m,X_\Sigma;g)$. Moreover, if we consider the space $\mathbb{RP}^m$ as a $\mathbb{Z}/2$-space of the trivial action, the $\mathbb{Z}/2$-action X_Σ also induces the $\mathbb{Z}/2$-actions on the spaces $\mathrm{Map}^*_D(\mathbb{RP}^m,X_\Sigma)$, $\mathrm{Map}_D(\mathbb{RP}^m,X_\Sigma)$, $F(\mathbb{RP}^m,X_\Sigma;g)$.

Corollary 1.4 ([8], [13]). *Under the same assumptions as Theorem 1.1, if* $2 \leq m \leq r_{\min}-2$ *and* $D=(d_1,\cdots,d_r)\in(\mathbb{Z}_{\geq 1})^r$, *the maps*

$$\begin{cases} i'_{D,\mathbb{C}} : A_D(m,X_\Sigma;g) \to F(\mathbb{RP}^m,X_\Sigma;g) \simeq \Omega^m X_\Sigma \\ j_{D,\mathbb{C}} : \widetilde{A_D}(m,X_\Sigma) \to \mathrm{Map}_D(\mathbb{RP}^m,X_\Sigma) \\ i_{D,\mathbb{C}} : A_D(m,X_\Sigma) \to \mathrm{Map}^*_D(\mathbb{RP}^m,X_\Sigma) \end{cases}$$

are $\mathbb{Z}/2$-equivariant homology equivalences through dimension $D(d_1,\cdots,d_r;m)$. □

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