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Weak and Strong Convergence Theorems for Semigroups of Not Necessarily Continuous Mappings

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Abstract. In this article, using the concept of strongly asymptotically invariant nets, we first introduce a broad semigroup of not necessarily continuous mappings in a Hilbert space. Furthermore, we consider such a semigroup in a Banach space which contains discrete semigroups generated by generalized nonspreading mappings [22] and semigroups of ϕ-nonexpansive mappings [40]. Then we prove weak convergence theorems of Mann's type iteration and strong convergence theorems of Halpern's type iteration for the semigroups of mappings in a Hilbert space. Furthermore, we obtain a weak convergence theorem of Mann's type iteration in a Banach space. Using these results, we obtain well-known and new theorems which are connected with weak and strong convergence theorems in a Hilbert space and a Banach space.

1 Introduction

Let H be a real Hilbert space and let C be a nonempty subset of H. We denote by $\mathbb{R}$ the set of real numbers. Kocourek, Takahashi and Yao [21] defined a class of nonlinear mappings containing nonexpansive mappings, nonspreading mappings and hybrid mappings in a Hilbert space. A mapping $T : C \to C$ is called *generalized hybrid* [21] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2$$

for all $x, y \in C$; see also [2]. We call such a mapping (α, β)*-generalized hybrid.* A (1, 0)-generalized hybrid mapping is nonexpansive. It is nonspreading [25] for $\alpha = 2$ and $\beta = 1$. It is hybrid [35] for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$. They proved a fixed point theorem and a mean convergence theorem for the mappings. Takahashi and Takeuchi [36] introduced the concept of attractive points of nonlinear mappings in a Hilbert space and then proved attractive point and mean convergence theorems without convexity for generalized hybrid mappings; see also [1, 26, 27, 37, 39]. In general, nonspreading and hybrid mappings are not continuous. We also know the concept of one-parameter nonexpansive semigroups in a Hilbert space. Let H be a Hilbert space and let C be a nonempty subset of H. Let $S = \mathbb{R}^+ = \{t \in \mathbb{R} : 0 \leq t < \infty\}$. A family $\mathcal{S} = \{S(t) : t \in \mathbb{R}^+\}$ of mappings of C into itself is called a *one-parameter nonexpansive semigroup* on C if $\mathcal{S}$ satisfies the following:

(1) $S(t+s)x = S(t)S(s)x, \quad \forall x \in C, \quad t, s \in \mathbb{R}^+$;
(2) $S(0)x = x, \quad \forall x \in C$;

(3) for each $x \in C$, the mapping $t \mapsto S(t)x$ from $\mathbb{R}^+$ into C is continuous;
(2) for each $t \in \mathbb{R}^+$, $S(t)$ is nonexpansive.

Of course, $S(t)$ are continuous. Such one-parameter nonexpansive semigroups are used in the theory of nonlinear evolution equations [7]. Recently, using the concept of means and invariant means, Takahashi, Wong and Yao [38] introduced the concept of semigroups of not necessarily continuous mappings in a Hilbert space which contains discrete semigroups generated by generalized hybrid mappings and semigroups of nonexpansive mappings. They proved a fixed point theorem and a mean convergence theorem of Baillon's type [5] which generalize simultaneously the results [21] and [6] for generalized hybrid mappings and one-parameter nonexpansive semigroups in a Hilbert space. They also generalized such results to Banach spaces; see [40]. It is natural to consider weak convergence theorems of Mann's type iteration [28] and strong convergence theorems of Halpern's type iteration [9] for semigroups of not necessarily continuous mappings.

In this article, using the concept of strongly asymptotically invariant nets, we first introduce a broad semigroup of not necessarily continuous mappings in a Hilbert space. Furthermore, we consider such a semigroup in a Banach space which contains discrete semigroups generated by generalized nonspreading mappings [22] and semigroups of ϕ-nonexpansive mappings [40]. Then we prove weak convergence theorems of Mann's type iteration and strong convergence theorems of Halpern's type iteration for the semigroups of mappings in a Hilbert space. Furthermore, we obtain a weak convergence theorem of Mann's type iteration in a Banach space. Using these results, we obtain well-known and new theorems which are connected with weak and strong convergence theorems in a Hilbert space and a Banach space.

2 Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, respectively. Let A be a nonempty subset of H. We denote by $\overline{co}A$ the closure of the convex hull of A. In a Hilbert space, it is known [34] that for all $x, y \in H$ and $\alpha \in \mathbb{R}$,

$$\|y\|^2 - \|x\|^2 \leq 2\langle y - x, y \rangle; \tag{2.1}$$

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha \|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x - y\|^2. \tag{2.2}$$

Furthermore, we have that

$$2\langle x - y, z - w \rangle = \|x - w\|^2 + \|y - z\|^2 - \|x - z\|^2 - \|y - w\|^2 \tag{2.3}$$

for all $x, y, z, w \in H$. From (2.3), we have that

$$2\langle x - y, z - y \rangle - \|z - y\|^2 = \|x - y\|^2 - \|x - z\|^2 \tag{2.4}$$

for all $x, y, z \in H$. Let E be a real Banach space and let E^* be the dual space of E. For a sequence $\{x_n\}$ of E and a point $x \in E$, the weak convergence of $\{x_n\}$ to x and the strong convergence of $\{x_n\}$ to x are denoted by $x_n \rightharpoonup x$ and $x_n \to x$, respectively. The *duality* mapping J from E into E^* is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}, \quad \forall x \in E.$$

Let $S(E)$ be the unit sphere centered at the origin of E, where $\langle x, x^* \rangle$ is the value of $x^* \in E^*$ at $x \in E$. The norm of E is said to be *Gâteaux differentiable* if for each $x, y \in S(E)$, the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.5}$$

exists. In this case, E is called *smooth.* The norm of E is said to be *Fréchet differentiable* if for each $x \in S(E)$, the limit (2.5) is attained uniformly for $y \in S(E)$. A Banach space E is said to be *strictly convex* if $\|\frac{x+y}{2}\| < 1$ whenever $x, y \in S(E)$ and $x \neq y$. It is said to be *uniformly convex* if for each $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that $\|\frac{x+y}{2}\| < 1 - \delta$ whenever $x, y \in S(E)$ and $\|x - y\| \geq \varepsilon$. It is known that if E uniformly convex, then E is strictly convex and reflexive. Furthermore, we know from [33] that

(i) if E is smooth, then J is single-valued;
(ii) if E is reflexive, then J is onto;
(iii) if E is strictly convex, then J is one-to-one;
(iv) if E is strictly convex, then J is strictly monotone;
(v) if E has a Fréchet differentiable norm, then J is continuous.

Let E be a smooth Banach space and let J be the duality mapping on E. Throughout this paper, define a function $\phi : E \times E \to \mathbb{R}$ by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \quad \forall x, y \in E.$$

Observe that, in a Hilbert space H, $\phi(x, y) = \|x - y\|^2$ for all $x, y \in H$. Furthermore, we know that for each $x, y, z, w \in E$,

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2; \tag{2.6}$$

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy \rangle; \tag{2.7}$$

$$2\langle x - y, Jz - Jw \rangle = \phi(x, w) + \phi(y, z) - \phi(x, z) - \phi(y, w). \tag{2.8}$$

If E is additionally assumed to be strictly convex, then

$$\phi(x, y) = 0 \quad \text{if and only if} \quad x = y. \tag{2.9}$$

The following lemmas are in Xu [42] and Kamimura and Takahashi [20].

Lemma 2.1 ([42]). *Let E be a uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous, and convex function $g : [0, 2r] \to [0, \infty)$ such that $g(0) = 0$ and*

$$\|ax + (1 - a)y\|^2 \leq a\|x\|^2 + (1 - a)\|y\|^2 - a(1 - a)g(\|x - y\|)$$

for all $x, y \in B_r$ and $a \in [0, 1]$, where $B_r = \{z \in E : \|z\| \leq r\}$.

Lemma 2.2 ([20]). *Let E be a uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous, and convex function $g : [0, 2r] \to [0, \infty)$ such that $g(0) = 0$ and*

$$g(\|x - y\|) \leq \phi(x, y)$$

for all $x, y \in B_r$, where $B_r = \{z \in E : \|z\| \leq r\}$.

Let E be a smooth Banach space and let C be a nonempty subset of E. A mapping $T : C \to E$ is called *generalized nonexpansive* [16] if $F(T) \neq \emptyset$ and $\phi(Tx, y) \leq \phi(x, y)$ for all $x \in C$ and $y \in F(T)$. Let D be a nonempty subset of a Banach space E. A mapping $R : E \to D$ is said to be *sunny* if $R(Rx + t(x - Rx)) = Rx$ for all $x \in E$ and $t \geq 0$. A mapping $R : E \to D$ is said to be a *retraction* or a *projection* if $Rx = x$ for all $x \in D$. A nonempty subset D of a smooth Banach space E is said to be a *generalized nonexpansive retract* (resp. *sunny generalized nonexpansive retract*) of E if there exists a generalized nonexpansive retraction (resp. sunny generalized nonexpansive retraction) R from E onto D; see [16, 15] for more details. The following results are in Ibaraki and Takahashi [16].

Lemma 2.3 ([16]). *Let C be a nonempty closed sunny generalized nonexpansive retract of a smooth and strictly convex Banach space E. Then the sunny generalized nonexpansive retraction from E onto C is uniquely determined.*

Lemma 2.4 ([16]). *Let C be a nonempty closed subset of a smooth and strictly convex Banach space E such that there exists a sunny generalized nonexpansive retraction R from E onto C and let $(x, z) \in E \times C$. Then the following hold:*

(i) $z = Rx$ *if and only if* $\langle x - z, Jy - Jz \rangle \leq 0$ *for all* $y \in C$;
(ii) $\phi(Rx, z) + \phi(x, Rx) \leq \phi(x, z)$.

In 2007, Kohsaka and Takahashi [23] proved the following results:

Lemma 2.5 ([23]). *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed subset of E. Then the following are equivalent:*

(a) C is a sunny generalized nonexpansive retract of E;
(b) C is a generalized nonexpansive retract of E;
(c) JC is closed and convex.

Lemma 2.6 ([23]). *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed sunny generalized nonexpansive retract of E. Let R be the sunny generalized nonexpansive retraction from E onto C and let $(x, z) \in E \times C$. Then the following are equivalent:*

(i) $z = Rx$;
(ii) $\phi(x, z) = \min_{y \in C} \phi(x, y)$.

Inthakon, Dhompongsa and Takahashi [19] obtained the following result concerning the set of fixed points of a generalized nonexpansive mapping in a Banach space; see also Ibaraki and Takahashi [18].

Lemma 2.7 ([19]). *Let E be a smooth, strictly convex and reflexive Banach space and let C be a closed subset of E such that $J(C)$ is closed and convex. Let T be a generalized nonexpansive mapping from C into itself. Then, $F(T)$ is closed and $JF(T)$ is closed and convex.*

The following is a direct consequence of Lemmas 2.5 and 2.7.

Lemma 2.8 ([19]). *Let E be a smooth, strictly convex and reflexive Banach space and let C be a closed subset of E such that $J(C)$ is closed and convex. Let T be a generalized nonexpansive mapping from C into itself. Then, $F(T)$ is a sunny generalized nonexpansive retract of E.*

Let l^∞ be the Banach space of bounded sequences with supremum norm. Let μ be an element of $(l^\infty)^*$ (the dual space of l^∞). Then, we denote by $\mu(f)$ the value of μ at $f = (x_1, x_2, x_3, \dots) \in l^\infty$. Sometimes, we denote by $\mu_n(x_n)$ the value $\mu(f)$. A linear functional μ

on l^∞ is called a *mean* if $\mu(e) = \|\mu\| = 1$, where $e = (1, 1, 1, \dots)$. A mean μ is called a *Banach limit* on l^∞ if $\mu_n(x_{n+1}) = \mu_n(x_n)$. We know that there exists a Banach limit on l^∞. If μ is a Banach limit on l^∞, then for $f = (x_1, x_2, x_3, \dots) \in l^\infty$,

$$\liminf_{n\to\infty} x_n \le \mu_n(x_n) \le \limsup_{n\to\infty} x_n.$$

In particular, if $f = (x_1, x_2, x_3, \dots) \in l^\infty$ and $x_n \to a \in \mathbb{R}$, then we have $\mu(f) = \mu_n(x_n) = a$. See [33] for the proof of existence of a Banach limit and its other elementary properties.

3 Attractive Point Theorems for Families of Mappings

Let S be a semitopological semigroup, i.e., S is a semigroup with a Hausdorff topology such that for each $a \in S$ the mappings $s \mapsto a \cdot s$ and $s \mapsto s \cdot a$ from S to S are continuous. In the case when S is commutative, we denote st by $s + t$. Let $B(S)$ be the Banach space of all bounded real-valued functions on S with supremum norm and let $C(S)$ be the subspace of $B(S)$ of all bounded real-valued continuous functions on S. Let μ be an element of $C(S)^*$ (the dual space of $C(S)$). We denote by $\mu(f)$ the value of μ at $f \in C(S)$. Sometimes, we denote by $\mu_t(f(t))$ or $\mu_t f(t)$ the value $\mu(f)$. For each $s \in S$ and $f \in C(S)$, we define two functions $l_s f$ and $r_s f$ as follows:

$$(l_s f)(t) = f(st) \quad \text{and} \quad (r_s f)(t) = f(ts)$$

for all $t \in S$. An element μ of $C(S)^*$ is called a *mean* on $C(S)$ if $\mu(e) = \|\mu\| = 1$, where $e(s) = 1$ for all $s \in S$. We know that $\mu \in C(S)^*$ is a mean on $C(S)$ if and only if

$$\inf_{s\in S} f(s) \le \mu(f) \le \sup_{s\in S} f(s), \quad \forall f \in C(S).$$

A mean μ on $C(S)$ is called *left invariant* if $\mu(l_s f) = \mu(f)$ for all $f \in C(S)$ and $s \in S$. Similarly, a mean μ on $C(S)$ is called *right invariant* if $\mu(r_s f) = \mu(f)$ for all $f \in C(S)$ and $s \in S$. A left and right invariant invariant mean on $C(S)$ is called an *invariant* mean on $C(S)$. If $S = \mathbb{N}$, an invariant mean on $C(S) = B(S)$ is a Banach limit on l^∞. The following theorem is in [33, Theorem 1.4.5].

Theorem 3.1 ([33]). *Let S be a commutative semitopological semigroup. Then there exists an invariant mean on $C(S)$, i.e., there exists an element $\mu \in C(S)^*$ such that $\mu(e) = \|\mu\| = 1$ and $\mu(r_s f) = \mu(f)$ for all $f \in C(S)$ and $s \in S$.*

Let E be a Banach space and let C be a nonempty subset of E. Let S be a semitopological semigroup and let $\mathcal{S} = \{T_s : s \in S\}$ be a family of mappings of C into itself. Then $\mathcal{S} = \{T_s : s \in S\}$ is called a *continuous representation* of S as mappings on C if $T_{st} = T_s T_t$ for all $s, t \in S$ and $s \mapsto T_s x$ is continuous for each $x \in C$. We denote by $F(\mathcal{S})$ the set of common fixed points of T_s, $s \in S$, i.e.,

$$F(\mathcal{S}) = \cap\{F(T_s) : s \in S\}.$$

The following definition [31] is crucial in the nonlinear ergodic theory of abstract semigroups; see also [10]. Let E be a reflexive Banach space and let E^* be the dual space of E. Let

$u : S \to E$ be a continuous function such that $\{u(s) : s \in S\}$ is bounded and let μ be a mean on $C(S)$. Then there exists a unique point $z_0 \in \overline{co}\{u(s) : s \in S\}$ such that

$$\mu_s\langle u(s), y^*\rangle = \langle z_0, y^*\rangle, \quad \forall y^* \in E^*. \tag{3.1}$$

We call such z_0 the *mean vector* of u for μ. In particular, let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as mappings on C such that $\{T_s x : s \in S\}$ is bounded for some $x \in C$. Putting $u(s) = T_s x$ for all $s \in S$, we have that there exists $z_0 \in E$ such that

$$\mu_s\langle T_s x, y^*\rangle = \langle z_0, y^*\rangle, \quad \forall^* y \in E^*.$$

We denote such z_0 by $T_\mu x$. A net $\{\mu_\alpha\}$ of means on $C(S)$ is said to be *strongly asymptotically invariant* if for each $s \in S$,

$$\|\ell_s^*\mu_\alpha - \mu_\alpha\| \to 0 \quad \text{and} \quad \|r_s^*\mu_\alpha - \mu_\alpha\| \to 0,$$

where ℓ_s^* and r_s^* are the adjoint operators of ℓ_s and r_s, respectively. See [8] and [33] for more details.

Let E be a smooth Banach space and let C be a nonempty subset of E. For a mapping T from C into C, we denote by $A(T)$ the set of *attractive points* [26, 36] of T, i.e.,

$$A(T) = \{u \in E : \phi(u, Tx) \le \phi(u, x), \quad \forall x \in C\}.$$

We know from Lin and Takahashi [26] that $A(T)$ is always closed and convex. Let S be a commutative semitopological semigroup with identity. For a continuous representation $\mathcal{S} = \{T_s : s \in S\}$ of S as mappings of C into itself, we denote the set $A(\mathcal{S})$ of *common attractive points* [4, 40] of $\mathcal{S} = \{T_s : s \in S\}$ by

$$A(\mathcal{S}) = \cap\{A(T_t) : t \in S\}.$$

It is obvious from Lin and Takahashi [26] that $A(\mathcal{S})$ is closed and convex. Using the technique developed by Takahashi [31], Takahashi, Wong and Yao [40] also proved the following attractive point theorem for a family of mappings in a Banach space.

Theorem 3.2 ([40]). *Let E be a smooth and reflexive Banach space with the duality mapping J and let C be a nonempty subset of E. Let S be a commutative semitopological semigroup with identity. Let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as mappings of C into itself such that $\{T_s x : s \in S\}$ is bounded for some $x \in C$. Let μ be a mean on $C(S)$. Suppose that*

$$\mu_s\phi(T_s x, T_t y) \le \mu_s\phi(T_s x, y)$$

for all $y \in C$ and $t \in S$. Then, $A(\mathcal{S}) = \cap\{A(T_t) : t \in S\}$ is nonempty. In particular, if E is strictly convex and C is closed and convex, then $F(\mathcal{S}) = \cap\{F(T_t) : t \in S\}$ is nonempty.

Let E be a smooth Banach space and let C be a nonempty subset of E. Let T be a mapping from C into C. We denote by $B(T)$ the set of *skew-attractive points* [26] of T, i.e.,

$$B(T) = \{z \in E, \phi(Tx, z) \le \phi(x, z), \quad \forall x \in C\}.$$

Lin and Takahashi [26] proved that $B(T)$ is always closed. Using the duality theory of nonlinear mappings [41] and [12], they also proved that $JB(T)$ is closed and convex. We can also define by $B(\mathcal{S})$ the set of all *common skew-attractive points* of a family $\mathcal{S} = \{T_s : s \in S\}$ of mappings of C into itself, i.e., $B(\mathcal{S}) = \cap\{B(T_s) : s \in S\}$. Takahashi, Wong and Yao [40] obtained the following skew-attractive point theorem for semigroups of not necessarily continuous mappings in a Banach space.

Theorem 3.3 ([40]). *Let E be a strictly convex and reflexive Banach space with a Fréchet differentiable norm and let C be a nonempty subset of E. Let S be a commutative semitopological semigroup with identity. Let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as mappings of C into itself such that $\{T_s x : s \in S\}$ is bounded for some $x \in C$. Let μ be a mean on $C(S)$. Suppose that*

$$\mu_s \phi(T_t y, T_s x) \leq \mu_s \phi(y, T_s x)$$

for all $y \in C$ and $t \in S$. Then, $B(\mathcal{S}) = \cap\{B(T_t) : t \in S\}$ is nonempty. In particular, if C is closed and JC is closed and convex, then $F(\mathcal{S}) = \cap\{F(T_t) : t \in S\}$ is nonempty.

4 Weak Convergence Theorems in Hilbert Spaces

In this section, we prove a weak convergence theorem of Mann's type iteration for semigroups of not necessarily continuous mappings in a Hilbert space.

Theorem 4.1 ([13]). *Let H be a Hilbert space and let C be a nonempty, bounded, closed and convex subset of H. Let S be a commutative semitopological semigroup with identity. Let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as mappings of C into itself. Suppose that*

$$\limsup_{\alpha} \sup_{x,y \in C} (\mu_\alpha)_s (\|T_s x - T_t y\|^2 - \|T_s x - y\|^2) \leq 0, \quad \forall t \in S \tag{4.1}$$

for all strongly asymptotically invariant nets $\{\mu_\alpha\}$ of means on $C(S)$. Let $\{\mu_n\}$ be a strongly asymptotically invariant sequence of means on $C(S)$, i.e.,

$$\|\mu_n - \ell_s^* \mu_n\| \to 0, \quad \forall s \in S.$$

Define a sequence $\{x_n\}$ in C as follows: $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T_{\mu_n} x_n, \quad \forall n \in \mathbb{N},$$

where $0 \leq \alpha_n \leq 1$ and $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$. Then, $\{x_n\}$ converges weakly to a point $z \in F(\mathcal{S})$ and $z = \lim_{n\to\infty} P_{F(\mathcal{S})} x_n$, where $P_{F(\mathcal{S})}$ is the metric projection of H onto $F(\mathcal{S})$.

Using Theorem 4.1, we obtain the following weak convergence theorem for generalized hybrid mappings in a Hilbert space.

Theorem 4.2. *Let C be a nonempty, closed and convex subset of a Hilbert space H. Let T be a generalized hybrid mapping of C into itself such that $F(T)$ is nonempty. Let $\{\mu_n\}$ be a strongly asymptotically invariant sequence of means on $B(\mathbb{N})$. Define a sequence $\{x_n\}$ in C as follows: $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T_{\mu_n} x_n, \quad \forall n \in \mathbb{N},$$

where $0 \leq \alpha_n \leq 1$ and $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$. Then $\{x_n\}$ converges weakly to $z \in F(T)$ and $z = \lim_{n\to\infty} P_{F(T)} x_n$, where $P_{F(T)}$ is the metric projection of H onto $F(T)$.

Using Theorem 4.1, we obtain the following weak convergence theorem for semigroups of nonexpansive mappings in a Hilbert space; see also [3].

Theorem 4.3. *Let H be a Hilbert space, let C be a nonempty, closed and convex subset of H. Let S be a commutative semitopological semigroup with identity and let $\mathcal{S} = \{T_t : t \in S\}$ be a nonexpansive semigroup on C such that $\{T_t x : t \in S\}$ is bounded for some $x \in C$. Let $\{\mu_n\}$ be a strongly asymptotically invariant sequence of means on $C(S)$, i.e., a sequence of means on $C(S)$ such that*

$$\lim_{n\to\infty} \|\mu_n - \ell_s^* \mu_n\| = 0, \quad \forall s \in S.$$

Define a sequence $\{x_n\}$ in C as follows: $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) T_{\mu_n} x_n, \quad \forall n \in \mathbb{N},$$

where $0 \le \alpha_n \le 1$ and $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$. Then, $\{x_n\}$ converges weakly to a point $z \in F(\mathcal{S})$ and $z = \lim_{n\to\infty} P_{F(\mathcal{S})} x_n$, where $P_{F(\mathcal{S})}$ is the metric projection of H onto $F(\mathcal{S})$.

5 Strong Convergence Theorems in Hilbert Spaces

In this section, we prove a strong convergence theorem of Halpern's type iteration for semigroups of not necessarily continuous mappings in a Hilbert space.

Theorem 5.1 ([13]). *Let H be a Hilbert space and let C be a nonempty, bounded, closed and convex subset of H. Let S be a commutative semitopological semigroup with identity. Let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as mappings of C into itself. Suppose that*

$$\limsup_{\alpha} \sup_{x,y\in C} (\mu_\alpha)_s(\|T_s x - T_t y\|^2 - \|T_s x - y\|^2) \le 0, \quad \forall t \in S \tag{5.1}$$

for all strongly asymptotically invariant nets $\{\mu_\alpha\}$ of means on $C(S)$. Let $\{\mu_n\}$ be a strongly asymptotically invariant sequence of means on $C(S)$, i.e.,

$$\|\mu_n - \ell_s^* \mu_n\| \to 0, \quad \forall s \in S.$$

Let $u \in C$ and define a sequence $\{x_n\}$ in C as follows: $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n u + (1-\alpha_n) T_{\mu_n} x_n, \quad \forall n \in \mathbb{N},$$

where $0 \le \alpha_n \le 1$, $\alpha_n \to 0$ and $\sum_{n=1}^\infty \alpha_n = \infty$. Then, $\{x_n\}$ converges strongly to a point $z \in F(\mathcal{S})$, where $z = P_{F(\mathcal{S})} u$.

Using Theorem 5.1, we can prove the following strong convergence theorem for generalized hybrid mappings in a Hilbert space.

Theorem 5.2. *Let C be a nonempty, closed and convex subset of a Hilbert space H. Let T be a generalized hybrid mapping of C into itself such that $F(T)$ is nonempty. Let $\{\mu_n\}$ be a strongly asymptotically invariant sequence of means on $B(\mathbb{N})$. Let $u \in C$ and define two sequences $\{x_n\}$ and $\{z_n\}$ in C as follows: $x_1 = x \in C$ and*

$$\begin{cases} x_{n+1} = \alpha_n u + (1-\alpha_n) z_n, \\ z_n = T_{\mu_n} x_n \end{cases}$$

for all $n \in \mathbb{N}$, where $0 \le \alpha_n \le 1$, $\alpha_n \to 0$ and $\sum_{n=1}^\infty \alpha_n = \infty$. Then $\{x_n\}$ and $\{z_n\}$ converge strongly to Pu, where P is the metric projection of H onto $F(T)$.

In particular, we obtain the following strong convergence theorem [11] from Theorem 5.2.

Theorem 5.3 ([11]). *Let C be a nonempty, closed and convex subset of a Hilbert space H. Let T be a generalized hybrid mapping of C into itself. Let $u \in C$ and define two sequences $\{x_n\}$ and $\{z_n\}$ in C as follows: $x_1 = x \in C$ and*

$$\begin{cases} x_{n+1} = \alpha_n u + (1-\alpha_n) z_n, \\ z_n = \dfrac{1}{n} \displaystyle\sum_{k=0}^{n-1} T^k x_n \end{cases}$$

for all $n \in \mathbb{N}$, where $0 \le \alpha_n \le 1$, $\alpha_n \to 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. If $F(T)$ is nonempty, then $\{x_n\}$ and $\{z_n\}$ converge strongly to Pu, where P is the metric projection of H onto $F(T)$.

Using Theorem 5.1, we also have a strong convergence theorem for semigroups of nonexpansive mappings in a Hilbert space.

Theorem 5.4 ([30]). *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let S be a commutative semitopological semigroup with identity. Let $\mathcal{S} = \{T_s : s \in S\}$ be a nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$. Let $\{\mu_n\}$ be a strongly asymptotically invariant sequence of means on $C(S)$, i.e.,*

$$\|\mu_n - \ell_s^* \mu_n\| \to 0, \quad \forall s \in S.$$

Let $u \in C$ and define a sequence $\{x_n\}$ in C as follows: $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n u + (1-\alpha_n) T_{\mu_n} x_n, \quad \forall n \in \mathbb{N},$$

where $0 \le \alpha_n \le 1$, $\alpha_n \to 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then, $\{x_n\}$ converges strongly to a point $z \in F(\mathcal{S})$, where $z = P_{F(\mathcal{S})} u$.

6 Weak Convergence Theorems in Banach Spaces

In this section, using the results in Sections 2 and 3, we prove a weak convergence theorem of Mann's type iteration [28] for a commutative family of not necessarily continuous mappings in a Banach space. The following lemma is crucial in the proof of our theorem.

Lemma 6.1. *Let E be a smooth and reflexive Banach space and let C be a nonempty subset of E. Let S be a commutative semitopological semigroup with identity. Let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as mappings of C into itself such that $B(\mathcal{S}) \neq \emptyset$. Let μ be a mean on $C(S)$. Then*

$$\phi(T_\mu x, m) \le \phi(x, m), \quad \forall x \in C, \quad m \in B(\mathcal{S}),$$

where $T_\mu x$ is a mean vector of $\{T_s x : s \in S\}$ and μ.

Using Lemma 6.1, we have the following result.

Lemma 6.2. *Let E be a uniformly convex and smooth Banach space and let C be a nonempty, closed and convex subset of E. Let S be a commutative semitopological semigroup with identity. Let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as mappings of C into itself such* *that $B(\mathcal{S}) \neq \emptyset$. Let $\{\mu_n\}$ be a sequence of means on $C(S)$. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n < 1$ and let $\{x_n\}$ be a sequence in E generated by $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T_{\mu_n} x_n, \quad \forall n \in \mathbb{N}.$$

If $R_{B(\mathcal{S})}$ is a sunny generalized nonexpansive retraction of E onto $B(\mathcal{S})$, then $\{R_{B(\mathcal{S})} x_n\}$ converges strongly to $z \in B(\mathcal{S})$.

Now, we can prove the following weak convergence theorem for semigroups of not necessarily continuous mappings in a Banach space.

Theorem 6.3 ([14]). *Let E be a uniformly convex Banach space with a Fréchet differentiable norm and let C be a nonempty, closed and convex subset of E. Let S be a commutative semitopological semigroup with identity. Let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as mappings of C into itself such that $A(\mathcal{S}) = B(\mathcal{S}) \neq \emptyset$ and let $R_{B(\mathcal{S})}$ be the sunny generalized nonexpansive retraction of E onto $B(\mathcal{S})$. Suppose that*

$$\limsup_{\alpha} \sup_{x,y \in D} (\mu_\alpha)_s (\phi(T_s x, T_t y) - \phi(T_s x, y)) \leq 0, \quad \forall t \in S \tag{6.1}$$

for every strongly asymptotically invariant net $\{\mu_\alpha\}$ of means on $C(S)$ and every bounded subset D of C. Let $\{\mu_n\}$ be a strongly asymptotically invariant sequence of means on $C(S)$, i.e., a sequence of means on $C(S)$ such that

$$\|\mu_n - \ell_s^* \mu_n\| \to 0, \quad \forall s \in S.$$

Define a sequence $\{x_n\}$ in C as follows: $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T_{\mu_n} x_n, \quad \forall n \in \mathbb{N},$$

where $0 \leq \alpha_n \leq 1$ and $\liminf_{n \to \infty} \alpha_n (1 - \alpha_n) > 0$. Then, $\{x_n\}$ converges weakly to a point $z \in F(\mathcal{S})$ and $z = \lim_{n \to \infty} R_{B(\mathcal{S})} x_n$.

Using Theorem 6.3, we obtain well-known and new theorems which are connected with weak convergence results in Banach spaces. Let E be a smooth Banach space and let C be a nonempty subset of E. A mapping $T : C \to C$ is called *generalized nonspreading* [22] if there exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$\begin{aligned} \alpha\phi(Tx, Ty) + &(1 - \alpha)\phi(x, Ty) + \gamma\{\phi(Ty, Tx) - \phi(Ty, x)\} \\ &\leq \beta\phi(Tx, y) + (1 - \beta)\phi(x, y) + \delta\{\phi(y, Tx) - \phi(y, x)\} \end{aligned} \tag{6.2}$$

for all $x, y \in C$. Putting $\alpha = \beta = \gamma = 1$ and $\delta = 0$ in (6.2), we obtain that

$$\phi(Tx, Ty) + \phi(Ty, Tx) \leq \phi(Tx, y) + \phi(Ty, x), \quad \forall x, y \in C.$$

Such a mapping T is *nonspreading* in the sense of Kohsaka and Takahashi [25]. In the case of $\alpha = 1$ and $\beta = \gamma = \delta = 0$ in (6.2), we obtain that

$$\phi(Tx, Ty) \leq \phi(x, y), \quad \forall x, y \in C.$$

Such a mapping T is called *ϕ-nonexpansive.* Using Theorem 6.3, we obtain the following weak convergence theorem of Mann's type iteration for generalized nonspreading mappings in a Banach space.

Theorem 6.4. *Let E be a uniformly convex Banach space with a Fréchet differentiable norm and let C be a nonempty, closed and convex subset of E. Let $T : C \to C$ be a generalized nonspreading mapping such that $A(T) = B(T) \neq \emptyset$. Let $R_{B(T)}$ be the sunny generalized nonexpansive retraction of E onto $B(T)$. Let $\{\mu_n\}$ be a strongly asymptotically invariant sequence of means on l^∞, i.e., a sequence of means on l^∞ such that*

$$\|\mu_n - \ell_1^* \mu_n\| \to 0.$$

Define a sequence $\{x_n\}$ in C as follows: $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T_{\mu_n} x_n, \quad \forall n \in \mathbb{N},$$

where $0 \leq \alpha_n \leq 1$ and $\liminf_{n \to \infty} \alpha_n (1 - \alpha_n) > 0$. Then the sequence $\{x_n\}$ converges weakly to a point $z \in F(T)$, where $z = \lim_{n \to \infty} R_{B(T)} x_n$.

Using Theorem 6.4, we obtain the following theorem.

Theorem 6.5. *Let E be a uniformly convex Banach space with a Fréchet differentiable norm. Let $T : E \to E$ be an $(\alpha, \beta, \gamma, \delta)$-generalized nonspreading mapping such that $\alpha > \beta$ and $\gamma \leq \delta$. Assume that $F(T) \neq \emptyset$ and let $R_{F(T)}$ be the sunny generalized nonexpansive retraction of E onto $F(T)$. Let $\{\mu_n\}$ be a strongly asymptotically invariant sequence of means on l^∞, i.e., a sequence of means on l^∞ such that*

$$\|\mu_n - \ell_1^* \mu_n\| \to 0.$$

Define a sequence $\{x_n\}$ in C as follows: $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T_{\mu_n} x_n, \quad \forall n \in \mathbb{N},$$

where $0 \leq \alpha_n \leq 1$ and $\liminf_{n \to \infty} \alpha_n (1 - \alpha_n) > 0$. Then the sequence $\{x_n\}$ converges weakly to a point $z \in F(T)$, where $z = \lim_{n \to \infty} R_{F(T)} x_n$.

Let E be a smooth Banach space and let C be a nonempty subset of E. Let S be a semitopological semigroup. A continuous representation $\mathcal{S} = \{T_s : s \in S\}$ of S as mappings on C is a *ϕ-nonexpansive semigroup* on C if each T_s, $s \in S$ is ϕ-nonexpansive. Using Theorem 6.3, we also have the following weak convergence theorem for ϕ-nonexpansive semigroups in a Banach space.

Theorem 6.6. *Let E be a uniformly convex Banach space with a Fréchet differentiable norm and let C be a nonempty closed and convex subset of E. Let S be a commutative semitopological semigroup with identity. Let $\mathcal{S} = \{T_s : s \in S\}$ be a ϕ-nonexpansive semigroup on C such that $A(\mathcal{S}) = B(\mathcal{S}) \neq \emptyset$ and let $R_{B(\mathcal{S})}$ be the sunny generalized nonexpansive retraction of E onto $B(\mathcal{S})$. Let $\{\mu_n\}$ be a strongly asymptotically invariant sequence of means on $C(S)$, i.e., a sequence of means on $C(S)$ such that*

$$\|\mu_n - \ell_s^* \mu_n\| \to 0, \quad \forall s \in S.$$

Define a sequence $\{x_n\}$ in C as follows: $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T_{\mu_n} x_n, \quad \forall n \in \mathbb{N},$$

where $0 \leq \alpha_n \leq 1$ and $\liminf_{n \to \infty} \alpha_n (1 - \alpha_n) > 0$. Then the sequence $\{x_n\}$ converges weakly to a point $z \in F(\mathcal{S})$, where $z = \lim_{n \to \infty} R_{B(\mathcal{S})} x_n$.

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