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ON THE EXISTENCE OF THE MEAN VALUES FOR CERAIN ORDER-PRESERVING OPERATORS IN $L^1$.

HIROMICHI MIYAKE (三宅啓道)

1. Introduction

Let $(\Omega, \mathcal{A}, \mu)$ be a positive measure space with $\sigma$-algebra $\mathcal{A}$ and measure $\mu$. It is known that if $T$ is a linear contraction on $L^1 = L^1(\Omega, \mathcal{A}, \mu)$ which does not increase $L^\infty$-norm (so called a Dunford-Schwartz operator on $L^1$) and $\mu$ is finite, then $T$ is weakly almost periodic, that is, for each $f \in L^1$, the orbit $\{T^n f : n = 0, 1, \ldots\}$ of $f$ under $T$ is a relatively weakly compact subset of $L^1$. This is, however, not the case when $\mu$ is infinite and $\sigma$-finite. Indeed, in this case, there exists a Dunford-Schwartz operator $T$ on $L^1$ which is not weakly almost periodic, but for each $f \in L^1$, the Cesàro means $n^{-1} \sum_{k=0}^{n-1} T^k f$ of $f$ converge strongly to a fixed point of $T$. Then, assigning to each $f \in L^1$ the limit of the Cesàro means $n^{-1} \sum_{k=0}^{n-1} T^k f$ of $f$, the linear operator on $L^1$ is a unique projection $P$ of $L^1$ onto the subspace of $L^1$ consisting of fixed points of $T$ such that $PT = P = TP$ and for each $f \in L^1$, $Pf$ is contained in the closure of convex hull of the orbit of $f$ under $T$. Such a projection $P$ is said to be ergodic; see Takahashi [21] and also Hirano, Kido and Takahashi [8]. Therefore, it is natural to ask a question of whether every Dunford-Schwartz operator on $L^1$ has the mean values on $L^1$ (in the sense defined in the following section) if $\mu$ is $\sigma$-finite.

Recently, we [15] discussed a method of constructing a separated locally convex topology $\tilde{\tau}$ on $L^1$ such that the weak topology of $L^1$ associated with $\tilde{\tau}$ is coarser than the weak topology on $L^1$ generated by $L^\infty = L^\infty(\Omega, \mathcal{A}, \mu)$ without the assumption that $\mu$ is finite. A sufficient and necessary condition was shown for a bounded subset of $L^1$ relative to $L^1$-norm to be relatively weakly compact in $(L^1, \tilde{\tau})$. We applied it to show the existence of the mean values for commutative semigroups of Dunford-Schwartz operators on $L^1$. This result also gives an identification of the limit function in almost everywhere convergence of the Cesàro means $n^{-1} \sum_{k=0}^{n-1} T^k f$ of an $f \in L^1$ for such an operator $T$ on $L^1$.

In this paper, we summarize those arguments presented in [15] about weak compactness in $(L^1, \tilde{\tau})$ and the existence of the mean values for commutative semigroups of Dunford-Schwartz operators on $L^1$. We also apply them to show the existence of the mean values for certain
order-preserving operators $T$ in $L^1$, for which it seems to be still unknown whether for each $f \in L^1_+$, the Cesàro means $n^{-1} \sum_{k=0}^{n-1} T^k f$ of $f$ converge weakly in $L^1$ in the case when $\mu$ is infinite and $\sigma$-finite.

2. Preliminaries

Throughout the paper, let $\mathbb{N}_+$ and $\mathbb{R}$ denote the set of non-negative integers and the set of real numbers, respectively. Let $\langle E, F \rangle$ be the duality between vector spaces $E$ and $F$ over $\mathbb{R}$. If $A$ is a subset of $E$, then $A^o = \{ y \in F : \langle x, y \rangle \leq 1 \ (x \in A) \}$ is a subset of $F$, called the polar of $A$. For each $y \in F$, we define a linear form $f_y$ on $E$ by $f_y(x) = \langle x, y \rangle$. Then, $\sigma(E, F)$ denotes the weak topology on $E$ generated by the family $\{ f_y : y \in F \}$. Let $\tau(E, F)$ and $\beta(E, F)$ denote the Mackey topology on $E$ with respect to $\langle E, F \rangle$ and the strong topology on $E$ with respect to $\langle E, F \rangle$, respectively. Let $(E, \mathfrak{T})$ is a locally convex space. Then, the topological dual of $E$ is denoted by $E'$. The bilinear form $(x, f) \mapsto f(x)$ on $E \times E'$ defines a duality $\langle E, E' \rangle$. The weak topology $\sigma(E, E')$ on $E$ generated by $E'$ is called the weak topology of $E$ (associated with $\mathfrak{T}$ if this distinction is necessary). The topological dual of $E$ under the strong topology $\beta(E', E)$ with respect to $\langle E, E' \rangle$ is denoted by $E'_\beta$, called the strong dual of $E$.

Let $S$ be a semigroup. We denote by $l^\infty(S)$ the vector space of real-valued bounded functions defined on $S$; under the norm $f \mapsto \|f\| = \sup_{s \in S} |f(s)|$, $l^\infty(S)$ is a Banach space. For each $s \in S$, we define operators $l(s)$ and $r(s)$ on $l^\infty(S)$ by $(l(s)f)(t) = f(st)$ and $(r(s)f)(t) = f(ts)$ for each $t \in S$ and $f \in l^\infty(S)$, respectively. Then, a linear functional $m$ on $l^\infty(S)$ is said to be a mean on $S$ if $\|m\| = m(e) = 1$, where $e(s) = 1$ for each $s \in S$. For each $s \in S$, we define a point evaluation $\delta_s$ by $\delta_s(f) = f(s)$ for each $f \in l^\infty(S)$. A convex combination of point evaluations is called a finite mean on $S$. As is well known, a linear functional $m$ on $l^\infty(S)$ is a mean on $S$ if and only if $\inf_{s \in S} f(s) \leq m(f) \leq \sup_{s \in S} f(s)$ for each $f \in l^\infty(S)$. We often write $m_s(f(s))$ for the value $m(f)$ of a mean $m$ on $S$ at an $f \in l^\infty(S)$. A mean $m$ on $S$ is said to be left (or right) invariant if $m = l(s)'m$ (or $m = r(s)'m$) for each $s \in S$, where $l(s)'$ and $r(s)'$ are the adjoint operators of $l(s)$ and $r(s)$, respectively. If a mean $m$ on $S$ is left and right invariant, then $m$ is said to be invariant. In particular, an invariant mean on $\mathbb{N}_+$ is called a Banach limit. If there exists a left (or right) invariant mean on $S$, then $S$ is said to be left (or right) amenable. If $S$ is left and right amenable, then $S$ is said to be amenable. It is known that if $S$ is commutative, then $S$ is amenable, due to the fixed point theorem of Kakutani and Markov; for more details, see Day [4].

We denote by $l^\infty_c(S, E)$ the vector space of vector-valued functions $f$ defined on a semigroup $S$ with values in a locally convex space $E$ for which the closure of convex hull of $f(S)$ is weakly compact. For each $s \in S$, we define the operators $L(s)$ and $R(s)$ on $l^\infty_c(S, E)$
by $(L(s)f)(t) = f(st)$ and $(R(s)f)(t) = f(ts)$ for each $t \in S$ and $f \in l^\infty_c(S, E)$, respectively. Motivated by an original work of Takahashi [21], we introduce a notion of the mean values for vector-valued functions in $l^\infty_c(S, E)$. Let $m$ be a mean on $S$. For each $f \in l^\infty_c(S, E)$, we define a linear functional $\tau(m)f$ on the strong dual $E'_\beta$ of $E$ by $\tau(m)f : x' \mapsto m_s(f(s), x')$ for each $x' \in E'$. Then, it follows from the separation theorem that $\tau(m)f$ is an element of $E$, which is contained in the closure of convex hull of $f(S)$. We denote by $\tau(m)$ the linear operator of $l^\infty_c(S, E)$ into $E$ that assigns to each $f \in l^\infty_c(S, E)$ a unique element $\tau(m)f$ of $E$ such that $m_s(f(s), x') = \langle \tau(m)f, x' \rangle$ for each $x' \in E'$. The operator $\tau(m)$ is called the vector-valued mean on $S$ (generated by $m$ if explicit reference to the mean $m$ is needed); for more details, see Kada and Takahashi [9]. Note that it is also a vector-valued mean in the sense of Goldberg and Irwin [7]. Whenever $S$ is left amenable, an $f \in l^\infty_c(S, E)$ is said to have the mean value if there exists an element $p$ of $E$ such that $p = \tau(m)f$ for each left invariant mean $m$ on $S$. The element $p$ is called the mean value of $f$; see Lorentz [13], Day [4] and Miyake [14]. It is shown in [14] that an $f \in l^\infty_c(S, E)$ has the mean value if and only if the closure of convex hull of the right orbit $\mathcal{R}\mathcal{O}(f) = \{R(s)f \in l^\infty_c(S, E) : s \in S\}$ of $f$ contains exactly one constant function, where $l^\infty_c(S, E)$ is endowed with the topology of weakly pointwise convergence, for which the family of finite intersections of sets of the form $U(s; x'; \epsilon) = \{f \in l^\infty_c(S, E) : |\langle f(s), x' \rangle| < \epsilon\}$ $(s \in S, x' \in E'$ and $\epsilon > 0)$ is a neighborhood base of $0$. It is also known that whenever $S$ is an amenable semigroup with identity, if a vector-valued function $f$ defined on $S$ with values in a bounded subset of a complete locally convex space is weakly almost periodic in the sense of Eberlein, then $f$ has the mean value in the sense herein defined; see also von Neumann [17], Bochner and von Neumann [2], Eberlein [6], Ruess and Summers [19] and Miyake and Takahashi [16].

The notion of the mean values for vector-valued functions is applied to semigroups of transformations in the following way. Let $C$ be a closed convex subset of a locally convex space $(E, \Sigma)$ and let $S$ be a left amenable semigroup acting on $C$. We assume that for each $x \in C$, the closure of convex hull of the orbit $\mathcal{O}(x) = \{s(x) : s \in S\}$ of $x$ under $S$ is weakly compact. Let $m$ be a mean on $S$. We define a mapping $\phi_S$ of $C$ into $l^\infty_c(S, E)$ by $\phi_S(x)(s) = s(x)$ for each $x \in C$ and $s \in S$. We simply write $S(m)x$ in place of $\tau(m)\phi_S(x)$. We denote by $S(m)$ the mapping of $C$ into itself that assigns to each $x \in C$ a unique element $S(m)x$ of $C$ such that $m_s(s(x), x') = \langle S(m)x, x' \rangle$ for each $x' \in E'$. An element $p$ of $E$ is said to be the mean value of an $x \in C$ under $S$ (with respect to $\Sigma$ if this distinction is necessary) if $p$ is the mean value of $\phi_S(x)$, that is, $p = S(m)x$ for each left invariant mean $m$ on $S$. If there exists the mean value of $x$ under $S$ for each $x \in C$, then $S$ is said to have the mean values on $C$ (with respect to $\Sigma$). If $S$ is a semigroup
generated by a single element $\sigma \in S$, then we often write $\sigma(m)x$ (or $\sigma(m)$) instead of $S(m)x$ (or $S(m)$). Accordingly, the mean value of an $x \in C$ under $S$ is simply called the mean value of $x$ under $\sigma$. Moreover, if $S$ has the mean values on $C$, then $\sigma$ is also said to have the mean values on $C$; see Ruess and Summers [19], Miyake and Takahashi [16] and Miyake [14].

3. ON WEAK COMPACTNESS IN A SEPARATED LOCALLY CONVEX TOPOLOGY ON $L^1$

Throughout the paper, let $(\Omega, \mathcal{A}, \mu)$ denote a positive measure space with $\sigma$-algebra $\mathcal{A}$ and measure $\mu$ and let $\mathcal{F}$ denote the family of measurable subsets of $\Omega$ with finite measure. Then, $\mathcal{F}$ is ordered by set inclusion in the sense that for $E, F \in \mathcal{F}$, $E \subseteq F$ if and only if $E \subset F$, so that each finite subset of $\mathcal{F}$ has an upper bound. Let $E \in \mathcal{A}$. If $\mathcal{A}_E$ denotes the family of intersections of members of $\mathcal{A}$ with $E$ and $\mu_E$ denotes the restriction of $\mu$ to $\mathcal{A}_E$, then the triple $(E, \mathcal{A}_E, \mu_E)$ is a positive measure space. For $1 \leq p < \infty$, let $\mathcal{L}^p(E)$ be the vector space of measurable functions $f$ defined on $E$ for which $\|f\|_{E,p} = (\int_E |f|^p \, d\mu)^{\frac{1}{p}} < \infty$ and let $\mathcal{L}^\infty(E)$ be the vector space of measurable functions $f$ defined on $E$ for which $\|f\|_{E,\infty} = \inf_N \sup_{w \in E \setminus N} |f(w)| < \infty$, where $N$ ranges over the null subsets of $E$. If $\mathcal{N}_E$ denotes the set of null functions defined on $E$ and $[f]$ denotes the equivalence class of an $f \in \mathcal{L}^p(E)$ mod $\mathcal{N}_E$ ($1 \leq p \leq \infty$), then $[f] \mapsto \|f\|_{E,p}$ is a norm on the quotient space $\mathcal{L}^p(E)/\mathcal{N}_E$, which thus becomes a Banach space, usually denoted by $L^p(E)$. For an $f \in L^p(\Omega)$, $\|f\|_{\Omega,p}$ is called the $L^p$-norm of $f$, simply denoted by $\|f\|_p$. A measurable function $f$ defined on $\Omega$ is called essentially-bounded if $\|f\|_{\infty} < \infty$. Every element of $L^p(\Omega)$ is considered as a measurable function $f$ defined on $E$ with $\|f\|_{E,p} < \infty$, if no confusion will occur. We note that $L^p(\Omega)$ is ordered by defining $f \leq g$ ($f, g \in L^p(\Omega)$) to mean that $f(x) \leq g(x)$ almost everywhere on $\Omega$, so that $L^p(\Omega)$ is a Banach lattice. We call a function $f \in L^p(\Omega)$ non-negative if $f \geq 0$. The set of non-negative functions in $L^p(\Omega)$ will be denoted by $L^p_+(\Omega)$. For each $E \in \mathcal{A}$, the bilinear form on $L^1(E) \times L^\infty(E)$ that is defined by $\langle f, h \rangle = \int_E fh \, d\mu$ for each $f \in L^1(E)$ and $h \in L^\infty(E)$ places $L^1(E)$ and $L^\infty(E)$ in duality. For $E, F \in \mathcal{F}$ with $E \subseteq F$, let $i_{EF}$ denote the mapping of $L^1(F)$ onto $L^1(E)$ that assigns to each $f \in L^1(F)$ the restriction $f|_E \in L^1(E)$ of $f$ to $E$. Then, the canonical imbedding of $L^\infty(E)$ into $L^\infty(F)$ is the adjoint operator of $i_{EF}$, denoted by $j_{FE}$.

Let $\mathcal{L}^1_{loc}(\Omega)$ be the vector space of measurable functions defined on $\Omega$ which are locally integrable, that is, integrable on each $E \in \mathcal{F}$ and let $\mathcal{N}_{loc}$ be the vector subspace of $\mathcal{L}^1_{loc}(\Omega)$ consisting of measurable functions $f$ defined on $\Omega$ for which $\mu\{w \in E : f(w) \neq 0\} = 0$ for each $E \in \mathcal{F}$. If $[f]$ denotes the equivalence class of an $f \in \mathcal{L}^1_{loc}(\Omega)$
mod $\mathcal{N}_{loc}$, then $[f] = [g]$ ($f, g \in L^1_{loc}(\Omega)$) means that for each $E \in \mathcal{F}$, $f|_E(x) = g|_E(x)$ almost everywhere on $E$, where $f|_E$ and $g|_E$ are the restrictions of $f$ and $g$ to $E$, respectively. In particular, if $\mu$ is $\sigma$-finite, then $\mathcal{N}_{loc}$ equals the set $\mathcal{N}_{\Omega}$ of null functions defined on $\Omega$ and hence for $f, g \in L^1_{loc}(\Omega)$, $[f] = [g]$ if and only if $f(x) = g(x)$ almost everywhere on $\Omega$. For each $E \in \mathcal{F}$, $[f] \mapsto \|f\|_{E,1}$ is a semi-norm on the quotient space $L^1_{loc}(\Omega)/\mathcal{N}_{loc}$, which becomes a locally convex space, denoted by $L^1_{loc}(\Omega)$, under the separated locally convex topology $\tau$ generated by the semi-norms $[f] \mapsto \|f\|_{E,1}$ ($E \in \mathcal{F}$). Every element of $L^1_{loc}(\Omega)$ is also considered as a measurable, locally integrable function defined on $\Omega$, if no confusion will occur.

In the sequel, we shall assume that the measure space $(\Omega, \mathcal{A}, \mu)$ is $\sigma$-finite. The product space $\mathcal{L}$ of $(L^1(E), \| \cdot \|_{E,1})$, $E \in \mathcal{F}$ is the Cartesian product $L = \prod_{E \in \mathcal{F}} L^1(E)$ endowed with the product topology. Then, $L^1_{loc}(\Omega)$ is identified as a closed (and hence complete) subspace of $\mathcal{L}$ by the isomorphism of $L^1_{loc}(\Omega)$ into $\mathcal{L}$ that is defined by $f \mapsto (f|_E)_{E \in \mathcal{F}}$, where $f|_E$ is the restriction of an $f \in L^1_{loc}(\Omega)$ to $E$. Let $D = \bigoplus_{E \in \mathcal{F}} L^\infty(E)$ be the direct sum of $L^\infty(E)$, $E \in \mathcal{F}$. The vector spaces $L$ and $D$ are placed in duality by the bilinear form on $L \times D$ that is defined by $\langle f, g \rangle = \sum_{E} \langle f_E, g_E \rangle$ for each $f = (f_E) \in L$ and $g = (g_E) \in D$, where $f_E \in L^1(E)$ and $g_E \in L^\infty(E)$ for each $E \in \mathcal{F}$ and the sum is taken over at most a finite number of non-zero terms of $g$. Then, the topological dual of $\mathcal{L}$ is $D$ and the topological dual of $L^1_{loc}(\Omega)$ is the quotient space $D/(L^1_{loc}(\Omega))^{\circ}$, which is algebraically isomorphic to the vector subspace $L^\infty_{loc}(\Omega)$ of $L^\infty(\Omega)$ consisting of measurable, essentially-bounded functions $f$ defined on $\Omega$ for which $\mu\{w \in \Omega : f(w) \neq 0\} < \infty$. Note that $L^1_{loc}(\Omega)$ is identified as the reduced projective limit $\lim \inf_{EF} i_{EF}(L^1(E))$ of the family $\{(L^1(E), \| \cdot \|_{E,1}) : E \in \mathcal{F}\}$ with respect to the mappings $i_{EF}$ ($E, F \in \mathcal{F}$ and $E \leq F$). If $D = \bigoplus_{E \in \mathcal{F}} L^\infty(E)$ is the locally convex direct sum of $(L^\infty(E), \tau(L^\infty(E), L^1(E)))$, $E \in \mathcal{F}$, then the quotient space $D/(L^1_{loc}(\Omega))^{\circ}$ is the inductive limit $\lim \inf_{EF} j_{FE} L^\infty(E)$ of the family $\{(L^\infty(E), \tau(L^\infty(E), L^1(E))) : E \in \mathcal{F}\}$ with respect to the mappings $j_{FE}$ ($E, F \in \mathcal{F}$ and $E \leq F$).

Proposition 1. $L^1_{loc}(\Omega)$ is a complete locally convex space. The topological dual of $L^1_{loc}(\Omega)$ is algebraically isomorphic to $L^\infty_{loc}(\Omega)$.

It is clear that if $\mu$ is finite, then $L^1_{loc}(\Omega)$ equals $L^1(\Omega)$ and hence, $\tau$ is just the topology on $L^1(\Omega)$ generated by the metric $(f, g) \mapsto \|f - g\|_1$. We note that if $C$ is a bounded subset of $L^1(\Omega) \cap L^p(\Omega)$ relative to $L^p$-norm, i.e. $\sup_{f \in C} \|f\|_p < \infty$, then the weak topology on $C$ generated by $L^q(\Omega)$ is the relative topology of the weak topology of $L^1_{loc}(\Omega)$ to $C$, where $p$ and $q$ are a pair of conjugate exponents, that is, $1 < p < \infty$ and $p^{-1} + q^{-1} = 1$. 
A subset $A$ of $L^1_{\text{loc}}(\Omega)$ is said to be locally uniformly integrable if for each $E \in \mathcal{F}$, the set $\{f|_E \in L^1(E) : f \in A\}$ of the restrictions of the functions in $A$ to $E$ is uniformly integrable, that is, for each $E \in \mathcal{F}$ and $\epsilon > 0$, there exists a $\delta > 0$ such that for each $F \in \mathcal{A}$ with $F \subset E$ and $\mu(F) < \delta$, $\sup_{f \in A} \int_F |f| \, d\mu < \epsilon$. It follows from the theorem of Tychonoff that if $A$ is a locally uniformly integrable, bounded subset of $L^1_{\text{loc}}(\Omega)$, then $A$ is relatively weakly compact, since $L^1_{\text{loc}}(\Omega)$ is a complete subspace of $\mathcal{L}$. The converse holds.

**Proposition 2.** Let $C$ be a subset of $L^1_{\text{loc}}(\Omega)$. Then, $C$ is relatively weakly compact if and only if $C$ is bounded and locally uniformly integrable.

We apply Cantor's diagonal argument to obtain a characterization of an adherent point of a subset $C$ of $L^1_{\text{loc}}(\Omega)$ as the limit function in almost everywhere convergence of some sequence of functions in $C$.

**Lemma 1.** Let $C$ be a subset of $L^1_{\text{loc}}(\Omega)$ and let $f$ be a function in the closure of $C$. Then, there exists a sequence $\{f_n\}$ of functions in $C$ such that $f_n(x)$ converges to $f(x)$ almost everywhere on $\Omega$.

Let $\tilde{\tau}$ denote the relative topology of $\tau$ on $L^1_{\text{loc}}(\Omega)$ to $L^1(\Omega)$, which is the locally convex topology on $L^1(\Omega)$ generated by the semi-norms $f \mapsto \|f\|_{E,1} (E \in \mathcal{F})$. In the sequel, $L^1(\Omega)$ will be considered as a locally convex space under this topology $\tilde{\tau}$, if $L^1(\Omega)$ is not specified explicitly as a Banach space $(L^1(\Omega), \|\cdot\|_1)$ under the norm $f \mapsto \|f\|_1$. Then, the topological dual of $L^1(\Omega)$ is algebraically isomorphic to $L^\infty_{\text{loc}}(\Omega)$. It follows from Lemma 1 that if a subset $C$ of $L^1(\Omega)$ is bounded relative to $L^1$-norm, i.e. $\sup_{f \in C} \|f\|_1 < \infty$, then the closure in $L^1_{\text{loc}}(\Omega)$ of $C$ is contained in $L^1(\Omega)$.

**Proposition 3.** If $C$ is a bounded subset of $L^1(\Omega)$ relative to $L^1$-norm, then the closure in $L^1(\Omega)$ of $C$ is complete.

A sufficient and necessary condition is also given by Lemma 1 for a bounded subset of $L^1(\Omega)$ relative to $L^1$-norm to be relatively weakly compact.

**Proposition 4.** Let $C$ be a bounded subset of $L^1(\Omega)$ relative to $L^1$-norm. Then, $C$ is relatively weakly compact if and only if $C$ is locally uniformly integrable.

**Remark 1.** Let $\Omega = \mathbb{R}$, let $\mathcal{A}$ be the $\sigma$-algebra of Lebesgue measurable subsets of $\mathbb{R}$ and let $\mu$ be Lebesgue measure on $\mathbb{R}$. Then, for each $f \in L^1(\mathbb{R})$, the subset $\{f_x : x \in \mathbb{R}\}$ of $L^1(\mathbb{R})$ is relatively weakly compact (or relatively compact relative to the weak topology of $L^1(\mathbb{R})$ associated with $\tilde{\tau}$), where $f_x(y) = f(y - x)$ for each $x, y \in \mathbb{R}$. For example, let $f$ be the real-valued function on $\mathbb{R}$ which is defined by $f(x) = e^{-|x|} (x \in \mathbb{R})$. Then, the subset $\{f_x : x \in \mathbb{R}\}$ of $L^1(\mathbb{R})$ is
not relatively weakly compact in \((L^1(\mathbb{R}), \| \cdot \|_1)\), but relatively weakly compact.

**Remark 2.** Let \(\Omega = \mathbb{R}^n\), i.e. \(n\)-dimensional Euclidean space, let \(\mathcal{A}\) be the \(\sigma\)-algebra of Lebesgue measurable subsets of \(\mathbb{R}^n\) and let \(\mu\) be Lebesgue measure on \(\mathbb{R}^n\). Then, by considering \(\mathcal{F}\) as the family \(\mathcal{K}\) of compact subsets of \(\mathbb{R}^n\), we can apply those arguments presented in this section to obtain similar results to the propositions in it, which concern weak compactness in the separated locally convex topology \(\tilde{\tau}_K\) on \(L^1(\mathbb{R}^n)\) generated by the semi-norms \(f \mapsto \|f\|_{K,1} (K \in \mathcal{K})\). The topological dual of \((L^1(\mathbb{R}^n), \tilde{\tau}_K)\) is algebraically isomorphic to the vector subspace of \(L^\infty(\mathbb{R}^n)\) consisting of Lebesgue measurable, essentially-bounded functions defined on \(\mathbb{R}^n\) with compact support. Note that, in this case, a Lebesgue measurable function \(f\) defined on \(\mathbb{R}^n\) is called locally integrable if \(f\) is Lebesgue integrable on each \(K \in \mathcal{K}\), and a subset \(A\) of \(L^1(\mathbb{R}^n)\) is said to be locally uniformly integrable if for each \(K \in \mathcal{K}\), the set \(\{f|_K \in L^1(K) : f \in A\}\) of the restrictions of the functions in \(A\) to \(K\) is uniformly integrable.

4. On existence of the mean values for operators

We apply the result about weak compactness in the separated locally convex topology \(\tilde{\tau}\) on \(L^1(\Omega)\) in the previous section to show the existence of the mean values for commutative semigroups of Dunford-Schwartz operators on \(L^1(\Omega)\). Similar results are also obtained for (commutative semigroups of) certain order-preserving operators in \(L^1(\Omega)\).

A linear operator \(T\) on \(L^1(\Omega)\) is said to be a Dunford-Schwartz operator on \(L^1(\Omega)\) if \(\|T\|_1 \leq 1\) and \(\|Tf\|_\infty \leq \|f\|_\infty\) for each \(f \in L^1(\Omega) \cap L^\infty(\Omega)\). In this section, \(T\) will denote such an operator on \(L^1(\Omega)\), if \(T\) is not specified explicitly. For each \(f \in L^1(\Omega)\), the orbit \(\{T^n f : n = 0, 1, \ldots\}\) of \(f\) under \(T\) (denoted by \(\mathcal{O}(f)\)) is a uniformly integrable, bounded subset of \(L^1(\Omega)\) relative to \(L^1\)-norm.

**Lemma 2.** For each \(f \in L^1(\Omega)\), the orbit \(\mathcal{O}(f)\) of \(f\) under \(T\) is relatively weakly compact. Moreover, if \(\mu\) is finite, then \(T\) is weakly almost periodic, that is, for each \(f \in L^1(\Omega)\), the orbit \(\mathcal{O}(f)\) of \(f\) under \(T\) is relatively weakly compact in \((L^1(\Omega), \| \cdot \|_1)\).

Let \(m\) be a mean on \(\mathbb{N}_+\). It follows from this lemma that for each \(f \in L^1(\Omega)\), there exists a unique function \(T(m)f\) in \(L^1(\Omega)\) such that \(m_h(\int_0^1 (T^k f) h \, d\mu) = \int_0^1 (T(m)f) h \, d\mu\) for each \(h \in L^\infty_{loc}(\Omega)\). Then, \(f \mapsto T(m)f\) is a linear operator on \(L^1(\Omega)\), denoted by \(T(m)\). For each \(f \in L^1(\Omega)\), \(T(m)f\) is contained in the closure of convex hull of the orbit \(\mathcal{O}(f)\) of \(f\) under \(T\).

**Lemma 3.** For each mean \(m\) on \(\mathbb{N}_+\), \(T(m)\) is a Dunford-Schwartz operator on \(L^1(\Omega)\).
Recall that a function $p$ in $L^1(\Omega)$ is the mean value of an $f \in L^1(\Omega)$ under $T$ with respect to $\tilde{\tau}$ if and only if $\int_{\Omega} pf \, d\mu = m_k \left( \int_{\Omega} (T^k f) h \, d\mu \right) = \int_{\Omega} (T^k f) h \, d\mu$ for each $h \in L^\infty(\Omega)$ and Banach limit $m$. It is known that $T$ can be regarded as a linear contraction on $L^p(\Omega)$ ($1 < p < \infty$), that is, a linear operator on $L^p(\Omega)$ whose norm is less than or equal to 1, due to Riesz-Thorin convexty theorem. It follows from the ergodic theorem of Yosida and Kakutani that for each $f \in L^1(\Omega) \cap L^2(\Omega)$, $n^{-1} \sum_{k=0}^{n-1} T^{k+h} f$ converges strongly to a fixed point of $T$ in $L^2(\Omega)$ uniformly in $h \in \mathbb{N}_+$. In other words, $T$ has the mean values on $L^1(\Omega) \cap L^2(\Omega)$ with respect to $\tilde{\tau}$; see Lorentz [13].

**Theorem 1.** Every Dunford-Schwartz operator on $L^1(\Omega)$ has the mean values on $L^1(\Omega)$ with respect to $\tilde{\tau}$.

The notion of the mean values for $T$ allows us to give an identification of the limit function in almost everywhere convergence of the Cesàro means $n^{-1} \sum_{k=0}^{n-1} T^k f$ of an $f \in L^1(\Omega)$ by virtue of the convergence theorem of Vitali.

**Proposition 5.** If the Cesàro means $n^{-1} \sum_{k=0}^{n-1} T^k f$ of an $f \in L^1(\Omega)$ converge almost everywhere on $\Omega$, then the limit function is the mean value of $f$ under $T$ with respect to $\tilde{\tau}$.

By the work of Takahashi [21], we are allowed to extend Theorem 1 to commutative semigroups of Dunford-Schwartz operators on $L^1(\Omega)$. It follows from Riesz-Thorin convexity theorem that every semigroup $S$ of Dunford-Schwartz operators on $L^1(\Omega)$ can be regarded as a semigroup of linear contractions on $L^p(\Omega)$ ($1 < p < \infty$). Moreover, if $S$ is commutative, then $S$ has the mean values on $L^2(\Omega)$ and also has the mean values on $L^1(\Omega) \cap L^2(\Omega)$ with respect to $\tilde{\tau}$; see also Kido and Takahashi [11].

**Theorem 2.** If $S$ is a commutative semigroup of Dunford-Schwartz operators on $L^1(\Omega)$, then $S$ has the mean values on $L^1(\Omega)$ with respect to $\tilde{\tau}$.

An operator $T$ on $L^1_+(\Omega)$ is said to be order-preserving if $f \leq g$ ($f, g \in L^1_+(\Omega)$) implies $T f \leq T g$. Similar results to the above proposition and theorems in this section can be obtained for order-preserving operators $T$ on $L^1_+(\Omega)$ for which $T(0) = 0$ and $T$ is nonexpansive with respect to $L^1$-norm and $L^\infty$-norm, that is, $\|T f - T g\|_1 \leq \|f - g\|_1$ for each $f, g \in L^1_+(\Omega)$ and $\|T f - T g\|_\infty \leq \|f - g\|_\infty$ for each $f, g \in L^1_+(\Omega) \cap L^\infty(\Omega)$, by means of the nonlinear interpolation theorem of Browder, which implies that such an operator on $L^1_+(\Omega)$ can be regarded as an operator $W$ on $L^p_+(\Omega)$ ($1 < p < \infty$) such that $\|W f - W g\|_p \leq \|f - g\|_p$ for each $f, g \in L^p_+(\Omega)$; see Krengel and Lin [12].

**Theorem 3.** If $T$ is an order-preserving operator on $L^1_+(\Omega)$ and $T(0) = 0$ and if $T$ is nonexpansive with respect to $L^1$-norm and $L^\infty$-norm, then $T$ has the mean values on $L^1_+(\Omega)$ with respect to $\tilde{\tau}$.
Finally, we note that the last theorem can be also generalized to commutative semigroups of such operators on $L^1_+(\Omega)$.

REFERENCES