GENERALIZED MIXED EQUILIBRIUM PROBLEMS FOR AN INFINITE FAMILY OF QUASI-ϕ-NONEXPANSIVE MAPPINGS IN BANACH SPACES

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Abstract. The purpose of this paper is to find a common element of the set of solutions for a generalized mixed equilibrium problem, the set of solutions for a variational inequality problem and the set of common fixed points for an infinite family of quasi-ϕ-nonexpansive mappings in a uniformly smooth and uniformly convex Banach space, by using a hybrid algorithm. As applications, we study the optimization problem.

Keywords: Generalized mixed equilibrium problem; mixed equilibrium problem; variational inequality; quasi-ϕ-nonexpansive mapping; maximal monotone operator; monotone mapping.

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1. Introduction

Let C is a nonempty closed convex subset of real Banach space E, E^* be the dual space of E, and $\langle \cdot, \cdot \rangle$ be the pairing between E and E^*. we denote by N and R the sets of positive integers and real numbers, respectively.

Let $G : C \times C \to R$ be a bifunction, $\psi : C \to R$ be a real-valued function and $A : C \to E^*$ be a nonlinear mapping. The *generalized mixed equilibrium problem* is to find $u \in C$ such that

$$G(u, y) + \langle Au,\ y - u\rangle + \psi(y) - \psi(u) \geq 0, \quad \forall y \in C. \tag{1.1}$$

The set of solutions of the problem (1.1) is denoted by Ω, i.e.,

$$\Omega = \{u \in C : G(u, y) + \langle Au,\ y - u\rangle + \psi(y) - \psi(u) \geq 0,\ \ \forall y \in C\}.$$

Special cases of the problem (1.1) are as follows:

(I) If $A = 0$, then the problem (1.1) is equivalent to find $u \in C$ such that

$$G(u, y) + \psi(y) - \psi(u) \geq 0, \quad \forall y \in C, \tag{1.2}$$

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which is called *the mixed equilibrium problem* [4]. The set of solutions of the problem (1.2) is denoted by $MEP(G)$.

(II) If $G = 0$, then the problem (1.1) is equivalent to find $u \in C$ such that

$$\langle Au,\ y - u\rangle + \psi(y) - \psi(u) \geq 0, \quad \forall y \in C, \tag{1.3}$$

which is called *the mixed variational inequality of Browder type* [3]. The set of solution of the problem (1.3) is denoted by $VI(C, A, \psi)$.

(III) If $A = 0$ and $\psi = 0$, then the problem (1.1) is equivalent to find $u \in C$ such that

$$G(u, y) \geq 0, \quad \forall y \in C, \tag{1.4}$$

which is called *the equilibrium problem for* G [2]. The set of solutions of the problem (1.4) is denoted by $EP(G)$

(IV) If $\psi = 0$, then the problem (1.1) is equivalent to find $u \in C$ such that

$$G(u, y) + \langle Au,\ y - u\rangle \geq 0, \quad \forall y \in C, \tag{1.5}$$

which is called *the generalized equilibrium problem* [16]. The set of solutions of the problem (1.5) is denoted by $GEP(G)$.

(V) If $G = 0$ and $\psi = 0$, then the problem (1.1) is equivalent to find $u \in C$ such that

$$\langle Au,\ y - u\rangle \geq 0, \quad \forall y \in C, \tag{1.6}$$

which is called the *Hartmann-Stampacchia variational inequality* [7]. The set of solutions of the problem (1.6) is denoted by $VI(C, A)$.

Recently, many authors studied the problems of finding a common element of the set of fixed points for a nonexpansive mapping and the set of solutions for an equilibrium problem in the setting of Hilbert spaces, uniformly smooth and uniformly convex Banach spaces, respectively (see, for instance, [9, 10, 14, 15, 17] and the references therein).

The purpose of this paper is to find a common element of the set of solutions for the generalized mixed equilibrium problem, the set of solutions for the variational inequality problem and the set of common fixed points for an infinite family of quasi-ϕ-nonexpansive mappings in a uniformly smooth and uniformly convex Banach space, by using a hybrid algorithm. As applications, we utilize our results to study the optimization problems. Our results improve and extend the corresponding results given in [4, 14, 15, 17].

2. Preliminaries

The mapping $J : E \to 2^{E^*}$ defined by

$$J(x) = \{x^* \in E^* : \langle x,\ x^*\rangle = ||x||^2 = ||x^*||^2\}, \ \ x \in E.$$

is called the *normalized duality mapping*. By the Hahn-Banach theorem, $J(x) \neq \emptyset$ for each $x \in E$.

In the sequel, we denote the strong convergence and weak convergence of a sequence $\{x_n\}$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively.

A Banach space E is said to be *strictly convex* if $\frac{||x+y||}{2} < 1$ for all $x, y \in U = \{z \in E : ||z|| = 1\}$ with $x \neq y$. E is said to be *uniformly convex* if, for each $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that $\frac{||x+y||}{2} \leq 1 - \delta$ for all $x, y \in U$ with $||x - y|| \geq \epsilon$. E is said to be *smooth* if the limit

$$\lim_{t \to 0} \frac{||x + ty|| - ||x||}{t} \tag{2.1}$$

exists for all $x, y \in U$. E is said to be *uniformly smooth* if the limit (2.1) exists uniformly in $x, y \in U$.

Remark 2.1. It is well-known that, if E is a smooth, strictly convex and reflexive Banach space, then the normalized duality mapping $J : E \to 2^{E^*}$ is single-valued, one-to-one and onto (see [5]).

Let E be a smooth, strictly convex and reflexive Banach space and C be a nonempty closed convex subset of E. The Lyapunov functional $\phi : E \times E \to R^+$ is defined by

$$\phi(x, y) = ||x||^2 - 2\langle x,\ Jy\rangle + ||y||^2, \quad \forall x, y \in E.$$

It is obvious from the definition of ϕ that

$$(||x|| - ||y||)^2 \leq \phi(x, y) \leq (||x|| + ||y||)^2, \quad \forall x, y \in E. \tag{2.2}$$

Following Alber [1], the generalized projection $\Pi_C : E \to C$ is defined by

$$\Pi_C(x) = \inf_{y \in C} \phi(y, x), \quad \forall x \in E. \tag{2.3}$$

Lemma 2.1. ([1,8]) Let E be a smooth, strictly convex and reflexive Banach space and C be a nonempty closed convex subset of E. Then the following conclusions hold:

(1) $\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \leq \phi(x, y)$ for all $x \in C$ and $y \in E$.

(2) If $x \in E$ and $z \in C$, then

$$z = \Pi_C x \Leftrightarrow \langle z - y,\ Jx - Jz\rangle \geq 0, \quad \forall y \in C.$$

Remark 2.2. (1) If E is a real Hilbert space H, then $\phi(x, y) = ||x - y||^2$ and Π_C is the metric projection P_C of H onto C.

(2) If E is a smooth, strictly convex and reflexive Banach space, then, for all $x, y \in E$, $\phi(x, y) = 0$ if and only if $x = y$ (see [5]).

Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E, $T : C \to C$ be a mapping and $F(T)$ be the set of fixed points of T. A point $p \in C$ is said to be an *asymptotic fixed point* of T if there exists a sequence $\{x_n\} \subset C$ such that $x_n \rightharpoonup p$ and $||x_n - Tx_n|| \to 0$. We denoted the set of all asymptotic fixed points of T by $\tilde{F}(T)$.

A mapping $T : C \to C$ is said to be *relatively nonexpansive* [12] if $F(T) \neq \emptyset$, $F(T) = \tilde{F}(T)$ and

$$\phi(p, Tx) \leq \phi(p, x), \quad \forall x \in C,\ p \in F(T).$$

A mapping $T : C \to C$ is said to be *closed* if, for any sequence $\{x_n\} \subset C$ with $x_n \to x$ and $Tx_n \to y$, then $Tx = y$.

Definition 2.1. ([13]) A mapping $T : C \to C$ is said to be *quasi-ϕ-nonexpansive* if $F(T) \neq \emptyset$ and

$$\phi(p, Tx) \leq \phi(p, x), \quad \forall x \in C\ p \in F(T).$$

Next, we give some examples which are closed and quasi-ϕ-nonexpansive mappings.

Example 2.1. ([11]) Let E be a uniformly smooth and strictly convex Banach space and $A \subset E \times E^*$ be a maximal monotone mapping such that $A^{-1}0$ (: the set of zero points of A) is nonempty. Then the mapping $J_r = (J + rA)^{-1}J$ is a closed and quasi-ϕ-nonexpansive from E onto $D(A)$ and $F(J_r) = A^{-1}0$.

Example 2.2. Let Π_C be the generalized projection from a smooth, strictly convex and reflexive Banach space E onto a nonempty closed convex subset $C \subset E$. Then Π_C is a closed and quasi-ϕ-nonexpansive mappings.

Lemma 2.2. ([8]) Let E be a smooth and uniformly convex Banach space. Let $\{x_n\}$ and $\{y_n\}$ be sequences in E such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If $\lim_{n\to\infty} \phi(x_n, y_n) = 0$, then $\lim_{n\to\infty} ||x_n - y_n|| = 0$.

Lemma 2.3. ([13]) Let E be a reflexive, strictly convex and smooth Banach space, C a closed convex subset of E and $T : C \to C$ be a quasi-ϕ-nonexpansive mapping. Then $F(T)$ is a closed convex subset of C.

For solving the generalized mixed equilibrium problem, let us assume that the function $\psi : C \to R$ is convex and lower semi-continuous, the nonlinear mapping $A : C \to E^*$ is continuous monotone and the bifunction $G : C \times C \to R$ satisfies the following conditions:

(A_1) $G(x, x) = 0, \ \forall\, x \in C$;
(A_2) G is monotone, i.e., $G(x, y) + G(y, x) \leq 0, \ \forall\, x, y \in C$;
(A_3) $\limsup_{t\downarrow 0} G(x + t(z - x), y) \leq G(x, y)$ for all $x, y, z \in C$;
(A_4) The function $y \mapsto G(x, y)$ is convex and lower semi-continuous.

Lemma 2.4. ([2, 6, 17]) Let E be a smooth, strictly convex and reflexive Banach space and C be a nonempty closed convex subset of E. Let $G : C \times C \to R$ be a bifunction satisfying the conditions (A_1)-(A_4). Let $r > 0$ and $x \in E$. Then we have the following:

(1) There exists $z \in C$ such that

$$G(z, y) + \frac{1}{r}\langle y - z,\ Jz - Jx\rangle \geq 0, \quad \forall y \in C. \tag{2.4}$$

(2) If we define a mapping $T_r : E \to C$ by

$$T_r(x) = \{z \in C :\ G(z, y) + \frac{1}{r}\langle y - z,\ Jz - Jx\rangle \geq 0,\ \forall y \in C\}, \ \forall x \in E,$$

then the following conclusions hold:

(a) T_r is single-valued;
(b) T_r is a firmly nonexpansive-type mapping, i.e., $\forall\, z, y \in E$

$$\langle T_r z - T_r y,\ JT_r z - JT_r y\rangle \leq \langle T_r z - T_r y,\ Jz - Jy\rangle;$$

(c) $F(T_r) = EP(G) = \hat{F}(T_r)$;

(d) $EP(G)$ is closed and convex;
(e) $\phi(q, T_r x) + \phi(T_r x, x) \leq \phi(q, x), \quad \forall q \in F(T_r)$

Lemma 2.5. Let E be a smooth, strictly convex and reflexive Banach space and C be a nonempty closed convex subset of E. Let $A : C \to E^*$ be a continuous monotone mapping, $\psi : C \to R$ a lower semi-continuous and convex function and $G : C \times C \to R$ be a bifunction satisfying the conditions (A_1)-(A_4). Let $r > 0$ be any given number and $x \in E$ be any given point. Then we have the following:

(I) There exists $u \in C$ such that

$$G(u, y) + \langle Au, \ y - u\rangle + \psi(y) - \psi(u) + \frac{1}{r}\langle y - u, \ Ju - Jx\rangle \geq 0, \quad \forall\, y \in C. \tag{2.5}$$

(II) If we define a mapping $K_r : C \to C$ by

$$\begin{aligned} K_r(x) = \{u \in C : G(u, y) + \langle Au, \ y - u\rangle + \psi(y) - \psi(u) \\ + \frac{1}{r}\langle y - u, \ Ju - Jx\rangle \geq 0, \ \forall\, y \in C\}, \ \forall x \in C, \end{aligned} \tag{2.6}$$

then the mapping K_r has the following properties:

(1) K_r is single-valued;
(2) K_r is a firmly nonexpansive type mapping, i.e.,

$$\langle K_r z - K_r y, JK_r z - JK_r y\rangle \leq \langle K_r z - K_r y, Jz - Jy\rangle, \quad \forall z, y \in E;$$

(3) $F(K_r) = \Omega = \hat{F}(K_r)$;
(4) Ω is a closed convex set of C;
(5) $\phi(p, K_r z) + \phi(K_r z, z) \leq \phi(p, x), \quad \forall p \in F(K_r), \ z \in E.$ (2.7)

Proof. Define a bifunction $H : C \times C \to R$ as follows:

$$H(x, y) = G(x, y) + \langle Ax, \ y - x\rangle + \psi(y) - \psi(x), \quad \forall x, y \in C.$$

It is easy to prove that H satisfies the conditions (A_1)-(A_4). Hence the conclusions (I) and (II) of Lemma 2.5 can be obtained from Lemma 2.4, immediately.

Remark 2.3. It follows from Lemma 2.4 that the mapping $K_r : C \to C$ defined by (12) is a relatively nonexpansive mapping and so it is quasi-ϕ-nonexpansive.

Lemma 2.6. ([18]) Let E be a uniformly convex Banach space, $r > 0$ be a positive number and $B_r(0)$ be a closed ball of E. Then, for any given subset $\{x_1, x_2, \cdots, x_N\} \subset B_r(0)$ and any positive numbers $\lambda_1, \lambda_2, \cdots, \lambda_N$ with $\sum_{n=1}^{N} \lambda_n = 1$, there exists a continuous, strictly increasing and convex function $g : [0, 2r) \to [0, \infty)$ with $g(0) = 0$ such that for any $i, j \in \{1, 2, \cdots, N\}$ with $i < j$,

$$\Big\|\sum_{n=1}^{N} \lambda_n x_n\Big\|^2 \leq \sum_{n=1}^{N} \lambda_n ||x_n||^2 - \lambda_i \lambda_j g(||x_i - x_j||). \tag{2.8}$$

Lemma 2.7. Let E be a uniformly convex Banach space, $r > 0$ be a positive number and $B_r(0)$ be a closed ball of E. Then, for any given sequence $\{x_i\}_{i=1}^{\infty} \subset B_r(0)$ and any given sequence $\{\lambda_i\}_{i=1}^{\infty}$ of positive numbers with $\sum_{n=1}^{\infty} \lambda_n = 1$, there exists a continuous, strictly increasing and convex function $g : [0, 2r) \to [0, \infty)$ with $g(0) = 0$

such that for any positive integers i, j with $i < j$,

$$\Big\|\sum_{n=1}^{\infty}\lambda_n x_n\Big\|^2 \leq \sum_{n=1}^{\infty}\lambda_n||x_n||^2 - \lambda_i\lambda_j g(||x_i - x_j||). \tag{2.9}$$

Proof. Since $\{x_i\}_{i=1}^{\infty} \subset B_r(0)$ and $\lambda_i > 0$ for all $i \geq 1$ with $\sum_{n=1}^{\infty}\lambda_n = 1$, we have

$$\Big\|\sum_{i=1}^{\infty}\lambda_i x_i\Big\| \leq \sum_{i=1}^{\infty}\lambda_i||x_i|| \leq r. \tag{2.10}$$

Hence, for any given $\epsilon > 0$ and any given positive integers i, j with $i < j$, it follows from (2.10) that there exists a positive integer $N > j$ such that $||\sum_{i=N+1}^{\infty}\lambda_i x_i|| \leq \epsilon$. Letting $\sigma_N = \sum_{i=1}^{N}\lambda_i$, by Lemma 2.6, we have

$$\begin{aligned}
\Big\|\sum_{i=1}^{\infty}\lambda_i x_i\Big\|^2 &= \Big\|\sigma_N\sum_{i=1}^{N}\frac{\lambda_i x_i}{\sigma_N} + \sum_{i=N+1}^{\infty}\lambda_i x_i\Big\|^2 \\
&\leq \sigma_N^2\Big\|\sum_{i=1}^{N}\frac{\lambda_i x_i}{\sigma_N}\Big\|^2 + \epsilon^2 + 2\epsilon\sigma_N\Big\|\sum_{i=1}^{N}\frac{\lambda_i x_i}{\sigma_N}\Big\| \\
&\leq \sigma_N^2\sum_{i=1}^{N}\frac{\lambda_i}{\sigma_N}||x_i||^2 - \lambda_i\lambda_j g(||x_i - x_j||) + \epsilon\Big(\epsilon + 2\sigma_N\Big\|\sum_{i=1}^{N}\frac{\lambda_i x_i}{\sigma_N}\Big\|\Big) \\
&\leq \sum_{i=1}^{N}\lambda_i||x_i||^2 - \lambda_i\lambda_j g(||x_i - x_j||) + \epsilon(\epsilon + 2||\sum_{i=1}^{N}\lambda_i x_i||) \\
&\leq \sum_{i=1}^{\infty}\lambda_i||x_i||^2 - \lambda_i\lambda_j g(||x_i - x_j||) + \epsilon\Big(\epsilon + 2\Big\|\sum_{i=1}^{N}\lambda_i x_i\Big\|\Big).
\end{aligned}$$

Therefore, since $\epsilon > 0$ is arbitrary, the conclusion of Lemma 2.7 hold.

3. Main Results

In this section, we shall use the hybrid method to prove a strong convergence theorem for finding a common element of the set of solutions for the generalized mixed equilibrium problem (1.1) and the set of common fixed points for an infinite family of quasi-ϕ-nonexpansive mappings in Banach spaces.

Theorem 3.1. Let C be a nonempty closed convex subset of a uniformly smooth and uniformly convex Banach space E. Let $A : C \to E^*$ be a continuous and monotone mapping, $\psi : C \to R$ be a lower semi-continuous and convex function and $G : C \times C \to R$ be a bifunction satisfying the conditions (A_1)-(A_4). Let $\{S_i\}_{i=1}^{\infty}$ be an infinite family of closed quasi-ϕ-nonexpansive mappings from C into itself with

$$\Gamma := \bigcap_{n=1}^{\infty} F(S_i)\bigcap\Omega \neq \emptyset,$$

where Ω is the set of solutions of the problem (1.1). Let $\{x_n\}$ be the sequence generated by

$$\begin{cases} x_0 \in C, \ \ C_0 = C, \\ y_n = J^{-1}(\alpha_{n0} J x_n + \sum_{i=1}^{\infty} \alpha_{ni} J S_i x_n), \\ u_n \in C \text{ such that} \\ G(u_n, y) + \langle A u_n, \ y - u_n \rangle + \psi(y) - \psi(u_n) + \dfrac{1}{r_n} \langle y - u_n, \ J u_n - J y_n \rangle \geq 0, \ \ \forall y \in C, \\ C_{n+1} = \{v \in C_n : \phi(v, u_n) \leq \phi(v, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \ \ \forall n \geq 0, \end{cases} \tag{3.1}$$

where $J : E \to E^*$ is the normalized duality mapping and, for each $i \geq 0$, $\{\alpha_{ni}\}$ is a sequence in $[0, 1]$ satisfying the following conditions:

(a) $\sum_{i=0}^{\infty} \alpha_{ni} = 1$ for all $n \geq 0$;

(b) $\liminf_{n \to \infty} \alpha_{n0} \cdot \alpha_{ni} > 0$ for all $i \geq 1$.

Then $\{x_n\}$ converges strongly to $\Pi_\Gamma x_0$, where Π_Γ is the generalized projection of E onto Γ.

Proof. First, we define two functions $H : C \times C \to R$ and $K_r : C \to C$ by

$$H(x, y) = G(x, y) + \langle Ax, \ y - x \rangle + \psi(y) - \psi(x), \quad \forall x, y \in C, \tag{3.2}$$

and

$$K_r(x) = \{u \in C : H(u, y) + \frac{1}{r} \langle y - u, \ Ju - Jx \rangle \geq 0, \ \ \forall y \in C\}, \ \ \forall x \in C. \tag{3.3}$$

By Lemma 2.5, we know that the function H satisfies the conditions (A_1)-(A_4) and K_r has the properties (1)-(5) as given in Lemma 2.5. Therefore, (3.1) is equivalent to the following:

$$\begin{cases} x_0 \in C, \ \ C_0 = C, \\ y_n = J^{-1}(\sum_{i=0}^{\infty} \alpha_{ni} J S_i x_n), \\ u_n \in C \text{ such that} \\ H(u_n, y) + \dfrac{1}{r_n} \langle y - u_n, \ J u_n - J y_n \rangle \geq 0, \ \ \forall y \in C, \\ C_{n+1} = \{v \in C_n : \phi(v, u_n) \leq \phi(v, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \ \ \forall n \geq 0, \end{cases} \tag{3.4}$$

where $S_0 = I$ (: the identity mapping).

Step (I): We prove that C_n is a closed and convex subset of C for all $n \geq 0$.

In fact, it is obvious that $C_0 = C$ is closed and convex. Suppose that C_k is closed and convex for some $k \geq 1$. For $v \in C_{k+1}$, we have $\phi(v, u_k) \leq \phi(v, x_k)$, which is equivalent to

$$2\langle v, Jx_k - Ju_k \rangle \leq ||x_k||^2 - ||u_k||^2.$$

Therefore, it follows that

$$C_{k+1} = \{v \in C_k : 2\langle v,\ Jx_k - Ju_k\rangle \le ||x_k||^2 - ||u_k||^2\}.$$

This implies that C_{k+1} is closed and convex. The desired conclusion is proved.

Step (II): We prove that $\{x_n\}$, $\{S_i x_n\}_{n=0}^{\infty}$ for all $i \ge 1$ and $\{y_n\}$ are bounded sequences in C.

By the definition of C_n, we have $x_n = \Pi_{C_n} x_0$ for all $n \ge 0$. It follows from Lemma 2.1 (1) that

$$\begin{aligned}\phi(x_n, x_0) = \phi(\Pi_{C_n} x_0, x_0) &\le \phi(u, x_0) - \phi(u, \Pi_{C_n} x_0)\\ &\le \phi(u, x_0), \quad \forall n \ge 0,\ u \in G.\end{aligned} \tag{3.5}$$

This implies that $\{\phi(x_n, x_0)\}$ is bounded. By virtue of (2.2), $\{x_n\}$ is bounded. Since $\phi(u, S_i x_n) \le \phi(u, x_n)$ for all $u \in G$ and $i \ge 1$, $\{S_i x_n\}$ is bounded for all $i \ge 1$ and so $\{y_n\}$ is bounded in C. Denote M by

$$M = \sup_{n\ge 0,\, i \ge 1} \{||x_n||,\ ||S_i x_n||,\ ||y_n||\} < \infty.$$

Step (III): Next, we prove that $\Gamma := \bigcap_{i=1}^{\infty} F(S_i) \bigcap \Omega \subset C_n$ for all $n \ge 0$.

Indeed, it is obvious that $\Gamma \subset C_0 = C$. Suppose that $\Gamma \subset C_k$ for some $k \in N$. Since $u_k = K_{r_k} y_k$, by Lemma 2.5 and Remark 2.3, K_{r_k} is quasi-ϕ- nonexpansive. Hence, for any given $u \in \Gamma \subset C_k$ and any positive integers m, j with $m < j$, it follows from Lemma 2.7 that

$$\begin{aligned}&\phi(u, u_k) = \phi(u, K_{r_k} y_k)\\
&\le \phi(u, y_k) = \phi\Big(u, J^{-1}\Big(\sum_{i=0}^{\infty} \alpha_{ki} J S_i x_k\Big)\Big)\\
&= ||u||^2 - \sum_{i=0}^{\infty} \alpha_{ki} 2\langle u,\ JS_i x_k\rangle + \Big\|\sum_{i=0}^{\infty} \alpha_{ki} J S_i x_k\Big\|^2\\
&\le ||u||^2 - \sum_{i=0}^{\infty} \alpha_{ki} 2\langle u,\ JS_i x_k\rangle + \sum_{i=0}^{\infty} \alpha_{ki} ||JS_i x_k||^2\\
&\quad - \alpha_{km}\alpha_{kj} g(||JS_m x_k - JS_j x_k||)\\
&\le ||u||^2 - \sum_{i=0}^{\infty} \alpha_{ki} 2\langle u,\ JS_i x_k\rangle + \sum_{i=0}^{\infty} \alpha_{ki} ||S_i x_k||^2\\
&\quad - \alpha_{km}\alpha_{kj} g(||JS_m x_k - JS_j x_k||)\\
&= \sum_{i=0}^{\infty} \alpha_{ki}\{||u||^2 - 2\langle u,\ JS_i x_k\rangle + ||S_i x_k||^2\} - \alpha_{km}\alpha_{kj} g(||JS_m x_k - JS_j x_k||)\\
&= \sum_{i=0}^{\infty} \alpha_{ki} \phi(u, S_i x_k) - \alpha_{km}\alpha_{kj} g(||JS_m x_k - JS_j x_k||)\\
&\le \phi(u, x_k) - \alpha_{km}\alpha_{kj} g(||JS_m x_k - JS_j x_k||).\end{aligned} \tag{3.6}$$

This implies that $u \in C_{k+1}$ and so $\Gamma \subset C_n$ for all $n \ge 0$.

Step (IV): Now, we prove $\{x_n\}$ is a Cauchy sequence and

$$||x_n - S_i x_n|| \to 0, \quad \forall i \ge 1.$$

Since $x_{n+1} = \Pi_{C_{n+1}}x_0 \in C_n$ and $x_n = \Pi_{C_n}x_0$, from the definition of Π_{C_n}, we have

$$\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0), \quad \forall n \geq 0.$$

Therefore, $\{\phi(x_n, x_0)\}$ is nondecreasing and bounded and so $\lim_{n\to\infty} \phi(x_n, x_0)$ exists. From Lemma 2.1(1), for any given $m \geq 1$, we have

$$\begin{aligned}\phi(x_{n+m}, x_n) &= \phi(x_{n+m}, \Pi_{C_n}x_0) \leq \phi(x_{n+m}, x_0) - \phi(\Pi_{C_n}x_0, x_0)\\ &= \phi(x_{n+m}, x_0) - \phi(x_n, x_0), \quad \forall n \geq 0.\end{aligned}$$

This implies that

$$\lim_{n\to\infty} \phi(x_{n+m}, x_n) = 0, \quad \forall m \geq 1.$$

By Lemma 2.2, it follows that

$$\lim_{n\to\infty} ||x_{n+m} - x_n|| = 0, \quad \forall m \geq 1, \tag{3.7}$$

which implies that $\{x_n\}$ is a Cauchy sequence in C. Without loss of generality, we can assume that

$$\lim_{n\to\infty} x_n = p \in C. \tag{3.8}$$

Since $x_{n+1} = \Pi_{C_{n+1}}x_0 \in C_n$, it follows from the definition of C_{n+1} that

$$\phi(x_{n+1}, u_n) \leq \phi(x_{n+1}, x_n), \quad \forall n \geq 0. \tag{3.9}$$

Since E is uniformly smooth and uniformly convex, it follows from (3.7)-(3.9) and Lemma 2.2 that

$$\lim_{n\to\infty} ||x_{n+1} - x_n|| = \lim_{n\to\infty} ||x_{n+1} - u_n|| = \lim_{n\to\infty} ||x_n - u_n|| = 0, \tag{3.10}$$

Taking $m = 0$ and $j = 1, 2, \cdots$ in (3.6), for any $u \in \Gamma$, we have

$$\phi(u, u_n) \leq \phi(u, x_n) - \alpha_{n0}\alpha_{nj}g(||Jx_n - JS_jx_n||), \quad \forall n \geq 0,$$

i.e.,

$$\alpha_{n0}\alpha_{nj}g(||Jx_n - JS_jx_n||) \leq \phi(u, x_n) - \phi(u, u_n). \tag{3.11}$$

Since we have

$$\begin{aligned}\phi(u, x_n) - \phi(u, u_n) &= ||x_n||^2 - ||u_n||^2 - 2\langle u,\ Jx_n - Ju_n\rangle\\ &\leq |||x_n||^2 - ||u_n||^2| + 2||u|| \cdot ||Jx_n - Ju_n|| \\ &\leq ||x_n - u_n||(||x_n|| + ||u_n||) + 2||u|| \cdot ||Jx_n - Ju_n||,\end{aligned} \tag{3.12}$$

it follows from (3.10) that $\phi(u, x_n) - \phi(u, u_n) \to 0$ as $n \to \infty$. Hence, from (3.11) and the condition (b), it follows that $g(||Jx_n - JS_jx_n||) \to 0$ as $n \to \infty$. Since g is continuous and strictly increasing with $g(0) = 0$ and J is uniformly continuous on any bounded subset of E, we have

$$||x_n - S_jx_n|| \to 0 \ (n \to \infty), \ \forall j \geq 1. \tag{3.13}$$

Thus the conclusion (IV) is proved.

Step (V): Now, we prove that $p \in \Gamma := \bigcap_{i=1}^{\infty} F(S_i) \bigcap \Omega$.

First, we prove that $p \in \bigcap_{i=1}^{\infty} F(S_i)$. In fact, by the assumption that, for each $j = 1, 2, \cdots$, S_j is closed, it follows from (3.13) and (3.8) that $p = S_jp$ for all $j \geq 1$, i.e., $p \in \bigcap_{i=1}^{\infty} F(S_i)$.

Next, we prove that $p \in \Omega$. In fact, since $x_n \to p$, it follows from (3.10) that $u_n \to p$. Again, since $u_n = K_{r_n} y_n$, it follows from (2.7), (3.5) and (3.12) that

$$\begin{aligned} \phi(u_n, y_n) &= \phi(K_{r_n} y_n, y_n) \leq \phi(u, y_n) - \phi(u, K_{r_n} y_n) \\ &\leq \phi(u, x_n) - \phi(u, K_{r_n} y_n) \\ &= \phi(u, x_n) - \phi(u, u_n) \to 0 \quad (n \to \infty). \end{aligned} \tag{3.14}$$

This implies that $||u_n - y_n|| \to 0$ and so $\lim_{n\to\infty} ||Ju_n - Jy_n|| = 0$. By the assumption that $r_n \geq a$ for all $n \geq 0$, we have

$$\lim_{n\to\infty} \frac{||Ju_n - Jy_n||}{r_n} = 0. \tag{3.15}$$

Again, since we have

$$H(u_n, y) + \frac{1}{r_n}\langle y - u_n,\ Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C,$$

by the condition A_1, it follows that

$$\frac{1}{r_n}\langle y - u_n,\ Ju_n - Jy_n\rangle \geq -H(u_n, y) \geq H(y, u_n), \quad \forall y \in C. \tag{3.16}$$

By the assumption that $y \mapsto H(x, y)$ is convex and lower semi-continuous, letting $n \to \infty$ in (3.16), it follows from (3.15) and (3.16) that

$$H(y, p) \leq 0, \quad \forall y \in C.$$

For any $t \in (0, 1]$ and $y \in C$, letting $y_t = ty + (1-t)p$, then $y_t \in C$ and $H(y_t, p) \leq 0$. By the condition (A_1) and (A_4), we have

$$0 = H(y_t, y_t) \leq tH(y_t, y) + (1-t)H(y_t, p) \leq tH(y_t, y).$$

Dividing by t, we have $H(y_t, y) \geq 0$ for all $y \in C$. Letting $t \downarrow 0$, it follows from the condition (A_3) that $H(p, y) \geq 0$ for all $y \in C$, i.e., $p \in \Omega$ and so

$$p \in \Gamma = \bigcap_{i=0}^{\infty} F(S_i) \bigcap \Omega.$$

Step (VI): Now, we prove that $x_n \to \Pi_\Gamma x_0$.

Let $w = \Pi_\Gamma x_0$. From $w \in \Gamma \subset C_{n+1}$ and $x_{n+1} = \Pi_{C_{n+1}} x_0$, we have

$$\phi(x_{n+1}, x_0) \leq \phi(w, x_0), \quad \forall n \geq 0.$$

This implies that

$$\phi(p, x_0) = \lim_{n\to\infty} \phi(x_n, x_0) \leq \phi(w, x_0). \tag{3.17}$$

By the definition of $\Pi_\Gamma x_0$ and (3.17), we have $p = w$. Therefore, it follows that $x_n \to \Pi_\Gamma x_0$. This completes the proof.

The following cCrollaries can be obtained from Theorem 3.1 immediately:

Crollary 3.2. Let E, C, ψ, G, $\{S_i\}$, $\{\alpha_{ni}\}$ and $\{r_n\}$ be the same as in Theorem 3.1. If

$$\bigcap_{i=1}^{\infty} F(S_i) \bigcap MEP(G) \neq \emptyset,$$

where $MEP(G)$ is the set of solutions of the mixed equilibrium problem (1.2), and $\{x_n\}$ is the sequence generated by

$$\begin{cases} x_0 \in C, \quad C_0 = C, \\ y_n = J^{-1}(\alpha_{n0} J x_n + \sum_{i=1}^{\infty} \alpha_{ni} J S_i x_n), \\ u_n \in C \text{ such that} \\ G(u_n, y) + \psi(y) - \psi(u_n) + \frac{1}{r_n}\langle y - u_n, \ J u_n - J y_n \rangle \geq 0, \ \ \forall y \in C, \\ C_{n+1} = \{v \in C_n : \phi(v, u_n) \leq \phi(v, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \ \ \forall n \geq 0, \end{cases} \tag{3.18}$$

then $\{x_n\}$ converges strongly to $\Pi_{\bigcap_{i=1}^{\infty} F(S_i) \bigcap MEP(G)} x_0$

Proof. Putting $A = 0$ in Theorem 3.1, then $\Omega = MEP(G)$. Hence the conclusion of Corollary 3.2 is obtained from Theorem 3.1, immediately.

Corollary 3.3. Let E, C, ψ, A, $\{S_i\}$, $\{\alpha_{ni}\}$ and $\{r_n\}$ be the same as in Theorem 3.1. If

$$\bigcap_{i=0}^{\infty} F(S_i) \bigcap VI(C, A, \psi) \neq \emptyset,$$

where $VI(C, A, \psi)$ is the set of solutions of the mixed variational inequality of Browder type (1.3), and $\{x_n\}$ is the sequence generated by

$$\begin{cases} x_0 \in C, \quad C_0 = C, \\ y_n = J^{-1}(\alpha_{n0} J x_n + \sum_{i=1}^{\infty} \alpha_{ni} J S_i x_n), \\ u_n \in C \text{ such that} \\ \langle A u_n, \ y - u_n \rangle + \psi(y) - \psi(u_n) + \frac{1}{r_n}\langle y - u_n, \ J u_n - J y_n \rangle \geq 0, \ \ \forall y \in C, \\ C_{n+1} = \{v \in C_n : \phi(v, u_n) \leq \phi(v, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \ \ \forall n \geq 0, \end{cases} \tag{3.19}$$

then $\{x_n\}$ converges strongly to $\Pi_{\bigcap_{i=1}^{\infty} F(S_i) \bigcap VI(C,A,\psi)} x_0$.

Proof. Putting $G = 0$ in Theorem 3.1, then $\Omega = VI(C, A, \psi)$. Hence the conclusion of Corollary 3.3 is obtained from Theorem 3.1.

Corollary 3.4. Let E, C, G, ψ, A, $\{\alpha_{ni}\}$ and $\{r_n\}$ be the same as in Theorem 3.1. If $\Omega \neq \emptyset$, where Ω is the set of solutions of the generalized mixed equilibrium problem (1.1), and $\{x_n\}$ is the sequence generated by

$$\begin{cases} x_0 \in C, \quad C_0 = C, \\ u_n \in C \text{ such that} \\ G(u_n, y) + \langle Au_n,\ y - u_n\rangle + \psi(y) - \psi(u_n) + \dfrac{1}{r_n}\langle y - u_n,\ Ju_n - Jx_n\rangle \ge 0, \quad \forall y \in C, \\ C_{n+1} = \{v \in C_n : \phi(v, u_n) \le \phi(v, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \ge 0, \end{cases} \tag{3.20}$$

then $\{x_n\}$ converges strongly to $\Pi_\Omega x_0$.

Proof. Taking $S_i = I$ for all $i \ge 1$ in Theorem 3.1, we have $y_n = x_n$. Hence the conclusion is obtained.

Corollary 3.5. Let E, C, $\{S_i\}$ and $\{\alpha_{ni}\}$ be the same as in Theorem 3.1. If

$$\bigcap_{i=1}^{\infty} F(S_i) \neq \emptyset$$

and $\{x_n\}$ is the sequence generated by

$$\begin{cases} x_0 \in C, \quad C_0 = C, \\ y_n = J^{-1}\Big(\alpha_{n0} Jx_n + \sum_{i=1}^{\infty} \alpha_{ni} JS_i x_n\Big), \\ u_n = \Pi_C y_n \\ C_{n+1} = \{v \in C_n : \phi(v, u_n) \le \phi(v, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall\, n \ge 0, \end{cases} \tag{3.21}$$

then $\{x_n\}$ converges strongly to $\Pi_{\bigcap_{i=1}^{\infty} F(S_i)} x_0$

Proof. Taking $G = A = \psi = 0$ and $r_n = 1$ for all $n \ge 0$ in Theorem 3.1, then $u_n = \Pi_C y_n$. Therefore, the conclusion of Corollary 3.5 is obtained from Theorem 3.1.

4. Applications to Optimization Problems

In this section, we will utilize the results presented in Section 3 to study the following *Optimization problem (OP)*:

$$\min_{x \in C} (h(x) + \psi(x)), \tag{4.1}$$

where C is a nonempty closed convex subset of a Hilbert space H and $h,\ \psi : C \to R$ are two convex and lower semi-continuous functionals. Denote by $Sol(OP) \subset C$ the set of solutions of the problem (4.1). It is easy to see that $Sol(OP)$ is a closed convex subset in C. Let $G : C \times C \to R$ be a bifunction defined by $G(x, y) = h(y) - h(x)$. Then we can consider the following mixed equilibrium problem:

Find $x^* \in C$ such that

$$G(x^*, y) + \psi(y) - \psi(x^*) \ge 0, \quad \forall y \in C. \tag{4.2}$$

It is easy to see that G satisfies the conditions (A_1)-(A_4) in Section 1 and $MEP(G) = Sol(OP)$, where $MEP(G)$ is the set of solutions of the mixed equilibrium problem (4.2). Let $\{x_n\}$ be the iterative sequence generated by

$$\begin{cases} x_0 \in C, \quad C_0 = C, \\ u_n \in C \text{ such that} \\ G(u_n, y) + \psi(y) - \psi(u_n) + \dfrac{1}{r_n}\langle y - u_n,\ u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{v \in C_n : ||v - u_n|| \leq ||v - x_n||\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad \forall n \geq 0, \end{cases} \tag{4.3}$$

where P_C is the projection operator from H onto C and $\{r_n\}$ is a sequence in $[a, \infty)$ for some $a > 0$. Then $\{x_n\}$ converges strongly to $P_K x_0$.

In fact, Taking $A = 0$ and $S_i = I$ for all $i = 1, 2, \cdots, N$ in Corollary 3.2, then we have $x_n = y_n$. Since H is a Hilbert space, it follows that $J = I$, $\phi(x, y) = ||x - y||^2$ and $\Pi_{C_{n+1}} = P_{C_{n+1}}$, where $P_{C_{n+1}}$ is the projection of H onto C_{n+1}. Thus the desired conclusion can be obtained from Corollary 3.2, immediately.

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