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Destabilization/Stabilization of Diffusion Systems by Diffusion and Boundary Flux

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1. PROBLEM AND BACKGROUND

In this section, we state the problem to be investigated and give some background motivations.

1.1. **Problem.** Let $\Omega \subset \mathbb{R}^m$ $(m \geq 1)$ be a bounded domain with smooth boundary $\partial\Omega$. We consider the following system of diffusion equations under the *Robin type boundary conditions* for $\boldsymbol{u}(t,x) = (u_1(t,x), \ldots, u_N(t,x))$;

$$\partial_t \boldsymbol{u} = D\triangle \boldsymbol{u} \quad \text{in } \Omega, \qquad D\partial_{\boldsymbol{n}}\boldsymbol{u} = J\boldsymbol{u} \quad \text{on } \partial\Omega, \tag{1}$$

where

- $\partial_t = \partial/\partial t$ is the partial derivative with respect to *time* t;
- $\triangle$ is the m-dimensional Laplace operator;
- $D = \mathrm{diag}(d_1, \ldots, d_N)$ is a diagonal *diffusion matrix* with $d_j > 0$;
- $\boldsymbol{n}$ is the unit outward normal vector field on $\partial\Omega$;
- $\partial_{\boldsymbol{n}} = \partial/\partial\boldsymbol{n}$ is the derivative along $\boldsymbol{n}$ on the boundary;
- $D\partial_{\boldsymbol{n}}\boldsymbol{u}$ stands for (the vector of) fluxes at the boundary, and
- J is an $N \times N$ *real* matrix called the *mass transfer matrix.*

We emphasize that for *diagonal* J the system (1) completely decouples. This means that (1) reduces to N sets of independent scalar problems;

$$\partial_t u_i = d_i \triangle u_i \ \text{ in } \Omega \ \text{ and } \ d_i \partial_{\boldsymbol{n}} u_i = J_{ii} u_i \ \text{ on } \partial\Omega \quad (i = 1, 2, \ldots, N).$$

Therefore, interesting dynamic behavior of the solutions of (1), due to its multicomponent nature, will arise only if J is *not diagonal.*

The purpose of this article is to determine stability/instability of the trivial solution $\boldsymbol{u} \equiv \mathbf{0} \in \mathbb{R}^N$ of (1), in terms of diffusion and mass transfer matrices D and J. We also pay attention to the role the geometry of Ω plays. The relevant eigenvalue problem for our purpose is

$$\lambda \boldsymbol{\phi} = D \triangle \boldsymbol{\phi} \quad \text{in } \Omega, \qquad D \partial_{\boldsymbol{n}} \boldsymbol{\phi} = J \boldsymbol{\phi} \quad \text{on } \partial\Omega. \tag{2}$$

The complex number $\lambda \in \mathbb{C}$ is called an *eigenvalue* when (2) has a nontrivial solution $\boldsymbol{\phi} \not\equiv \mathbf{0}$. The eigenvalue problem (2) is obtained from (1) by substituting the ansatz $\boldsymbol{u}(t,x) = \mathrm{e}^{\lambda t} \boldsymbol{\phi}(x)$. Therefore, if the eigenvalues of (2) have negative real part then the trivial solution of (1) is *asymptotically stable.* If, on the other hand, (2) has an eigenvalue with $\operatorname{Re}\lambda > 0$ then the system (1) is *unstable.*

1.2. **Background Motivations.** The first motivation to study (1) dates back to the period from late 1960's to early 1970's (see, [4, 7] and reference therein). In chemical engineering community around that period, the dynamics of inert materials diffusing in a container Ω whose boundary $\partial\Omega$ is the site of catalytic reactions was of great interest. Such a chemical setting is modeled by

$$\partial_t \boldsymbol{u} = D \triangle \boldsymbol{u} \quad \text{in } \Omega, \qquad D \partial_{\boldsymbol{n}} \boldsymbol{u} = \boldsymbol{f}(\boldsymbol{u}) \quad \text{on } \partial\Omega, \tag{3}$$

in which the *flux* $D\nabla \boldsymbol{u}$ on the boundary is given by a nonlinear mapping $\boldsymbol{f} : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ modeling the *mass transfer mechanisms.* For the well-posedness of (3), we refer to [1, 5]. The reaction terms in the bulk are absent in this model, because the materials are non-reactive without catalysts. It is easy to see:

- If $\boldsymbol{f}(\boldsymbol{u}^*) = \mathbf{0}$, then $\boldsymbol{u}(x) \equiv \boldsymbol{u}^*$ is a uniform steady solution of (3) for any diffusion matrix $D > 0$.
- The linearization of (3) around $\boldsymbol{u} = \boldsymbol{u}^*$ gives rise to (1) in which $J = \partial_{\boldsymbol{u}} \boldsymbol{f}(\boldsymbol{u}^*)$.

The experimental community was interested in the stability properties of the permanent concentration profiles (steady states) produced by (3), and investigations of the dynamic behavior of (3), posed in 1-dimensional intervals with $N = 1$, flourished in 1970's (see, [4, 7] and reference therein).

Another motivation comes from models for biological cell activity.

For example, in an attempt to explain observations of oscillatory behavior in biological cell activity, Levine and Rappel [15] proposed a mathematical model, in which nonlinear chemical reactions take place on the boundary of domain and reagents transport in the bulk dominantly by diffusion and decay at certain rates. When relevant parameters are chosen appropriately, they have shown numerically that oscillatory and non-oscillatory spatially inhomogeneous modes destabilize even when diffusion rates of reagents are equal. Equal diffusivity or near equal diffusivity is important in molecular biological systems such as cells.

To be specific, the model in [15] is a two-component hypothetical reaction-diffusion system where the reactions take place on the bounding surface of some bulk region (presumably representing a biological cell):

$$\begin{aligned} \dot{u}_m &= -r_d u_m + r_a u + a(u_m^2 v_m - u_m), \\ \dot{v}_m &= -p_d v_m + p_a v + 1 - u_m^2 v_m. \end{aligned} \tag{4}$$

In (4), (u, v) and (u_m, v_m), respectively, stand for the concentrations of bulk species and surface-resident ones, and the first two terms on the right hand sides represent the exchange process between the bulk and surface-resident species, while the remaining terms are nonlinear chemical interactions on the bounding surface. The equation in the bulk (with no decay) is given by

$$\dot{u} = D_u \nabla^2 u, \qquad \dot{v} = D_v \nabla^2 v \tag{5}$$

supplemented by the boundary conditions for the normal derivatives

$$\begin{aligned} D_u \partial_{\boldsymbol{n}} u &= r_d u_m - r_a u, \\ D_v \partial_{\boldsymbol{n}} v &= p_d v_m - p_a v. \end{aligned} \tag{6}$$

Substituting (6) into (4), and neglecting the distinction between (u, v) and (u_m, v_m), we obtain, together with (5), the *linear diffusion system under dynamic boundary conditions*:

$$\left\{ \begin{array}{l} \partial_t u = D_u \Delta u \\ \partial_t v = D_v \Delta v \end{array} \right. \text{ in } \Omega, \quad \left\{ \begin{array}{l} \partial_t u + D_u \partial_{\boldsymbol{n}} u = f(u, v) \\ \partial_t v + D_v \partial_{\boldsymbol{n}} v = g(u, v) \end{array} \right. \text{ on } \partial\Omega. \tag{7}$$

Levine and Rappel [15] also developed a numerical scheme based on the phase-field method to investigate problem (4) - (6). With specific parameter values for r_d, r_a, p_d, p_a and a, they numerically solved the initial value problem for the model and found out that oscillatory patterns of Turing-type emerge even under equal diffusion case $D_u = D_v > 0$. One of the purposes of this article is to explore a possible mathematical explanation of such events which may be reckoned *unusual* from the viewpoint of the conventional Turing-instability mechanism.

Thus, we have been led to the following general model which incorporates *diffusion, bulk reaction* and *non-linear fluxes on the boundary,* together with *dynamic boundary conditions.*

$$\partial_t \boldsymbol{u} = D\triangle \boldsymbol{u} + \boldsymbol{g}(\boldsymbol{u},\boldsymbol{p}) \text{ in } \Omega, \quad B\partial_t \boldsymbol{u} + C\partial_{\boldsymbol{n}} \boldsymbol{u} = \boldsymbol{f}(\boldsymbol{u},\boldsymbol{p}) \text{ on } \partial\Omega$$

whose well-posedness we again refer to [1, 5].

In this article, however, we focus our attention to the spacial cace (3) where $\boldsymbol{g} \equiv \boldsymbol{0}, B \equiv 0$ and $C = D$.

2. Sufficient Conditions for Stability and Instability

In this section, sufficient conditions for the stability and instability of the system (1) are given. There are two situations where the analysis is rather elementary. These are presented in §2.1 and §2.2.

2.1. Variational Case.

When the mass transfer matrix is symmetric, the analysis simplifies substantially. In this situation, the eigenvalue problem (2) turns out to be *variational.*

Theorem 2.1. *Assume that J is a* real symmetric *matrix.*

(i) *For* any *positive diffusion matrix D, the eigenvalues of* (2) *are* real, *and the corresponding eigenfunctions are* real-valued.
(ii) *If the eigenvalues of J are* negative, *then the eigenvalues of* (2) *are* negative *for* any *positive diffusion matrix D.*
(iii) *If J has k* positive *eigenvalues and $N - k$* negative *eigenvalues (counting with multiplicity), then* (2) *has at least k* positive *eigenvalues (counting with multiplicity) for* any *positive diffusion matrix D. Moreover, there exists a positive constant $d^*(J,\Omega) > 0$, which depends only on J and Ω, so that* (2) *has exactly k positive eigenvalues for* any *diffusion matrix satisfying* $\min\{d_1,\ldots,d_N\} > d^*(J,\Omega)$.

Theorem 2.1 says that for a symmetric mass transfer matrix J the stability of the trivial solution $\boldsymbol{0}$ of (1) is completely determined by the eigenvalues of J. Namely,

the system (1) with symmetric J is stable (unstable)
if and only if
J is stable (unstable).

In this case, it turns out that the eigenvalue problem (2) is variational (cf. §2.4 below), and the eigenvalues of (2) behaves similarly to those for the reaction-diffusion system with the natural boundary value conditions:

$$\lambda\boldsymbol{\phi} = D\triangle\boldsymbol{\phi} + J\boldsymbol{\phi} \text{ in } \Omega, \quad D\partial_{\boldsymbol{n}}\boldsymbol{\phi} = \boldsymbol{0} \text{ on } \partial\Omega$$

whose eigenvalues are characterized in the variational manner by the functional

$$\mathfrak{G}(\phi) := \int_{\Omega} \phi(x) \cdot J\phi(x)\,\mathrm{d}x - \int_{\Omega} \sum_{k=1}^{N} d_k |\nabla \phi_k(x)|\,\mathrm{d}x.$$

If 0 is an eigenvalue of J with multiplicity k, then we can show that 0 is an eigenvalue of (2) with multiplicity at least k for any diagonal diffusion matrix D, which is true even for non-symmetric J (see Theorem 2.3 (i), below). However, there exist added multiplicities of the 0-eigenvalue for particular diagonal diffusion matrices D determined by J and Ω, to be explained below in Theorem 2.3 (cf. §2.3 below). This fact is related to the so called *Turing-type* destabilization for (1), to be elucidated in §3. We refer to [25, 19, 17, 16, 18, 11, 12, 3, 20, 21, 22, 23, 26, 24, 27, 28] for Turing destabilization for multicomponent reaction-diffusion systems under homogeneous Neumann boundary conditions.

2.2. **Non-Variational Case.** Another situation in which analysis goes rather easy is the case where the diffusion rates are all equal.

Theorem 2.2. *There exists a piecewise smooth curve in the complex ζ-plane, represented by $\operatorname{Re}\zeta = \mathcal{C}(\operatorname{Im}\zeta)$ such that $\mathcal{C}$ is even, satisfies*

$$\mathcal{C}(0) = 0, \quad \mathcal{C}'(0) = 0 \ \textit{ and } \ \mathcal{C}(s) > 0 \quad (s \neq 0)$$

and depends only on Ω. We then have the following:

Suppose that the diffusion rates are all equal; $D = d\mathbb{I}_N$, $d > 0$.

(a) *If all eigenvalues α of J satisfy $\operatorname{Re}\alpha < d\mathcal{C}(\operatorname{Im}\alpha/d)$, then all eigenvalues λ of (2) satisfy $\operatorname{Re}\lambda < 0$.*
(b) *If there exists an eigenvalue α of J that satisfies $\operatorname{Re}\alpha > d\mathcal{C}(\operatorname{Im}\alpha/d)$, then there exists an eigenvalue λ of (2) such that $\operatorname{Re}\lambda > 0$.*
(c) *If an eigenvalue α of J satisfies $\operatorname{Re}\alpha = d\mathcal{C}(\operatorname{Im}\alpha/d)$, then there exists an eigenvalue λ of (2) such that $\operatorname{Re}\lambda = 0$.*

Let us exhibit an example of the curve $\operatorname{Re}\zeta = \mathcal{C}(\operatorname{Im}\zeta)$ in the simplest case $\Omega = (-1, +1)$. In this case the function $\mathcal{C}(s)$ is defined by $\mathcal{C}(s) = \min\{\mathcal{C}_0(s), \mathcal{C}_1(s)\}$ where $\mathcal{C}_0, \mathcal{C}_1$ have the following parametric representations for $\operatorname{Im}\alpha \geq 0$.

$$\mathcal{C}_0 : \operatorname{Re}\zeta = \frac{\sqrt{2\tau}}{2}\,\frac{\sinh\sqrt{2\tau} - \sin\sqrt{2\tau}}{\cosh\sqrt{2\tau} + \cos\sqrt{2\tau}}, \quad \operatorname{Im}\zeta = \frac{\sqrt{2\tau}}{2}\,\frac{\sinh\sqrt{2\tau} + \sin\sqrt{2\tau}}{\cosh\sqrt{2\tau} + \cos\sqrt{2\tau}},$$

$$\mathcal{C}_1 : \operatorname{Re}\zeta = \frac{\sqrt{2\tau}}{2}\,\frac{\sinh\sqrt{2\tau} + \sin\sqrt{2\tau}}{\cosh\sqrt{2\tau} - \cos\sqrt{2\tau}}, \quad \operatorname{Im}\zeta = \frac{\sqrt{2\tau}}{2}\,\frac{\sinh\sqrt{2\tau} - \sin\sqrt{2\tau}}{\cosh\sqrt{2\tau} - \cos\sqrt{2\tau}},$$

where $\tau \geq 0$ (note that the curve is symmetric with respect to the real axis). These curves are depicted in Figure 1 (below), and one can also show that

$$\mathcal{C}_0(s) = \frac{1}{3}s^2 + O(s^4),\ \mathcal{C}_1(s) = 1 + \frac{1}{5}s^2 + O(s^4) \ \text{ for } \ s \approx 0$$

We note that the functions $\mathcal{C}_0(s), \mathcal{C}_1(s)$ are smooth. However, the two curves defined by these functions intersects infinitely many times on the complex ζ-plane, and $\mathcal{C}$ is defined as the minimum of these functions. This is the reason why we say in Theorem 2.2 that $\mathcal{C}$ is *piecewise smooth.* The intersection of these first two curves seems to originate from that the boundary of $\Omega = (-1, +1)$ has more than one connected components.

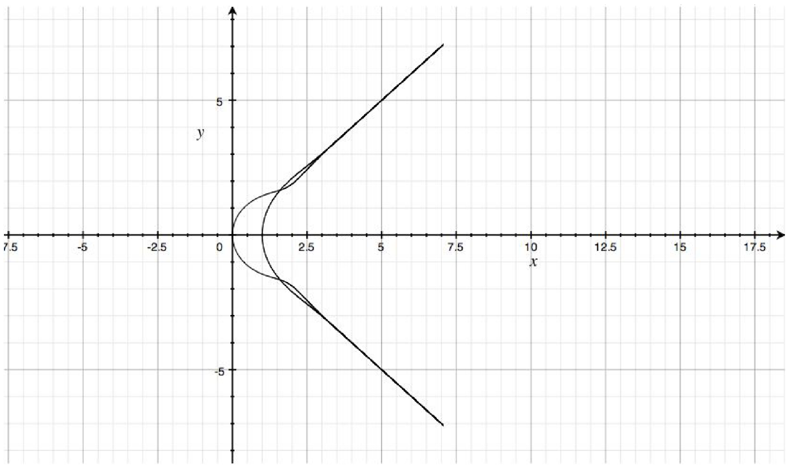

Figure 1 The Curves $\mathrm{Re}\,\zeta = \mathcal{C}_k(\mathrm{Im}\,\zeta)$ $(k = 0, 1)$ for $\Omega = (-1, 1)$.

We now make some comments on Theorem 2.2. When diffusion rates are equal, the stability property of (1) is summarize as follows.

- The trivial solution $\mathbf{0}$ is stable if the mass transfer matrix J is stable.
- However, the instability of J does not necessarily imply the instability of the trivial solution $\mathbf{0}$.
- Theorem 2.2 (a) says that the diffusion has a stabilizing effect to some extent, in the sense that the system (1) can be stable for *suitably unstable* J.
- In order for (1) to be unstable, the mass transfer matrix J has to be *sufficiently unstable* in the sense that one of the eigenvalues of J satisfies the condition in Theorem 2.2 (b).
- If $N \geq 2$ and eigenvalues of a family of mass transfer matrix $J = J_p$ cross *the critical curve* as the parameter p varies, then steady or oscillatory destabilizations of spatially heterogeneous

modes occur, which may help us to understand the origin of the numerical results obtained by Levine and Rappel [15].

If, on the other hand, we allow the diffusion rates $d_1, \dots, d_N$ to be different, then there exist stable mass transfer matrices for which (2) has eigenvalues with positive real part for suitable diffusion matrices D.

2.3. **Zero Eigenvalue.** To determine the stability or instability of (1), it is crucial to detect *critical eigenvalues* λ, i.e., those eigenvalues λ that satisfy $\operatorname{Re}\lambda = 0$, of (2). Thanks to Theorem 2.2, we are able, at least in abstract manners, to determine such critical eigenvalues when diffusion rates are all equal. In this subsection, we provide necessary and sufficient conditions for the existence of 0-eigenvalue, a special type of critical eigenvalues, in terms of the general diagonal diffusion matrix D, the mass transfer matrix J and the *Steklov eigenvalues* of the Laplacian on Ω.

Theorem 2.3 (Zero Eigenvalue $\lambda = 0$). *There exist positive constants $\nu_k > 0$ $(k \in \mathbb{N})$, depending on the domain Ω, and $\nu_0 = 0$ with*

$$\nu_0 = 0 < \nu_1 \le \nu_2 \le \dots \le \nu_k \le \dots, \quad \lim_{k\to\infty} \nu_k = \infty, \quad \text{if } m \ge 2,$$
$$\nu_0 = 0 < \nu_1 = \frac{2}{|\Omega|} < \infty, \quad \text{if } m = 1 \text{ and } \Omega \text{ is an interval},$$

such that $\lambda = 0$ is an eigenvalue of (2) if and only if

$$\det\bigl(\nu_k D - J\bigr) = 0 \quad \text{for some } k \in \mathbb{N} \cup \{0\}. \tag{8}$$

More precisely, with the notation

$$\Gamma_k := \{(d_1, \dots, d_N) \in (\mathbb{R}_{>0})^N \mid \det(\nu_k D - J) = 0,\ D = \operatorname{diag}(d_1, \dots, d_N)\},$$

the following properties hold.

(i) *If 0 is an eigenvalue of J with geometric multiplicity $\ell \ge 1$ and the corresponding eigenvectors being $\boldsymbol{e}_i$ $(i = 1, \dots, \ell)$, then for any diffusion matrix D, $\lambda = 0$ is an eigenvalue of (2) with multiplicity at least ℓ, and $\phi(x) \equiv \boldsymbol{e}_i$ $(i = 1, \dots, \ell)$ are the corresponding 0-eigenfunctions.*

(ii) *If the diffusion matrix satisfies $D \in \Gamma_k \neq \emptyset$ for some $k \in \mathbb{N}$, then $\lambda = 0$ is an eigenvalue of (2) and the corresponding eigenfunctions are of the form $v(x)\boldsymbol{\psi}$, where $\boldsymbol{\psi} \neq \mathbf{0}$ is a 0-eigenvector of $\nu_k D - J$ and $v(x)$ is a non-constant scalar function defined on Ω.*

(iii) *Assume that* $\Gamma_k \neq \emptyset$ *for some* $k \in \mathbb{N}$ *and that* $\lambda = 0$ *is a simple eigenvalue of* (2) *at a regular point* $(d_1^0, \dots, d_N^0) \in \Gamma_k$. *As* $D = \mathrm{diag}(d_1, \dots, d_N)$ *transversely crosses* Γ_k *at* $(d_1^0, \dots, d_N^0)$, *an eigenvalue* λ *of* (2) *transversely crosses* $\lambda = 0$ *along the real axis.*

Some comments on Theorem 2.3 now follow.

- The constants ν_k in Theorem 3 are actually the *Steklov eigenvalues* for the Laplacian on Ω.
- There are infinitely many Steklov eigenvalues and $\nu_k \to \infty$ as $k \to \infty$ for $m \geq 2$.
- On the other hand, when $m = 1$ and Ω is an interval, there are only two Steklov eigenvalues $0 = \nu_0 < \nu_1 = 2/|\Omega|$.
- The scalar function $v(x)$ in Theorem 2.3 (ii) is the harmonic extension of the *Steklov eigenfunction* ρ_k corresponding to the Steklov eigenvalue ν_k, i.e., $0 = \Delta v$ in Ω and $v = \rho_k$ on $\partial\Omega$. We denote the unique solution of this problem by $\hat{\rho}_k(x)$.
- It is also easy to verify the equality $\nu_k \Gamma_k = \nu_j \Gamma_j$ between the two sets for all pairs $j, k \in \mathbb{N}$.
- Therefore, if Γ_j is nonempty for some $j \in \mathbb{N}$ then Γ_k is also nonempty for all $k \in \mathbb{N}$.
- Theorem 2.3 (iii) implies that as the diagonal diffusion matrix crosses one of Γ_k $(k \in \mathbb{N})$, a steady destabilization of Turing type occurs in (1).

For our purpose in what follows, it is convenient to make a definition concerning matrix stability (see, [13, 10]).

Definition 2.1. (i) A square matrix J is said to be *strongly stable* if $J - D$ is stable for any nonnegative diagonal matrix D.

(ii) A square matrix J is called *Turing-stable*, if it is stable and there exists a positive diagonal matrix D such that $\det(D - J) = 0$.

Clearly, a strongly stable matrix is stable, and for a strongly stable J the sets Γ_k $(k \in \mathbb{N})$ are empty. This means that for strongly stable J, steady destabilization of Turing type never occurs.

Now, for a Turing-stable J, the sets Γ_k $(k \in \mathbb{N})$ are nonempty, and Theorem 2.2 implies that the system (1) is stable for $d_1 = d_2 = \cdots = d_N > 0$. Theorem 2.3 (iii), on the other hand, says that as $(d_1, \dots, d_N)$ transversely crosses one of the critical hyper-surfaces Γ_k at a regular point for the first time starting from the diagonal $\{d_1 = d_2 \cdots = d_N > 0\}$, an eigenvalue of (2) crosses $\lambda = 0$ from negative to positive, which renders the system (1) unstable. This observation is useful in §3.

2.4. **Outline of Proof.** In this section, we give main ideas to prove Theorems 2.1, 2.2 and 2.3.

2.4.1. *Proof of Theorem 2.1.* Let ϕ be a non-trivial solution of (2). The equation and the boundary conditions in (2) imply

$$\begin{aligned} \operatorname{Re}\lambda \int_\Omega |\phi|^2 \mathrm{d}x &= \frac{1}{2}\int_{\partial\Omega} \overline{\phi}\cdot (J+J^{\mathrm{T}})\phi\,\mathrm{d}\sigma - \int_\Omega \sum_{k=1}^N d_k |\nabla\phi_k|^2 \mathrm{d}x, \\ \operatorname{Im}\lambda \int_\Omega |\phi|^2 \mathrm{d}x &= \frac{1}{2\mathrm{i}}\int_{\partial\Omega} \overline{\phi}\cdot (J-J^{\mathrm{T}})\phi\,\mathrm{d}\sigma. \end{aligned}$$

Note that eigenvalues and eigenfunctions are in general complex valued. When J is symmetric, from the second equation, we immediately conclude $\lambda \in \mathbb{R}$, and hence, the corresponding eigenfunction ϕ is $\mathbb{R}$-valued, proving Theorem 2.1 (i). The real eigenpair (λ, ϕ) satisfies the following.

$$\text{(R)} \qquad \lambda \int_\Omega |\phi|^2 \mathrm{d}x = \int_{\partial\Omega} \phi\cdot J\phi\,\mathrm{d}\sigma - \int_\Omega \sum_{k=1}^N d_k |\nabla\phi_k|^2 \mathrm{d}x.$$

If the eigenvalues of J are negative, then there exists a constant $C > 0$ such that $\phi \cdot J\phi \le -C|\phi|^2$ for $\phi \in \mathbb{R}^N$, and hence (R) implies that

$$\lambda \int_\Omega |\phi|^2 \mathrm{d}x \le -C \int_{\partial\Omega} |\phi|^2\,\mathrm{d}\sigma - \int_\Omega \sum_{k=1}^N d_k |\nabla\phi_k|^2 \mathrm{d}x < 0,$$

establishing (ii).

To prove (iii), we use the variational characterizations of the k-th eigenvalue λ_{k-1} and the $(k+1)$-th eigenvalue λ_k. To begin with, we use the following *max-min* characterization [9, 14] of λ_{k-1};

$$\lambda_{k-1} = \sup_{Y \in \mathcal{H}_k^N} \left(\inf\left\{\mathcal{R}^{(N)}(\phi) \;\middle|\; \phi \in Y,\ \|\phi\|_{L^2(\Omega)} = 1\right\}\right),$$

where $\mathcal{H}_k^N$ is the set of k-dimensional subspaces of $[H^1(\Omega)]^N$ and

$$\mathcal{R}^{(N)}(\phi) = \int_{\partial\Omega} \phi\cdot J\phi\,\mathrm{d}\sigma - \int_\Omega \sum_{k=1}^N d_k |\nabla\phi_k|^2 \mathrm{d}x.$$

Let us denote the eigenvalues and eigenvectors of J, respectively, by

$$\alpha_1 \ge \ldots \ge \alpha_k > 0 > \alpha_{k+1} \ge \ldots \ge \alpha_N \text{ and } \boldsymbol{e}_1, \ldots, \boldsymbol{e}_k, \boldsymbol{e}_{k+1}, \ldots, \boldsymbol{e}_N,$$

in which $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_k, \boldsymbol{e}_{k+1}, \ldots, \boldsymbol{e}_N\}$ forms an orthonormal basis of $\mathbb{R}^N$. We choose $Y = Y_{\mathrm{c}} := \mathbb{R}\boldsymbol{e}_1 \oplus \cdots \oplus \mathbb{R}\boldsymbol{e}_k \in \mathcal{H}_k^N$ in the characterization above, and obtain

$$\lambda_{k-1} \geq \inf\left\{\mathcal{R}^{(N)}(\boldsymbol{\phi}) \mid \boldsymbol{\phi} \in Y_{\mathrm{c}}, \|\boldsymbol{\phi}\|_{L^2(\Omega)} = 1\right\} \geq \alpha_k|\partial\Omega|/|\Omega| > 0,$$

establishing that (2) has at least k positive eigenvalues.

To prove that there exist exactly k positive eigenvalues for large diffusion rates, we use the following *min-max* characterization [9] of λ_k.

$$\lambda_k = \inf_{Y \in \mathcal{H}_k^N} \left(\sup\left\{\mathcal{R}^{(N)}(\boldsymbol{\phi}) \mid \boldsymbol{\phi} \in Y^\perp, \|\boldsymbol{\phi}\|_{L^2(\Omega)} = 1\right\}\right),$$

where $Y^\perp = \{\boldsymbol{\phi} \in [H^1(\Omega)]^N \mid (\boldsymbol{\phi}, \boldsymbol{\psi})_{L^2(\Omega)} = 0 \quad \forall \boldsymbol{\psi} \in Y\}$ is the L^2-orthogonal complement of Y. We choose again $Y = Y_{\mathrm{c}}$. Its orthogonal complement $Y_{\mathrm{c}}^\perp$ is given by

$$Y_{\mathrm{c}}^\perp = \hat{H}^1 \boldsymbol{e}_1 \oplus \ldots \hat{H}^1 \boldsymbol{e}_k \oplus H^1 \boldsymbol{e}_{k+1} \oplus \ldots \oplus H^1 \boldsymbol{e}_N,$$

where

$$\hat{H}^1 = \left\{v \in H^1(\Omega) \;\middle|\; \int_{\partial\Omega} v \,\mathrm{d}\sigma = 0\right\}, \qquad H^1 = H^1(\Omega).$$

In order to maximize the first term in $\mathcal{R}^{(N)}(\boldsymbol{\phi})$, it is best to choose $\boldsymbol{\phi} \in \hat{H}^1 \boldsymbol{e}_1$, i.e., $\boldsymbol{\phi} = v\boldsymbol{e}_1$ for some $v \in \hat{H}^1$ with $\|v\|_{L^2(\Omega)} = 1$. For this choice of test function, we have

$$\begin{aligned} \lambda_k \leq \mathcal{R}^{(N)}(\boldsymbol{\phi}) =& \alpha_1 \int_{\partial\Omega} v^2 \,\mathrm{d}\sigma - \int_\Omega \sum_{j=1}^N d_j (e_1^{(j)})^2 |\nabla v|^2 \,\mathrm{d}x \\ leq& \alpha_1 \int_{\partial\Omega} v^2 \,\mathrm{d}\sigma - d_{\mathrm{min}} \int_\Omega |\nabla v|^2 \,\mathrm{d}x, \end{aligned}$$

where $d_{\mathrm{min}} = \min\{d_1, \ldots, d_N\}$ and $\boldsymbol{e}_k^{(j)}$ stands for the j-th component of $\boldsymbol{e}_k$. By using the characterization of the smallest non-zero Steklov eigenvalue ([14, 6])

$$\nu_1 = \min\left\{\int_\Omega |\nabla v|^2 \,\mathrm{d}x \Big/ \int_{\partial\Omega} v^2 \,\mathrm{d}\sigma \;\middle|\; v \in \hat{H}^1, \|v\|_{L^2} \neq 0\right\},$$

we find that $\mathcal{R}^{(N)}(\boldsymbol{\phi}) < 0$ if $d_{\mathrm{min}} > \alpha_1/\nu_1$. This implies that $\lambda_k < 0$ for diffusion matrices D with $d_{\mathrm{min}} > \alpha_1/\nu_1$. Hence, by choosing

$$d^*(J, \Omega) := \alpha_1/\nu_1,$$

we establish Theorem 2.1 (iii).

2.4.2. *Proof of Theorem 2.2.* Our proof crucially depends on the so called *Dirichlet-to-Neumann map.* Let us begin with recalling the definition and several properties of this map (see, [6]).

We consider Dirichlet boundary value problem for the $\mathbb{C}$-valued function $v(x)$:

$$\lambda v(x) = \Delta v(x) \quad x \in \Omega, \qquad v(x) = p(x) \quad x \in \partial\Omega, \tag{9}$$

where $p \in H^{3/2}(\partial\Omega)$ is a given function (Dirichlet data) and $\lambda \in \mathbb{C}$ is considered as a parameter. Dirichlet-to-Neumann map is defined by following the steps below.

Step 1: The problem (9) has a unique solution $v(\cdot;\lambda,p) \in H^2(\Omega)$ for each $p \in H^{3/2}(\partial\Omega)$ if $\lambda \in \mathbb{C}\backslash\sigma(\Delta_{\mathrm{Dir}})$.

Step2: The *Dirichlet-to-Neumann map* $\mathcal{T}(\lambda) : H^{3/2}(\partial\Omega) \longrightarrow H^{1/2}(\partial\Omega)$ is defined by

$$\big(\mathcal{T}(\lambda)p\big)(x) = \partial_{\boldsymbol{n}} v(x;\lambda,p) \quad x \in \partial\Omega, \tag{10}$$

Step 3: Eigenvalues of $\mathcal{T}(0)$: $\nu_0 = 0 < \nu_1 \leq \nu_2 \leq \ldots \leq \nu_j \to \infty$ are called the *Steklov eigenvalues for the Laplacian* Δ.

Lemma 2.1. *Let* $\Sigma^\theta_\mu := \{\lambda \in \mathbb{C} \mid \lambda \neq \mu, |\arg(\lambda-\mu)| \leq \pi - \theta\}$ *for* $\mu \in \mathbb{R}$, $0 < \theta < \pi$, *and* $\mu_0 := \max\sigma(\Delta_{\mathrm{Dir}}) < 0$.

(i) *The operators* $\mathcal{T}(\lambda) : H^{3/2}(\partial\Omega) \to H^{1/2}(\partial\Omega)$ *are bounded uniformly with respect to* $\lambda \in \Sigma^\theta_{\mu_0/2}$.

(ii) *For each* $\varepsilon \in (0,\pi)$, $d\mathcal{T}(\lambda/d) \to 0$ *as* $d \to 0$ *uniformly in* $\lambda \in \{\lambda \in \mathbb{C} \mid |\arg\lambda| < \pi - \varepsilon\} \cup \{0\}$.

(iii) *The operator* $\mathcal{T}(0)$ *is self-adjoint with respect to the* L^2*-inner product*

$$\langle u, v\rangle := \int_{\partial\Omega} u(x)v(x)\,\mathrm{d}\sigma(x).$$

(iv) *For each* $p \in H^{3/2}(\partial\Omega)$, $\mathcal{T}(\lambda)p$ *is analytic in* $\lambda \in \Sigma^\theta_{\mu_0/2}$. *In particular,* $\langle\mathcal{T}'(0)p,p\rangle > 0$ *for any* $p \in H^{3/2}(\partial\Omega), p \not\equiv 0$, *where the prime means the derivative with respect to* λ.

This Lemma will play roles in what follows.

Theorem 2.4. *There exists a piecewise smooth curve in the complex* ζ*-plane, represented by* $\operatorname{Re}\zeta = \mathcal{C}(\operatorname{Im}\zeta)$ *such that* $\mathcal{C}$ *is even, satisfies*

$$\mathcal{C}(0) = 0, \quad \mathcal{C}'(0) = 0 \quad \textit{and} \quad \mathcal{C}(s) > 0 \quad (s \neq 0)$$

and depends only on Ω. *We then have the following:*

Suppose that $\mathcal{T}(\lambda)v = \zeta v$ *has a nontrivial solution* $v \not\equiv 0$ *for* $\zeta \in \mathbb{C}$.

(a) *If* ζ *satisfies* $\operatorname{Re}\zeta < \mathcal{C}(\operatorname{Im}\zeta)$, *then* $\operatorname{Re}\lambda < 0$.

(b) *If* ζ *satisfies* $\operatorname{Re}\zeta > \mathcal{C}(\operatorname{Im}\zeta)$, *then there exists a* λ *such that* $\operatorname{Re}\lambda > 0$.
(c) *If* ζ *satisfies* $\operatorname{Re}\zeta = \mathcal{C}(\operatorname{Im}\zeta)$, *then there exists a* λ *such that* $\operatorname{Re}\lambda = 0$.

PROOF of Theorem 2.4. Our strategy for the proof is to characterize the eigenvalues $E(\lambda)$ of $\mathcal{T}(\lambda)$ for $\lambda \in \mathbb{C}\backslash\sigma(\Delta_{\mathrm{Dir}})$, as the analytic continuation of the eigenvalues $\{\nu_k\}_{k=0}^{\infty}$ of $\mathcal{T}(0)$. We then use the characterizations to identify the threshold curve defined by $\operatorname{Re}\zeta = \mathcal{C}(\operatorname{Im}\zeta)$.

$E \in \mathbb{C}$ is an eigenvalue of $\mathcal{T}(\lambda)$ if and only if $\mathcal{F}(\lambda, b, E) = 0$ has a nontrivial solution $b \neq 0$, where $\mathcal{F}(\lambda, b, E) := \mathcal{T}(\lambda)b - Eb$ which is defined for $\lambda \in \mathbb{C}\backslash\sigma(\Delta_{\mathrm{Dir}})$, $b \in H^{3/2}(\partial\Omega)$ with $b \not\equiv 0$.

We employ the Lyapunov-Schmidt method as developed in Chapter 14 of [8]. For each $j \in \{0\} \cup \mathbb{N}$, we decompose the space $H^{3/2}(\partial\Omega)$ and the equation as follows:

$$H^{3/2}(\partial\Omega) = [\rho_j] \oplus [\rho_j]^{\perp} \quad \text{and} \quad \left\{ \begin{array}{rcl} \mathcal{F}_1(\lambda, \hat{b}, E) & = & 0 \\ \mathcal{F}_2(\lambda, \hat{b}, E) & = & 0 \end{array} \right. \quad \text{for } \hat{b} \in [\rho_j]^{\perp}$$

in which

$$\mathcal{F}_1(\lambda, \hat{b}, E) := \langle \mathcal{T}(\lambda)(\rho_j + \hat{b}) - E(\rho_j + \hat{b}), \rho_j \rangle, \quad \mathcal{F}_2 := \mathcal{F} - \mathcal{F}_1\rho_j.$$

Implicit function theorem applied to $\mathcal{F}_2 = 0$ near $(\lambda, \hat{b}, E) = (0, 0, \nu_j)$ show that there exists $\hat{b}(\lambda, E)$ defined near $(\lambda, E) = (0, \nu_j)$ with the following properties: (1) $\mathcal{F}_2(\lambda, \hat{b}(\lambda, E), E) = 0$ for all (λ, E) in a neighborhood of $(\lambda, E) = (0, \nu_j)$; (2) $\hat{b}(\lambda, E)$ is analytic in (λ, E); (3) $\hat{b}$ satisfies $\hat{b}(0, \nu_j) = 0$ and $\partial_E\hat{b}(0, \nu_j) = 0$. Moreover, $\partial_\lambda\hat{b}(0, \nu_j)$ solves

$$[\mathcal{T}(0) - \nu_j]\,(\partial_\lambda\hat{b}(0, \nu_j)) + \mathcal{T}'(0)\rho_j - \langle \mathcal{T}'(0)\rho_j, \rho_j\rangle\rho_j = 0.$$

Substituting $\hat{b} = \hat{b}(\lambda, E)$, the equation $\mathcal{F}_1 = 0$ is equivalent to

$$0 = \mathcal{G}(\lambda, E) := \mathcal{F}_1(\lambda, \hat{b}(\lambda, E), E).$$

The equation has the unique solution $E = E(\lambda)$, since $\partial_E\mathcal{G}(0, \nu_j) = -1$. It is also easy to show that $E(\lambda)$ is analytic in λ, satisfies $E(0) = \nu_j$ and $E'(0) = \langle \mathcal{T}'(0)\rho_j, \rho_j\rangle > 0$ (cf. Lemma 2.1 (iv)). Substituting $E(\lambda)$ back into $\rho_j + \hat{b}(\lambda, E(\lambda))$, we obtain the analytic family of eigenpair $(E(\lambda), b(\lambda))$ for $\mathcal{T}(\lambda)$, i.e., $\mathcal{T}(\lambda)b(\lambda) = E(\lambda)b(\lambda)$ with $b(\lambda)$ normalize as $\|b(\lambda)\|_{L^2(\partial\Omega)} = 1$.

We now analytically continue this $E(\lambda)$ for $\lambda \in \mathbb{C}\backslash\sigma(\Delta_{\mathrm{Dir}})$ and call the extended function $E_j(\lambda)$. For any eigenvalue $E(\lambda)$ we continue it to $\lambda = 0$, then $E(0)$ must be one of the Steklov eigenvalues ν_j ($j = 0, 1, \ldots$).

For each $j \in \{0\} \cup \mathbb{N}$ and $\tau \in \mathbb{R}$, we define $\mathcal{C}_j$ by

$$\operatorname{Re} E_j(\mathrm{i}\tau) = \mathcal{C}_j(\operatorname{Im} E_j(\mathrm{i}\tau)).$$

This means that we map the imaginary axis of λ-plane into ζ-plane by the function $E_j(\cdot) : \lambda \mapsto E_j(\lambda)$. It is easy to see that $\overline{E_j(\lambda)} = E_j(\overline{\lambda})$, and hence $\mathcal{C}_j(s)$ $(j = 0, 1, \ldots)$ are even functions. By setting $\tau = 0$, we also have $\mathcal{C}_j(0) = E_j(0) = \nu_j$. To prove Theorem 2.4 , let us define the desired function by $\mathcal{C}(s) = \inf\{\mathcal{C}_j(s) \mid j \in \{0\} \cup \mathbb{N}\}$. Then it follows that statements (a), (b) and (c) of Theorem 2.4 hold for $\mathcal{C}$. □

By using the Theorem 2.4, we are now able to complete the proof of Theorem 2.2 as follows.

Note that $\lambda \in \mathbb{C}\backslash(-\infty, 0)$ is an eigenvalue of (2) if and only if

$$\begin{pmatrix} d_1\mathcal{T}(\frac{\lambda}{d_1}) & \mathbf{0} & 0 \\ \mathbf{0} & \ddots & \mathbf{0} \\ 0 & \mathbf{0} & d_N\mathcal{T}(\frac{\lambda}{d_N}) \end{pmatrix} \boldsymbol{\phi}(x) - J\boldsymbol{\phi}(x) = 0, \; x \in \partial\Omega \tag{11}$$

which is equivalent to $d\mathcal{T}(\lambda/d)\boldsymbol{\phi} = J\boldsymbol{\phi}$ for $D = d\mathbb{I}_N,\ d > 0$.

By using the spectral decomposition of J, one can show that λ is an eigenvalue of (2) if and only if

$$d\mathcal{T}(\lambda/d)b = \alpha b \quad \text{on } \partial\Omega$$

has a non-trivial solution $b \not\equiv 0$, where α is an eigenvalue of J. We now apply Theorem 2.4 with $(\lambda, \zeta) \leftrightarrow (\lambda/d, \alpha/d)$ to complete the proof. □

2.4.3. *Proof of Theorem 2.3.* We put $\lambda = 0$ in (11), and obtain

$$D\mathcal{T}(0)\boldsymbol{\phi} = J\boldsymbol{\phi} \qquad \text{on } \partial\Omega. \tag{12}$$

Expanding $\boldsymbol{\phi}$ in terms of the complete ortho-nomal system of eigenpairs $\{\rho_j, \nu_j\}_{j=1}^{\infty}$ of $\mathcal{T}(0)$, this equation decouples as follows:

$$\boldsymbol{\phi}(x) = \sum_{j=0}^{\infty} \rho_j(x)\boldsymbol{\psi}_j \qquad x \in \partial\Omega,$$

$$\sum_{j=0}^{\infty} \rho_j(x)\left(\nu_j D\boldsymbol{\psi}_j - J\boldsymbol{\psi}_j\right) = \mathbf{0} \qquad x \in \partial\Omega,$$

where $\boldsymbol{\psi}_j \in \mathbb{C}^N$ $(j \in \mathbb{N})$ are constant vectors. Therefore, (12) has nontrivial solutions if and only if there exists $k \in \{0\} \cup \mathbb{N}$ such that $\nu_k D\boldsymbol{\psi}_k - J\boldsymbol{\psi}_k = \mathbf{0}$ has a nontrivial solution $\boldsymbol{\phi}_k \neq \mathbf{0}$, which immediately proves (i) and (ii).

Let us now proceed to the proof of (iii). Since $\lambda = 0$ is a simple eigenvalue of (11) for $D = D^0$ by the assumption in Theorem 2.3 (iii), where $D^0 = \operatorname{diag}(d_1^0, \dots, d_N^0)$, Theorem 2.3 (ii) implies that there exists $\boldsymbol{\psi} \neq \mathbf{0} \in \mathbb{R}^N$ such that $(\nu_k D^0 - J)\boldsymbol{\psi} = \mathbf{0}$, and that the corresponding eigenfunction is given by $\boldsymbol{\phi} = \rho_k \boldsymbol{\psi}$. The simplicity of the eigenvalue $\lambda = 0$ means that $\ker(\nu_k D^0 - J)$ is spanned by $\boldsymbol{\psi}$ and ν_k is a simple eigenvalue of $\mathcal{T}(0)$. Moreover, the kernel of $(\nu_k D^0 - J)^{\mathrm{T}}$, the transpose of $\nu_k D^0 - J$, is also one-dimensional and spanned by a nonzero vector $\boldsymbol{\psi}^* \in \mathbb{R}^n$ such that $\boldsymbol{\psi} \cdot \boldsymbol{\psi}^* = 1$ (so normalized).

We will now show that the eigenvalue λ of (11) is continued from $\lambda = 0$ to $\lambda \in \mathbb{R}$ in a neighborhood of $\lambda = 0$, by using again the Lyapunov-Schmidt method [8]. Let us define an L^2-inner product on the two Hilbert spaces $X = \left[H^{3/2}(\partial\Omega)\right]^N$ and $Y = \left[H^{1/2}(\partial\Omega)\right]^N$ by

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle := \int_{\partial\Omega} \boldsymbol{u} \cdot \boldsymbol{v} \, \mathrm{d}\sigma \quad \text{for } \boldsymbol{u}, \boldsymbol{v} \in X, Y.$$

To ease presentation in the sequel, we denote the left hand side of (11), with the identification $D = \operatorname{diag}(d_1, \dots, d_N)$, as follows.

$$\mathcal{T}_D(\lambda) := \operatorname{diag}\left(d_1 \mathcal{T}\left(\frac{\lambda}{d_1}\right), d_2 \mathcal{T}\left(\frac{\lambda}{d_2}\right), \dots, d_N \mathcal{T}\left(\frac{\lambda}{d_N}\right)\right),$$

and hence (11) is rewritten concisely as: $\mathcal{T}_D(\lambda)\boldsymbol{\phi} - J\boldsymbol{\phi} = 0$. The kernel of $L = \mathcal{T}_{D^0}(0) - J = D^0\mathcal{T}(0) - J$ is spanned by $\rho_k\boldsymbol{\psi}$ and the kernel of its adjoint (with respect to the inner product introduced above) $L^* = D^0\mathcal{T}(0) - J^{\mathrm{T}}$ is spanned by $\rho_k \boldsymbol{\psi}^*$, since $\mathcal{T}(0)$ is self-adjoint (cf. Lemma 2.1 (iii)). We now decompose the spaces X, Y as follows.

$$X = [\ker L] \oplus X_1, \quad Y = [\ker L^*] \oplus Y_1$$
$$\text{with } X_1 = [\ker L]^{\perp}, \quad Y_1 = \operatorname{im} L = [\ker L^*]^{\perp}.$$

We also denote by $Q : Y \longrightarrow Y$ the orthogonal projection onto Y_1. In order to show the solvability of $\mathcal{T}_D(\lambda)\boldsymbol{\phi} - J\boldsymbol{\phi} = 0$ near $(\lambda, D, \boldsymbol{\phi}) = (0, D^0, \rho_k\boldsymbol{\psi})$, we define the analytic mapping $\mathcal{F} : (-\epsilon, \epsilon) \times V \times X_1 \longrightarrow Y$ for $(\lambda, D, \boldsymbol{\phi}) \in (-\epsilon, \epsilon) \times V \times X_1$ by

$$\mathcal{F}(\lambda, D, \boldsymbol{\phi}) := \mathcal{T}_D(\lambda)(\rho_k\boldsymbol{\psi} + \boldsymbol{\phi}) - J(\rho_k\boldsymbol{\psi} + \boldsymbol{\phi}) \in Y,$$

where $\epsilon > 0$ is a small constant and V is a neighborhood of $D = D^0$. In the sequel, we always identify the matrix $D = \operatorname{diag}(d_1, \dots, d_N)$ with the vector $\boldsymbol{d} = (d_1, \dots, d_N)^{\mathrm{T}}$. It is easy to see that $\mathcal{F}(\lambda, D, \boldsymbol{\phi}) = 0$ is equivalent to

$$Q\mathcal{F}(\lambda, D, \boldsymbol{\phi}) = 0 \tag{13}$$
$$\langle \mathcal{F}(\lambda, D, \boldsymbol{\phi}),\ \rho_k\boldsymbol{\psi}^* \rangle = 0. \tag{14}$$

Since $\mathcal{F}(0, D^0, \mathbf{0}) = 0$ and $\partial_{\boldsymbol{\phi}} Q\mathcal{F}(0, D^0, \mathbf{0}) = QL : X_1 \longrightarrow Y_1$ is an isomorphism, the implicit function theorem implies that (13) is solvable in $\boldsymbol{\phi} = \boldsymbol{\phi}(\lambda, D)$ with

$$\boldsymbol{\phi}(0, D^0) = \mathbf{0}, \quad \partial_\lambda \boldsymbol{\phi}(0, D^0) = -L_1^{-1} Q \mathcal{T}'(0) \rho_k \boldsymbol{\psi},$$
$$\partial_{d_i} \boldsymbol{\phi}(0, D^0) \hat{d}_i = -\, \nu_k \left(L_1^{-1} Q \rho_k \tilde{\boldsymbol{\psi}}^i \right) \hat{d}_i \quad (i = 1, \ldots, N),$$

where $L_1 = QL|_{X_1} : X_1 \longrightarrow Y_1$ is isomorphic and $\tilde{\boldsymbol{\psi}}^i \in \mathbb{R}^n$ is the vector whose i-th component is the same as that of $\boldsymbol{\psi}$ and all other components are 0. Substituting $\boldsymbol{\phi}(\lambda, D)$ into (14), we find that λ near $\lambda = 0$ is an eigenvalue of (11), if and only if $(\lambda, D) \in (-\epsilon, \epsilon) \times V$ satisfies

$$F(\lambda, D) := \langle \mathcal{F}(\lambda, D, \boldsymbol{\phi}(\lambda, D)),\ \rho_k \boldsymbol{\psi}^* \rangle = 0. \tag{15}$$

Since $F(0, D^0) = 0$ and $\partial_\lambda F(0, D^0) = \langle \mathcal{T}'(0) \rho_k, \rho_k \rangle \boldsymbol{\psi} \cdot \boldsymbol{\psi}^* = \langle \mathcal{T}'(0) \rho_k, \rho_k \rangle$ is positive thanks to Lemma 2.1 (iv), we apply the implicit function theorem to $F(\lambda, D) = 0$, and obtain the solution $\lambda = \lambda(D)$ of (15) which satisfies $\lambda(D^0) = 0$ and

$$\partial_{\boldsymbol{d}} \lambda(D^0) \hat{\boldsymbol{d}} = -\frac{\nu_k}{\langle \mathcal{T}'(0) \rho_k, \rho_k \rangle} \sum_{i=1}^{N} \psi^i \psi_i^* \hat{d}_i, \tag{16}$$

where $\boldsymbol{\psi} = (\psi^1, \ldots, \psi^N)^{\mathrm{T}}$, $\boldsymbol{\psi}^* = (\psi_1^*, \ldots, \psi_N^*)^{\mathrm{T}}$ and $\hat{\boldsymbol{d}} = (\hat{d}_1, \ldots, \hat{d}_N)^{\mathrm{T}}$. Notice that $\lambda(D) = 0$ for $D \in \Gamma_k \cap V$. This means that $\partial_{\boldsymbol{d}} \lambda(D^0) \hat{\boldsymbol{d}} = 0$ for all vectors $\hat{\boldsymbol{d}}$ in the tangent space to Γ_k at $D = D^0$. On the other hand, $\boldsymbol{\psi} \cdot \boldsymbol{\psi}^* = 1$ implies that there exists at least one $i \in \{1, \ldots, N\}$ so that $\psi^i \psi_i^* \neq 0$. Therefore, (16) implies that $\lambda = \lambda(D)$ transversely crosses $\lambda = 0$ as D transversely crosses Γ_k at $D = D^0$. This completes the proof of Theorem 2.3 (iii).

2.4.4. *Challenges.* The proof displayed above for Theorem 2.2 raises several challenges.

Our proof does not give explicit way to compute the function $\mathcal{C}$ for a given domain Ω. Since the main part of this function is determined by $E_0(\lambda)$, we would like, ideally, to find the analytic expression of $E_0(\lambda)$ by computing derivatives $E_0^{(n)}(0)$ for all integers $n \geq 0$. We are able to compute the first two derivatives as follows.

$$E_0'(0) = \frac{|\Omega|}{|\partial\Omega|},$$
$$E_0''(0) = \frac{2}{|\partial\Omega|} \left(\int_\Omega v_0' + v_0 \, \mathrm{d}x \right) < 0,$$

where v_0', v_0 are the unique solutions of

$$\Delta v_0' = 1 \quad \text{in } \Omega, \qquad v_0' = 0 \quad \text{on } \partial\Omega,$$
$$\Delta v_0 = 0 \quad \text{in } \Omega, \qquad v_0 = b_1 \quad \text{on } \partial\Omega,$$

and b_1 is such that

$$\mathcal{T}(0)b_1 = E_0'(0) - \partial_{\boldsymbol{n}} v_0' \text{ on } \partial\Omega \quad \text{and} \quad \int_{\partial\Omega} b_1 \, \mathrm{d}\sigma = 0.$$

Although continuing this kind of process seems to be rather intractable, it is worthwhile pursuing and we also have to take into account that the operator $\mathcal{T}(\lambda)$ has singularities on $\sigma(\triangle_{\mathrm{Dir}})$.

When the diagonal diffusion matrix D is a positive multiple of the identity matrix, we are able to conceptually determine critical eigenvalues in terms of the eigenvalues of the mass transfer matrix J. For nonidentical diffusion rates, the arguments employed in the proof of Theorem 2.2 will not work. Hence, we need to change our viewpoint, and ask how to determine the set of diffusion rates $d_j > 0$ for which

$$\begin{pmatrix} d_1\mathcal{T}(\frac{\mathrm{i}\tau}{d_1}) & \mathbf{0} & 0 \\ \mathbf{0} & \ddots & \mathbf{0} \\ 0 & \mathbf{0} & d_N\mathcal{T}(\frac{\mathrm{i}\tau}{d_N}) \end{pmatrix} \phi(x) - J\phi(x) = 0, \ x \in \partial\Omega$$

has a non-trivial solution $\phi(x)$ for some $\tau \in \mathbb{R}$. For each $\tau > 0$, such a set of diffusion rates, if not empty, constitutes an $N-2$ dimensional surfaces $\Gamma_{\mathrm{Hop}}(\tau)$, because such "determinant equals 0" condition will give rise to an equation for one complex variable, and hence to a set of two real valued equations. Subsequently, we need to find conditions on J which imply $\Gamma_{\mathrm{Hop}}(\tau) \neq \emptyset$ for some $\tau > 0$, and to describe the set $\Gamma_{\mathrm{Hop}} := \bigcup\{\Gamma_{\mathrm{Hop}}(\tau) \mid \tau \in \mathbb{R}\}$ which presumably is $N-1$ dimensional. In the simplest situation, we will exhibit this kind of argument, below in §3.

3. Destabilization

The purpose of this section is to find Turing type of destabilization mechanism in (1). The results obtained in Theorems 2.1 and 2.2 may be summarized as follows.

> The system (1) tends to be *stable* (resp. *unstable*) if J is *stable* (resp. *unstable*) and diffusion rates do not differ substantially.

We now want to seek analogy with reaction-diffusion case. In [3], Turing instability in the general N-component reaction-diffusion system

$$\text{(RD)} \qquad \partial_t \boldsymbol{u} = D\triangle\boldsymbol{u} + A\boldsymbol{u} \ \text{ in } \Omega, \qquad \partial_{\boldsymbol{n}}\boldsymbol{u} = \mathbf{0} \ \text{ on } \partial\Omega$$

is investigated under the natural boundary conditions, where A is an $N \times N$ real matrix. A *guiding principle* for the instability to occur is proposed, and its validity is rigorously confirmed for $N = 2, 3$ in [3], based upon and summarizing the previous results in [25, 19, 17, 16, 18, 11, 21, 22, 23, 26, 24, 27, 28] . To explain the main feature of the results of [3], we introduce some terminologies. For an $N \times N$ matrix A and a subset $I \subset \{1, \ldots, N\}$ of indices, we denote by A_I the principal sub-matrix obtained from A by choosing exactly rows and columns of indices belonging to I. If two sets of indices $I, K \subset \{1, \ldots, N\}$ satisfy $I \cup K = \{1, \ldots, N\}$ and $I \cap K = \emptyset$, then the corresponding sub-matrices A_I and A_K are called *complementary* in the full matrix A.

When A is stable and contains an unstable sub-matrix A_I with its complementary partner being A_K, the matrix A is rearranged as

$$A = \begin{pmatrix} A_I & B \\ C & A_K \end{pmatrix}.$$

Then (RD) is written as

$$\begin{pmatrix} \partial_t \boldsymbol{v} \\ \partial_t \boldsymbol{w} \end{pmatrix} = \begin{pmatrix} D_I \Delta + A_I & B \\ C & D_K \Delta + A_K \end{pmatrix} \begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{w} \end{pmatrix}, \quad \boldsymbol{u} = \begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{w} \end{pmatrix},$$

and the *guiding principle* says that the trivial solution of (RD) destabilizes if the diffusion rates in D_I are sufficiently small compared with the diffusion rates in D_K. In other words, it is summarized as follows.

> *If the diffusion effect of an unstable subsystem in a stable full system is sufficiently weak compared with the diffusion effect of its complementary partner, then the diffusion-induced instability (Turing instability) occurs.*

In §3.2 we will establish, to some extent, the validity of this *guiding principle* for (1). We also show in §3.3 that unstable systems for equal diffusivity are stabilized when the diffusion effects of a stable subsystem is weak compared with that of its complementary partner. Such a stabilization phenomenon does not occur in the reaction-diffusion system (RD).

3.1. **Reduced System.** To apply the analogy to (1), let us decompose J as follows.

$$J = \begin{pmatrix} J_I & B \\ C & J_K \end{pmatrix} \qquad I \cup K = \{1, 2, \ldots, N\}, \ I \cap K = \emptyset,$$

so that (J_I, J_K) constitutes a complementary pair in J. Using this, we find that (11) is equivalent to

$$\text{(11E)} \qquad \begin{pmatrix} \mathcal{T}_{D_I}(\lambda) & \mathbf{0} \\ \mathbf{0} & \mathcal{T}_{D_K}(\lambda) \end{pmatrix} \begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{w} \end{pmatrix} = \begin{pmatrix} J_I & B \\ C & J_K \end{pmatrix} \begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{w} \end{pmatrix}$$

where

$$\phi = \begin{pmatrix} v \\ w \end{pmatrix}, \quad \mathcal{T}_{D_L}(\lambda) = \operatorname{diag}\Big(d_\ell \mathcal{T}(\lambda/d_\ell)\Big)_{\ell \in L} \quad L = I, K.$$

If J is *stable* and the diffusion rates $d_1, \ldots, d_N$ do not differ substantially, then all the eigenvalues of (11E) have negative real part. If $\det J_I \neq 0$, then (11E) formally converges to

$$\text{(17)} \qquad \mathcal{T}_{D_K}(\lambda) \boldsymbol{w} = (J_K - CJ_I^{-1}B)\boldsymbol{w} \quad \text{as } D_I \to \mathbf{0}.$$

When J is *stable* and J_I is *unstable*, is $J_K^* = J_K - CJ_I^{-1}B$ *unstable*? If J_J^* is unstable, then the reduced system will be unstable, hence Turing type instability must have taken place in the process of taking limit $D_I \to 0$.

This *suggests* (but does not prove) that (2) has eigenvalues with positive real part when the diffusion rates in D_I are sufficiently small. This remind us of Turing instability. The line of reasoning so far is helpful to detect Turing type mechanism in (1), but it does not rigorously prove that such a mechanism exists. The reason is that (11E) does not necessarily converge to its reduced system (17) as $D_I \to 0$, since the process of taking the limit $D_I \to 0$ is a singular perturbation problem, and some critical pieces of information may be lost in the limit. Despite of this, we are still able to prove the existence of Turing type destabilization, by showing that $D = \operatorname{diag}(D_I, D_K)$ crosses the family of hyper-surfaces $\{\Gamma_k\}_{k=1}^{\infty}$ infinitely many times as $D_I \to 0$ in case the dimension of Ω is greater than 1.

When $m = 1$ and $\Omega = (-1, +1)$, we have a simple example of Turing type Instability.

Theorem 3.1. *Consider* (1) *with* $m = 1, N = 2, \Omega = (-1, +1)$, *and let* J *satisfy*

$$J = \begin{pmatrix} p & q \\ r & s \end{pmatrix}, \quad p < 0, s > 0, \operatorname{tr} J < 0, \det J > 0.$$

(i) *For all* $d_2 \geq d_1 > 0$, *the system* (1) *is stable.*
(ii) *As* $d_2 \to 0$, *the system* (2) *converges to the reduced system* $d_1 \mathcal{T}(\lambda/d_1)u = J_1^* u$ *with* $J_1^* = \det J/s > 0$.
(iii) *For each fixed* $d_1 \leq \det J/s$, *as* $d_2 \to 0$ (1) *becomes destabilized through an* oscillatory mode.
(iv) *For each fixed* $d_1 > \det J/s$, *as* $d_2 \to 0$ *the system* (1) *becomes destabilized through a* steady mode *or through an* oscillatory mode.

PROOF of Theorem 3.1. The Dirichlet-to-Neumann map $\mathcal{T}(\lambda)$ is explicitly given by the 2×2 matrix:

$$\mathcal{T}(\lambda)\begin{pmatrix} v(-1) \\ v(+1) \end{pmatrix} := \frac{\sqrt{\lambda}}{\sinh 2\sqrt{\lambda}}\begin{pmatrix} \cosh 2\sqrt{\lambda} & -2 \\ -2 & \cosh 2\sqrt{\lambda} \end{pmatrix}\begin{pmatrix} v(-1) \\ v(+1) \end{pmatrix}.$$

In terms of the matrix $\mathcal{T}(\lambda)$, the problem (11) with J being given as above has nontrivial solution if and only if the 4×4 matrix

$$\det\begin{pmatrix} d_1\mathcal{T}(\lambda/d_1) - p\mathbb{I}_2 & -qI_2 \\ -rI_2 & d_2\mathcal{T}(\lambda/d_2) - s\mathbb{I}_2 \end{pmatrix} = 0$$

is singular, where $\mathbb{I}_2$ is the 2×2 identity matrix. Applying elementary row (or column) operations on the 4×4 matrix, this condition is further recast as the singularity condition of the 2×2 matrix

$$\det\big[d_1d_2\mathcal{T}(\lambda/d_1)\mathcal{T}(\lambda/d_2) - \{pd_2\mathcal{T}(\lambda/d_2) + sd_1\mathcal{T}(\lambda/d_1)\} + (ps - qr)I_2\big] = 0.$$

This is equivalent to

$$\big[(\mathrm{t}_1^+ - p)(\mathrm{t}_2^+ - s) - qr\big]\big[(\mathrm{t}_1^- - p)(\mathrm{t}_2^- - s) - qr\big] = 0, \tag{18}$$

where

$$t_j^+ = d_j\sqrt{\lambda/d_j}\coth\sqrt{\lambda/d_j}, \quad t_j^- = d_j\sqrt{\lambda/d_j}\tanh\sqrt{\lambda/d_j},$$

from which we easily find

$$\lim_{d_j\to0}\mathrm{t}_j^\pm = 0, \quad \lim_{\lambda\to0}\mathrm{t}_j^- = 0 \text{ and } \lim_{\lambda\to0}\mathrm{t}_j^+ = d_j \text{ for } j = 1,2. \tag{19}$$

To determine the steady destabilization curve in the first quadrant of the d_1-d_2 plane, we let $\lambda \to 0$ in (18). By using (19), then, we obtain the equation

$$(d_1 - p)(d_2 - s) - qr = 0$$

for the steady destabilization curve. This curve intersects d_1-axis at $d_1 = \det J/s$ and is meaningful (i.e., lies in the first quadrant) only for $d_1 > \det J/s$ while d_2 ranges in $0 < d_2 < s$ and $d_2 \to s$ as $d_1 \to \infty$.

If we let $d_2 \to 0$ in (18) and use (19), then it follows that

$$[\mathrm{t}_1^- - \det J/s][\mathrm{t}_1^+ - \det J/s] = 0.$$

This equation for the eigenvalue λ has

- infinitely many negative solutions;
- one positive solution if $d_1 > \det J/s$;
- two positive solutions if $d_1 < \det J/s$;
- one positive solution and $\lambda = 0$ if $d_1 = \det J/s$.

In any event, the reduced system $d_1\mathcal{T}(\lambda/d_1)\phi = s\phi$ is unstable in the sense that it has at least one positive eigenvalue $\lambda > 0$.

The eigenvalues of the 2×2 matrix J have negative real part under the conditions in Theorem 3.1, and hence, Theorem 2.2 implies that all the eigenvalues λ of (2) satisfy $\operatorname{Re}\lambda < 0$ for $d_2 = d_1 > 0$. Therefore, as d_2 decreases from $d_2 = d_1$ to $d_2 = 0$, some eigenvalues cross the imaginary axis in $\mathbb{C}$ from $\operatorname{Re}\lambda < 0$ to $\operatorname{Re}\lambda > 0$.

If $d_1 > \det J/s$, the crossing across the imaginary axis $\operatorname{Re}\lambda = 0$ is either through $\lambda = 0$ or a pair of complex conjugate eigenvalues cross the imaginary axis, possibly several times back and forth, and eventually one pair of complex conjugate eigenvalues remain in the right half plane, which finally collide on the positive real axis and one of them return to the negative real axis through $\lambda = 0$ while the other remain on the positive real axis.

On the other hand, if $d_1 < \det J/s$, eigenvalues λ cannot cross the origin, and hence a pair of non-zero complex conjugate eigenvalues have to cross the imaginary axis at least once. This completes the proof of Theorem 3.1. $\square$

3.2. **Turing type destabilization.** Consider the 3×3 matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

There are three 1-component sub-systems;

$$A_1 = (a_{11}), \quad A_2 = (a_{22}), \quad A_3 = (a_{33}).$$

There are three 2-component sub-systems;

$$A_{12} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \ A_{13} = \begin{pmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{pmatrix}, \ A_{23} = \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix}.$$

Stability or instability of 1- or 2-component subsystems are characterized as follows.

For 1-component systems, A_i is

stable, if $a_{ii} < 0$; *neutral*, if $a_{ii} = 0$; *unstable*, if $a_{ii} > 0$.

For 2-component systems, A_{ij} is

stable, if $\operatorname{tr} A_{ij} < 0$, $\det A_{ij} > 0$;
type-1 unstable, if $\operatorname{tr} A_{ij} > 0$, $\det A_{ij} > 0$;
type-2 unstable, if $\operatorname{tr} A_{ij} < 0$, $\det A_{ij} < 0$;
type-3 unstable, if $\operatorname{tr} A_{ij} > 0$, $\det A_{ij} < 0$.

Thanks to the nomenclature, we now concisely state Turing type destabilization theorem for 3-component systems on general smooth bounded domains Ω.

Theorem 3.2 (Turing type mechanism). *When $\Omega \subset \mathbb{R}^m$ $(m \geq 2)$ is a bounded smooth domain, we have the following:*

(i) *Assume that*
   - *the 3×3 matrix J is* stable $(\det J < 0)$;
   - *the subsystem J_1 is* unstable $(\det J_1 = a_{11} > 0)$.

   Then for each $d_2, d_3 > 0$ fixed with d_2 and d_3 being not substantially different, (1) is stable *for $d_1 \approx d_2 \approx d_3$, and undergoes Turing type destabilization as $d_1 \to 0$.*

(ii) *Assume that*
   - *the 3×3 matrix J is* stable $(\det J < 0)$;
   - *the matrix J_{23} is* type-2 *or* Type-3 unstable $(\det J_{23} < 0)$.

   Then for each $d_1 > 0$ fixed, (1) is stable *for $d_1 \approx d_2 \approx d_3$, and undergoes Turing type destabilization as $D_{23} \to \mathbf{0}$.*

(iii) *Assume that*
   - *the 3×3 matrix J is* stable $(\det J < 0)$;
   - *the matrix J_{23} is* type-1 unstable $(\operatorname{tr} J_{23} > 0, \det J_{23} > 0)$ *and has two positive eigenvalues .*

   Then for each $d_1 > 0$ fixed, (1) is stable *for $d_1 \approx d_2 \approx d_3$, and undergoes Turing type destabilization as $D_{23} \to \mathbf{0}$.*

PROOF of Theorem 3.2. The idea of proof is the same for the statements (i) (ii) and (iii). We show that the surfaces $\{\Gamma_k\}_{k\in\mathbb{N}}$ in the first octant of the $d_1d_2d_3$-space are located between the diagonal $\{d_1 = d_2 = d_3\}$ and either the d_2d_3-plane (for (i)) or d_1-axis (for (ii) and (iii)). Therefore, (d_1, d_2, d_3) crosses infinitely many members of the family of *steady destabilization surfaces* $\{\Gamma_k\}_{k\in\mathbb{N}}$ as the appropriate diffusion rates (d_1 in (i) and d_2, d_3 in (ii) and (iii)) approach 0.

To prove (i) note that

$$\det J_{23}^* = \frac{\det J}{\det J_1} < 0.$$

Hence, J_{23}^* is either type-2 or type-3 unstable, which implies that one of the eigenvalues of J_{23}^* is positive. Therefore, the reduced system

$$\mathcal{T}_{D_{23}}(\lambda)\boldsymbol{w} = J_{23}^*\boldsymbol{w}$$

has eigenvalues with positive real part. By looking into the configuration of Γ_k, we conclude that (d_1, d_2, d_3) traverses infinitely many members of $\{\Gamma_k\}_{k\in\mathbb{N}}$ as $d_1 \to 0$.

(ii) In this case, the reduced system is $d_1\mathcal{T}(\lambda/d_1)\phi = J_1^*\phi$ where

$$J_1^* = \frac{\det J}{\det J_{23}} > 0,$$

which means that (1) is unstable due to Theorem 2.1. If $j \in \mathbb{N}$ is the largest number with the property $d_1 < J_1^*/\nu_j$, then the number of *positive eigenvalues* λ of the reduced system is $j+1$. Therefore, as $D_{23} \to \mathbf{0}$, (d_1, d_2, d_3) goes through all Γ_k with $k \geq j$.

(iii) In this case, the reduced system is $d_1\mathcal{T}(\lambda/d_1)\phi = J_1^*\phi$ where

$$J_1^* = \frac{\det J}{\det J_{23}} < 0.$$

The reduced system is stable and contains no sign of instability. Nevertheless, we are able to prove that (1) undergoes Turing type destabilization by closely examining the surfaces $\{\Gamma_k\}$.

3.3. **Anti-Turing Mechanism.** In this subsection, we present a stabilization mechanism in (1). The reaction-diffusion system (RD) does not possess such a mechanism.

In the system (1), assume that J is unstable and that J_I and J_J are complementary in J with J_I stable. Under this situation, we know from the results in §2 that (1) tends to be unstable for equal diffusion rates. If we let $D_I \to \mathbf{0}$ so that $\max\{d_i \;|i \in I\} \ll \min\{d_j \mid j \in J\}$, then does (1) becomes stable? We show that the answer to this question is affirmative for (1). The key to establish such result is that the full system (11E) converges to its reduced system (11), provided that J_i is *stable* (instead of being unstable). As regard to the last statement, we refer to Lemma 3.1 and its proof in [2].

Our first result is for the 2-component system ($N = 2$).

Theorem 3.3 (Anti-Turing mechanism for 2×2 systems). *Let J be a 2×2 matrix.*

(i) *Assume that*

- *J is* type-1 unstable *and has real eigenvalues;*
- *J_1 is stable (i.e., $a_{11} < 0$).*

Then for each $d_2 > 0$ fixed, (1) is unstable for $d_1 = d_2$ and becomes stabilized as $d_1 \to 0$.

(ii) *If , Assume that*

- *J is either* type-2 *or* type-3 unstable*;*
- *J_1 is stable (i.e., $a_{11} < 0$).*

Then for each $d_2 > 0$ fixed (1) is unstable for $d_1 = d_2$ and remains unstable as $d_1 \to 0$, i.e., there is no anti-Turing mechanism.

PROOF of Theorem 3.3. We use the fact stated above that (1) is stable (or unstable) for $D_i \approx \mathbf{0}$ if its reduced system (17) is stable (unstable).

(i) Since J is unstable and has real eigenvalues, Theorem 2.2 implies that (1) is unstable for $d_1 = d_2$. The reduced problem is $d_2\mathcal{T}(\lambda/d_2)w = J_2^* w$ with

$$J_2^* = \frac{\det J}{a_{11}} < 0,$$

hence, it is stable. According to Lemma 3.1 of [2], we conclude that (1) becomes stabilized as $d_1 \to 0$.

(ii) When J is type-2 or type-3 unstable, then $\det J < 0$. Therefore, J has at least one positive eigenvalue, and Theorem 2.2 implies that (1) is unstable for $d_1 = d_2$. Moreover, we have

$$J_2^* = \frac{\det J}{a_{11}} > 0$$

which implies that the reduced system (17) is unstable, hence (1) remains unstable for $d_1 \approx 0$. □

We now show that statements similar to Theorem 3.3 remains true for 3-component system (1), as follows.

Theorem 3.4 (Anti-Turing mechanism for 3×3 matrix). *Let J be a 3×3 real matrix.*

(i) *Assume that*
- *the matrix J has two positive and one negative eigenvalues* ($\det J < 0$)*;*
- *the submatrix J_{23} is stable.*

Then for each $d_1 > 0$ fixed, (1) is unstable for $d_1 \approx d_2 \approx d_3$ and becomes stabilized as $d \to 0$ in $D_{23} = d\mathbb{I}_2$.

(ii) *Assume that*
- *J has two positive and one negative eigenvalues* ($\det J < 0$)*;*
- *the subsystem J_1 is stable* ($a_{11} < 0$)*;*
- $\det J_{12} + \det J_{13} > 0$.

Then for each $d_2 \approx d_3 > 0$ fixed, (1) is unstable for $d_1 \approx d_2 \approx d_3$ and becomes stabilized as $d_1 \to 0$.

PROOF of Theorem 3.4. The idea of proof being identical to that of Theorem 3.3, we only show that the reduced system (17) is stable.

(i) In this case, the reduced system (17) is a scalar problem

$$d_1\mathcal{T}(\lambda/d_1)\phi = J_1^*\phi, \text{ where } J_1^* = \frac{\det J}{\det J_{23}} < 0.$$

Hence, the eigenvalues λ of the reduced system are negative.

(ii) In this case, the reduced matrix J_{23}^* satifies

$$\det J_{23}^* = \frac{\det J}{a_{11}} > 0,$$
$$\operatorname{tr} J_{23}^* = \frac{1}{a_{11}}(\det J_{12} + \det J_{13}) < 0,$$

which imply that J_{23}^* is stable. Hence, for $d_2 \approx d_3$ the eigenvalues λ of $\mathcal{T}_{D_{23}}(\lambda)\boldsymbol{w} = J_{23}^*\boldsymbol{w}$ have negative real part. □

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