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An $(N-2)$-dimensional surface with positive principal curvatures gives an N-dimensional traveling front in bistable reaction-diffusion equations

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Abstract

This paper is a preliminary report of the forthcoming paper [21]. This paper studies traveling fronts to the Allen-Cahn equation in $\mathbb{R}^N$ for $N \geq 3$. We consider $(N-2)$-dimensional smooth surfaces as boundaries of strictly convex compact sets in $\mathbb{R}^{N-1}$, and define an equivalence relation between them. We prove that there exists a traveling front associated with a given surface and that it is asymptotically stable for given initial perturbation. The associated traveling fronts coincide up to phase transition if and only if the given surfaces satisfy the equivalence relation.

AMS Mathematical Classifications: 35C07, 35B20, 35K57
Key words: traveling front, Allen-Cahn equation, non-symmetric

As a preliminary report of the forthcoming paper [21] we briefly state the results. We study the following reaction-diffusion equation

$$\begin{aligned} &\frac{\partial u}{\partial t} = \Delta u + f(u) && \boldsymbol{x} \in \mathbb{R}^N, t > 0, \\ &u(\boldsymbol{x}, 0) = u_0 && \boldsymbol{x} \in \mathbb{R}^N. \end{aligned} \tag{1}$$

Here $\Delta = \sum_{j=1}^N D_{jj}$ with $D_j = \partial/\partial x_j$ and $D_{jj} = (\partial/\partial x_j)^2$ for $1 \leq j \leq N$. Now $N \geq 3$ is a given integer, and u_0 is a given bounded and uniformly continuous function from $\mathbb{R}^N$ to $\mathbb{R}$.

The assumption on f is as follows.

(A1) $f \in C^1[-1,1]$ satisfies $f(1) = 0$, $f(-1) = 0$, $f'(1) < 0$, $f'(-1) < 0$ and

$$\int_{-1}^{1} f(s) ds > 0.$$

(A2) There exists $a_* \in (-1,1)$ such that

$$\begin{aligned} &f(s) < 0 && \text{for all } s \in (-1, -a_*), \\ &f(s) > 0 && \text{for all } s \in (-a_*, 1). \end{aligned}$$

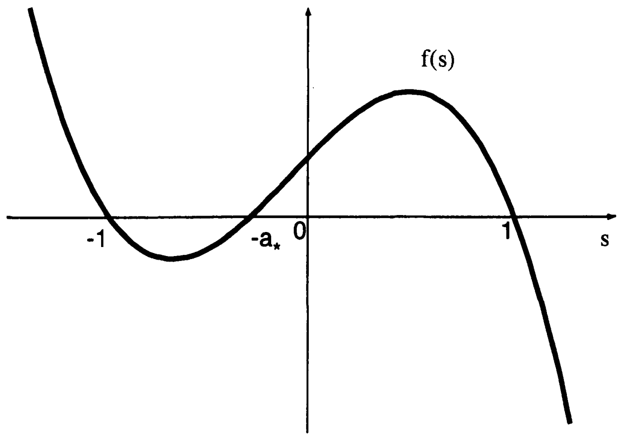

Figure 1: The graph of f.

See Figure 1. Equation (1) is called the Nagumo equation [15] or the unbalanced Allen-Cahn equation [1]. For this equation, multi-dimensional traveling fronts have been studied by many mathematicians. Two-dimensional V-form fronts are studied by Ninomiya and myself [16, 17], Hamel, Monneau and Roquejoffre [8, 9] and Haragus and Scheel [10] and so on. Cylindrically symmetric traveling fronts in $\mathbb{R}^N$ are studied by [8, 9]. Traveling fronts of pyramidal shapes and convex polyhedral shapes are studied by [18, 19, 13, 20]. See [14] for a related work. Traveling fronts associated with strictly convex compact domain in $\mathbb{R}^2$ with a smooth boundary are studied for the Allen-Cahn equation in $\mathbb{R}^3$ in [20]. The purpose of this paper is to show that a strictly convex compact set in $\mathbb{R}^{N-1}$ with a smooth boundary gives a traveling front in the Allen-Cahn equation in $\mathbb{R}^N$ by using a clear and concise argument. Since the Allen-Cahn equation is one of the simplest reaction-diffusion equations, the argument in this paper might be useful for studies on other reaction-diffusion equations or reaction-diffusion systems that admit comparison principles.

The profile equation of a one-dimensional traveling front with speed k is given by

$$\begin{gathered} -\Phi''(y) - k\Phi'(y) - f(\Phi(y)) = 0 \qquad -\infty < y < \infty, \\ \Phi(-\infty) = 1, \ \Phi(\infty) = -1. \end{gathered} \tag{2}$$

It is known that (2) has a solution Φ under (A1) and (A2), and it is unique up to translation. One can refer to [2, 3, 11, 12, 6, 4] for instance. See Figure 2. Now (A1) gives $k > 0$. Especially one has $k = \sqrt{2}a_*$ and $\Phi(x) = -\tanh(x/\sqrt{2})$ when $0 < a_* < 1$ and $f(u) = -(u+1)(u+a_*)(u-1)$.

The Allen-Cahn equation by a moving coordinate system with speed c toward the x_N-direction is given by

$$\begin{aligned} &(D_t - \Delta - cD_N)\, w - f(w) = 0 \qquad && \boldsymbol{x} \in \mathbb{R}^N, t > 0, \\ &w(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}) && \boldsymbol{x} \in \mathbb{R}^N. \end{aligned} \tag{3}$$

Here we assume $c > k$. We denote the solution of (3) by $w(\boldsymbol{x}, t; u_0)$. The profile equation of a traveling front in $\mathbb{R}^N$ is given by

$$(-\Delta - cD_N)\, v - f(v) = 0 \qquad \boldsymbol{x} \in \mathbb{R}^N. \tag{4}$$

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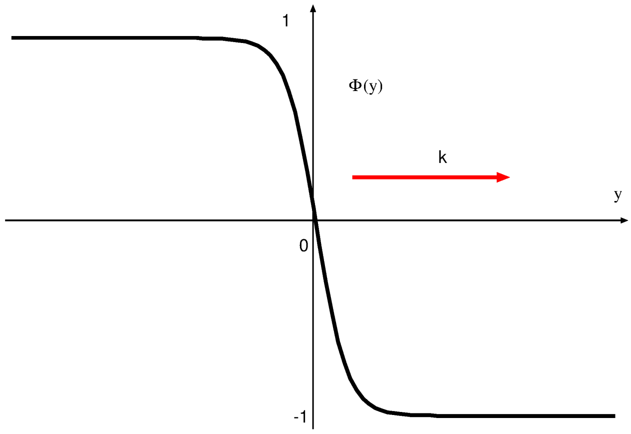

Figure 2: A one-dimensional traveling front Φ.

Here we put $\boldsymbol{x}' = (x_1, \ldots, x_{N-1}) \in \mathbb{R}^{N-1}$ and $\boldsymbol{x} = (\boldsymbol{x}', x_N)$.

We extend f as a function of class $C^1(\mathbb{R})$ with $f'(s) < 0$ for $|s| > 1$. Setting

$$\beta = \frac{1}{2} \min \{-f'(-1), -f'(1)\} > 0,$$

we choose $\delta_* \in (0, 1/4)$ with

$$-f'(s) > \beta \qquad \text{if } |s+1| \leq 2\delta_* \text{ or } |s-1| \leq 2\delta_*.$$

In this paper we assume $c > k$. Let

$$\begin{aligned} M &= \max_{|s| \leq 1+\delta_*} |f'(s)| > 0, \\ m_* &= \frac{\sqrt{c^2 - k^2}}{k}, \end{aligned}$$

and define $\theta_* \in (0, \pi/2)$ by

$$\tan \theta_* = m_*.$$

Let $n \geq 2$ be a given integer and let$\{\boldsymbol{a}_j\}_{j=1}^n$ be a set of unit vectors in $\mathbb{R}^{N-1}$ with $\boldsymbol{a}_i \neq \boldsymbol{a}_j$ for $i \neq j$. Then $\boldsymbol{a}_j = (a_j^1, \ldots, a_j^{N-1})$ satisfies

$$|\boldsymbol{a}_j|^2 = \sum_{i=1}^{N-1} (a_j^i)^2 = 1 \qquad \text{for all } 1 \leq j \leq n.$$

Here we put $\boldsymbol{x}' = (x_1, \ldots, x_{N-1}) \in \mathbb{R}^{N-1}$ and $\boldsymbol{x} = (\boldsymbol{x}', x_N) = (x_1, \ldots, x_N) \in \mathbb{R}^N$ with $|\boldsymbol{x}'| = \sqrt{\sum_{i=1}^{N-1} x_i^2}$ and $|\boldsymbol{x}| = \sqrt{\sum_{i=1}^{N} x_i^2}$, respectively. For $\boldsymbol{x}' \in \mathbb{R}^{N-1}$ we set

$$\begin{aligned} h_j(\boldsymbol{x}') &= m_*(\boldsymbol{a}_j, \boldsymbol{x}'), &(5)\\ h(\boldsymbol{x}') &= \max_{1 \leq j \leq n} h_j(\boldsymbol{x}') = m_* \max_{1 \leq j \leq n} (\boldsymbol{a}_j, \boldsymbol{x}'). &(6) \end{aligned}$$

Here $(\boldsymbol{a}_j, \boldsymbol{x}')$ denotes the inner product of vectors $\boldsymbol{a}_j$ and $\boldsymbol{x}'$. In this paper we call $\{(\boldsymbol{x}', x_N) \in \mathbb{R}^N \mid x_N \geq h(\boldsymbol{x}')\}$ a pyramid. Setting

$$\Omega_j = \{\boldsymbol{x}' \in \mathbb{R}^{N-1} \mid h(\boldsymbol{x}') = h_j(\boldsymbol{x}')\}$$

for $j = 1, \dots, n$, we have

$$\mathbb{R}^{N-1} = \cup_{j=1}^{n} \Omega_j.$$

We denote the boundary of Ω_j by $\partial\Omega_j$. Now we put

$$S_j = \{\boldsymbol{x} \in \mathbb{R}^N \mid x_N = h_j(\boldsymbol{x}') \quad \text{for } \boldsymbol{x}' \in \Omega_j\}$$

for each j, and call $\cup_j^n S_j \subset \mathbb{R}^N$ the lateral faces of a pyramid. We put

$$\Gamma_j = \{\boldsymbol{x} \in \mathbb{R}^N \mid x_N = h_j(\boldsymbol{x}') \quad \text{for } \boldsymbol{x}' \in \partial\Omega_j\}$$

for $j = 1, \dots, n$. Then $\cup_{j=1}^n \Gamma_j$ represents the set of all edges of a pyramid. For $\gamma > 0$ let

$$D(\gamma) = \left\{\boldsymbol{x} \mid \operatorname{dist}\left(\boldsymbol{x}, \cup_{j=1}^n \Gamma_j\right) > \gamma\right\}.$$

Now we define $\underline{v}(\boldsymbol{x})$ by

$$\underline{v}(\boldsymbol{x}) = \Phi\left(\frac{k}{c}(x_N - h(\boldsymbol{x}'))\right) = \max_{1\leq j\leq n} \Phi\left(\frac{k}{c}(x_N - h_j(\boldsymbol{x}'))\right).$$

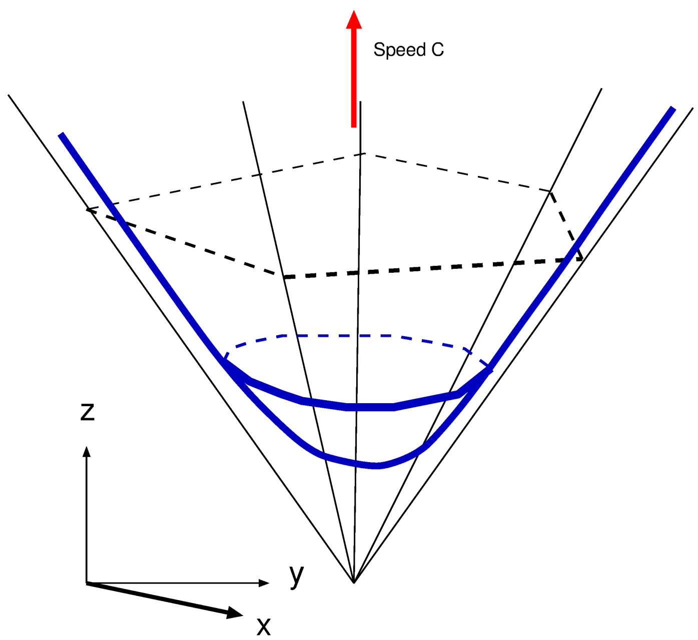

Figure 3: The graph of a level set of a pyramidal traveling front ([18, 19])

Pyramidal traveling fronts are stated as follows. See Figure 3. For the proof see [16] for $n = 2$ and see [13] for $n \geq 3$.

Theorem 1 ([16], [13]) *Let h be given in (6). Let V be defined by*

$$V(\boldsymbol{x}) = \lim_{t\to\infty} w(\boldsymbol{x}, t; \underline{v}) \qquad \text{for all } \boldsymbol{x} \in \mathbb{R}^N.$$

Then V satisfies

$$(-\Delta - cD_N)V - f(V) = 0 \qquad \boldsymbol{x} \in \mathbb{R}^N. \tag{7}$$

with

$$\lim_{\gamma\to\infty} \sup_{\boldsymbol{x}\in D(\gamma)} |V(\boldsymbol{x}) - \underline{v}(\boldsymbol{x})| = 0,$$
$$-1 < \underline{v}(\boldsymbol{x}) < V(\boldsymbol{x}) < 1 \qquad \text{for all } \boldsymbol{x} \in \mathbb{R}^N.$$

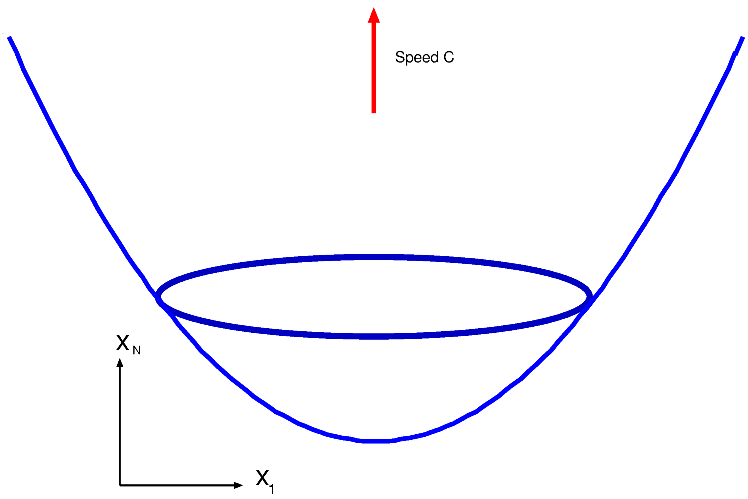

Figure 4: The graph of a level set of U.

Cylindrically symmetric traveling front $U(r, z)$ satisfies

$$\begin{aligned} \left(-D_{rr} - \frac{N-2}{r}D_r - D_{zz} - cD_z\right)U - f(U(r,z)) &= 0, && \text{for } r > 0,\ z \in \mathbb{R}, \\ U_r(0,z) &= 0 && \text{for } z \in \mathbb{R}, \\ U(0,0) &= 0. \end{aligned} \tag{8}$$

Here $D_rU = \partial U/\partial r$, $D_{rr}U = \partial^2 U/\partial r^2$, $D_zU = \partial U/\partial z$ and $D_{zz}U = \partial^2 U/\partial z^2$. See Figure 4.

The following is the main assertion in this paper.

Theorem 2 ([21]) *Let $g \in C^2(S^{N-2})$ satisfy $g(\boldsymbol{\xi}) > 0$ for all $\boldsymbol{\xi} \in S^{N-2}$. Assume that $D_g = \left\{r\boldsymbol{\xi} \,|\, 0 \le r \le g(\boldsymbol{\xi}), \boldsymbol{\xi} \in S^{N-2}\right\}$ is a convex compact set in $\mathbb{R}^{N-1}$ and all principal*

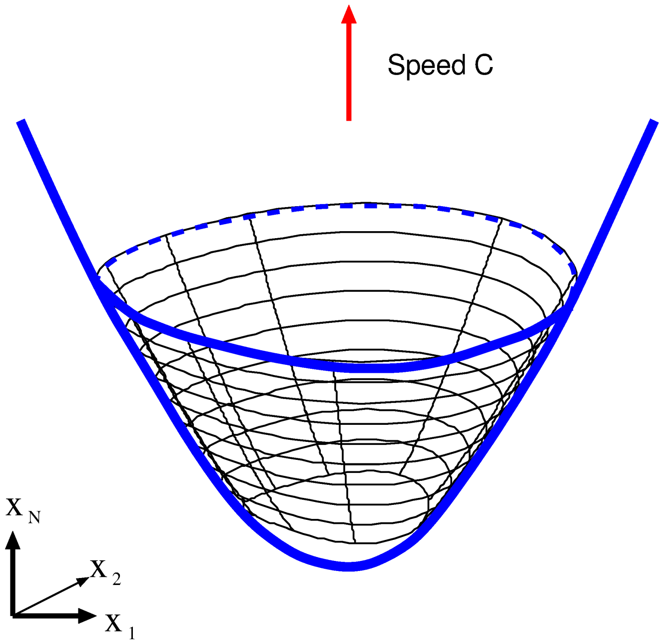

Figure 5: The graph of a level set of $\widetilde{U}$.

curvatures of $\partial D_g = \{g(\boldsymbol{\xi})\boldsymbol{\xi} \,|\, \boldsymbol{\xi} \in S^{N-2}\}$ *are positive at every point of* ∂D_g. *Then there exists a unique solution* $\widetilde{U}$ *to*

$$\left(-\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} - c\frac{\partial}{\partial x_N}\right)\widetilde{U} - f(\widetilde{U}) = 0 \quad in\ \mathbb{R}^N, \tag{9}$$

$$\lim_{s\to\infty} \sup_{|\boldsymbol{x}|\geq s} \left|\widetilde{U}(\boldsymbol{x}) - \min_{\boldsymbol{\xi}\in S^{N-2}} U(|\boldsymbol{x}' - g(\boldsymbol{\xi})\boldsymbol{\xi}|, x_N)\right| = 0. \tag{10}$$

Let g_j *satisfy the assumption stated above and let* $\widetilde{U}_j$ *be the associated solution for* $j = 1, 2$, *respectively. One has*

$$\widetilde{U}_2(x_1, \ldots, x_{N-1}, x_N) = \widetilde{U}_1(x_1, \ldots, x_{N-1}, x_N - \zeta) \tag{11}$$

for some $\zeta \in \mathbb{R}$ *if and only if* $g_1 \sim g_2$.

Let $\mathcal{G}$ be the set of all g that satisfies the assumption of Theorem 2. Let D_g be as in Theorem 2 for $g \in \mathcal{G}$. We define an equivalence relation in $\mathcal{G}$. Roughly speaking, we define $g_1 \sim g_2$ if and only if one can expand D_{g_1} with a constant width and the expanded one equals D_{g_2} or one can expand D_{g_2} with a constant width and the expanded one equals D_{g_1}. See Figure 6.

Let $g \in C^2(S^{N-2})$ satisfy $g(\boldsymbol{\xi}) > 0$ for all $\boldsymbol{\xi} \in S^{N-2}$. We set

$$\begin{aligned} C_g &= \left\{g(\boldsymbol{\xi})\boldsymbol{\xi} \,|\, \boldsymbol{\xi} \in S^{N-2}\right\}, \\ D_g &= \left\{r\boldsymbol{\xi} \,|\, 0 \leq r \leq g(\boldsymbol{\xi}), \boldsymbol{\xi} \in S^{N-2}\right\}, \end{aligned}$$

and have $C_g = \partial D_g \subset \mathbb{R}^{N-1}$. For some neighborhood of $g(\boldsymbol{\xi})\boldsymbol{\xi} \in C_g$ with $\boldsymbol{\xi} \in S^{N-2}$ we write C_g as $(\boldsymbol{y}, \psi(\boldsymbol{y}))$ with $\psi(\boldsymbol{y}^0) = 0$ and $\nabla\psi(\boldsymbol{y}^0) = \mathbf{0}$, where $\boldsymbol{y} = (y_1, \ldots, y_{N-2})$. Here we put $g(\boldsymbol{\xi})\boldsymbol{\xi} = (\boldsymbol{y}^0, \psi(\boldsymbol{y}^0))$ with $\boldsymbol{y}^0 \in \mathbb{R}^{N-2}$.

Let $\boldsymbol{\nu}(\boldsymbol{y})$ be the unit normal vector of C_g at $(\boldsymbol{y}, \psi(\boldsymbol{y}))$ pointing from D_g to $\mathbb{R}^{N-1}\backslash D_g$. We have

$$\boldsymbol{\nu}(\boldsymbol{y}) = \frac{1}{1 + |\nabla\psi(\boldsymbol{y})|^2} \begin{pmatrix} -\nabla\psi(\boldsymbol{y}) \\ 1 \end{pmatrix},$$

where

$$\nabla\psi(\boldsymbol{y}) = {}^t(D_1\psi(\boldsymbol{y}), \ldots, D_{N-2}\psi(\boldsymbol{y})).$$

The eigenvalues $\kappa_1(\boldsymbol{y}^0), \ldots, \kappa_{N-2}(\boldsymbol{y}^0)$ of the Hessian matrix

$$-D^2\psi(\boldsymbol{y}^0) = -\left(D_{ij}\psi(\boldsymbol{y}^0)\right)_{1 \le i,j \le N-2}$$

are the principal curvatures of C_g at $(\boldsymbol{y}^0, \psi(\boldsymbol{y}^0))$. We take the basis of $\mathbb{R}^{N-1}$ as the eigenvectors of the Hessian matrix. Using this principal coordinate system, we have

$$-D^2\psi(\boldsymbol{y}^0) = \operatorname{diag}\left(\kappa_1(\boldsymbol{y}^0), \ldots, \kappa_{N-2}(\boldsymbol{y}^0)\right)$$

and

$$D_j\nu_i(\boldsymbol{y}^0) = \kappa_i(\boldsymbol{y}^0)\delta_{ij} \qquad 1 \le i,j \le N-2.$$

We define $\mathcal{G}$ by

$$\left\{g \in C^2(S^{N-2}) \,|\, g \ge 0, \text{all principal curvature of } C_g \text{ are positive at every point of } C_g\right\}.$$

For any $g \in \mathcal{G}$ and $a \ge 0$ we define $g_1 = \tau_a g$ by

$$C_{g_1} = \left\{x' \in C_g \cup \left(\mathbb{R}^{N-1}\backslash D_g\right) \mid \operatorname{dist}(x', C_g) = a\right\}.$$

See Figure 6.

Then we have the following lemma.

Lemma 1 *For any $a \ge 0$, τ_a is a mapping in $\mathcal{G}$. Moreover one has*

$$\tau_b\left(\tau_a g\right) = \tau_{b+a} g \tag{12}$$

for any $a \ge 0$, $b \ge 0$ and $g \in \mathcal{G}$.

Now we define an equivalence relation $g_1 \sim g_2$ for $g_1, g_2 \in \mathcal{G}$. We define $g_1 \sim g_2$ if and only if one has either $g_1 = \tau_a g_2$ or $g_2 = \tau_a g_1$ for some $a \ge 0$. We will show that $\mathcal{G}/\sim$ gives a traveling front of (1).

Theorem 2 says that each element of a quotient set $\mathcal{G}/\sim$ gives an N-dimensional traveling front $\widetilde{U}$ in the Allen-Cahn equation. Figure 5 shows the graph of a level set $\{\boldsymbol{x} \in \mathbb{R}^N \,|\, \widetilde{U}(\boldsymbol{x}) = -a_*\}$.

We choose $\eta > 0$ large enough such that we have

$$\eta > \max_{1 \le j \le N-2} \max_{\boldsymbol{\xi} \in S^{N-2}} \frac{1}{\kappa_j(\boldsymbol{\xi})}$$

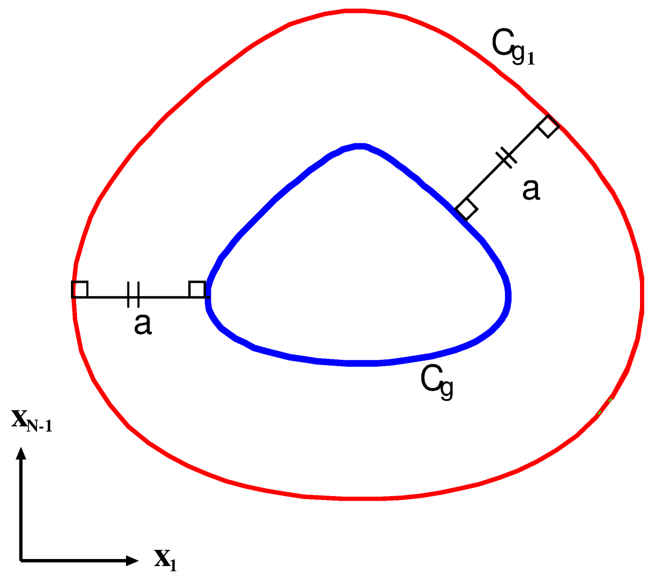

Figure 6: The graphs of C_g and C_{g_1}.

and D_g is included in the closure of a circumscribed ball of C_g at $g(\boldsymbol{\xi})\boldsymbol{\xi}$ with radius η for every $\boldsymbol{\xi} \in S^{N-2}$. Let $\boldsymbol{\nu}(\boldsymbol{\xi})$ be the unit normal vector of C_g at $g(\boldsymbol{\xi})\boldsymbol{\xi}$ pointing from D_g to $\mathbb{R}^{N-1}\backslash D_g$ for $\boldsymbol{\xi} \in S^{N-2}$.

Now we define a weak subsolution $\underline{v}(\boldsymbol{x})$ as

$$\underline{v}(\boldsymbol{x}', x_N) = \max_{\boldsymbol{\xi} \in S^{N-2}} U\left(|\boldsymbol{x}' - g(\boldsymbol{\xi})\boldsymbol{\xi} + \eta\boldsymbol{\nu}(\boldsymbol{\xi})|, x_N + m_* \eta\right) \qquad \text{for all } (\boldsymbol{x}', x_N) \in \mathbb{R}^N. \tag{13}$$

The stability of $\widetilde{U}$ is as follows.

Corollary 3 (Stability [21]) *Let $\underline{v}$ and $\widetilde{U}$ be as in (13) and Theorem 2, respectively. Let a bounded and uniformly continuous function u_0 satisfy*

$$\lim_{R \to \infty} \sup_{|\boldsymbol{x}| \geq R} |u_0(\boldsymbol{x}) - \widetilde{U}(\boldsymbol{x})| = 0,$$

$$\underline{v}(\boldsymbol{x}) \leq u_0(\boldsymbol{x}) \leq 1 \qquad \textit{for all } \boldsymbol{x} \in \mathbb{R}^N.$$

Then one has

$$\lim_{t \to \infty} \sup_{\boldsymbol{x} \in \mathbb{R}^N} |w(\boldsymbol{x}, t; u_0) - \widetilde{U}(\boldsymbol{x})| = 0.$$

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