On a Characterization of the Bilinear Forms Graphs $Bil_q(d \times d)$

Research on finite groups and their representations, vertex operator algebras, and algebraic combinatorics

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1 Introduction

Much attention has been paid to a problem of classification of all $Q$-polynomial distance-regular graphs with large diameter [1] (for the definitions, we refer the reader to Section 2). One of the steps towards solution of this problem is a characterization of known distance-regular graphs by their intersection arrays. For the current status of the classification of the $Q$-polynomial distance-regular graphs, we refer the reader to the survey paper [3] by Van Dam, Koolen and Tanaka.

The bilinear forms graph denoted here by $Bil_q(d \times n)$ is a graph defined on the set of $d \times n$-matrices over $\mathbb{F}_q$ with two matrices being adjacent if and only if the rank of their difference is 1. We refer to [2, Chapter 9.5.A] for the detailed description of these graphs.

In 1999, K. Metsch [5] obtained the following result.

Result 1.1 The bilinear forms graph $Bil_q(d \times n)$ is characterized by its intersection array if:

- $q = 2$ and $n \geq d + 4$,
- $q \geq 3$ and $n \geq d + 3$.

Thus, the open cases are:

- $q = 2$ and $n \in \{d, d + 1, d + 2, d + 3\}$,
- $q \geq 3$ and $n \in \{d, d + 1, d + 2\}$.

In this paper, we discuss a problem of characterization of the bilinear forms graphs $Bil_q(d, d)$, $d \geq 3$, by their intersection arrays.
This paper is based on a talk given at RIMS, and describes a sketch of the proof of our main result (see Section 3). The details of the proof will be given elsewhere.

2 Definitions and preliminaries

All the graphs considered in this paper are finite, undirected and simple. Suppose that $\Gamma$ is a connected graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$, where $E(\Gamma)$ consists of unordered pairs of adjacent vertices. The distance $d(x, y)$ between any two vertices $x, y$ of $\Gamma$ is the length of a shortest path connecting $x$ and $y$ in $\Gamma$.

For a subset $X$ of the vertex set of $\Gamma$, we will also write $X$ for the subgraph of $\Gamma$ induced by $X$. For a vertex $x \in V(\Gamma)$, define $\Gamma_i(x)$ to be the set of vertices which are at distance precisely $i$ from $x$ (0 $\leq$ $i$ $\leq$ $D$), where $D$ := max{$d(x, y) \mid x, y \in V(\Gamma)$} is the diameter of $\Gamma$. In addition, define $\Gamma_{-1}(x) = \Gamma_{D+1}(x) = \emptyset$. The subgraph induced by $\Gamma_1(x)$ is called the neighborhood or the local graph of a vertex $x$. The ball of radius 1 around $x$ is denoted by $x^+$, i.e. $x^+ = \{x\} \cup \Gamma_1(x)$. We write $\Gamma(x)$ instead of $\Gamma_1(x)$ for short, and we denote $x \sim y$ or simply $x \sim y$ if two vertices $x$ and $y$ are adjacent in $\Gamma$. For a graph $G$, a graph $\Gamma$ is called locally $G$ if any local graph of $\Gamma$ is isomorphic to $G$.

For a set of vertices $x_1, \ldots, x_n$, let $\Gamma(x_1, \ldots, x_n)$ denote $\bigcap_{i=1}^{n} \Gamma_1(x_i)$. Moreover, if $x$ and $y$ are at distance 2 in $\Gamma$, we call $\Gamma(x, y)$ the $\mu$-graph of $x, y$.

The eigenvalues of a graph are the eigenvalues of its adjacency matrix (recall that they are algebraic integers). If, for some eigenvalue $\eta$ of $\Gamma$, its eigenspace contains a vector orthogonal to the all ones vector, we say the eigenvalue $\eta$ is non-principal. If $\Gamma$ is regular with valency $k$ then all its eigenvalues are non-principal unless the graph is connected and then the only eigenvalue that is principal is its valency $k$.

For a graph $\Gamma$ and its vertex $x$, we say that $\eta$ is a local eigenvalue at $x$, if $\eta$ is an eigenvalue of $\Gamma_1(x)$.

A connected graph $\Gamma$ with diameter $D$ is called distance-regular if there exist integers $b_{i-1}, c_i$ (1 $\leq$ $i$ $\leq$ $D$) such that, for any two vertices $x, y \in V(\Gamma)$ with $d(x, y)$ = $i$, there are precisely $c_i$ neighbors of $y$ in $\Gamma_{i-1}(x)$ and $b_i$ neighbors of $y$ in $\Gamma_{i+1}(x)$. In particular, any distance-regular graph is regular with valency $k := b_0$. We define $a_i := k - b_i - c_i$ for notational convenience and note that $a_i = |\Gamma(y) \cap \Gamma_1(x)|$ holds for any two vertices $x, y$ with $d(x, y)$ = $i$ (1 $\leq$ $i$ $\leq$ $D$). The array $\{b_0, b_1, \ldots, b_{D-1}; c_1, c_2, \ldots, c_D\}$ is called the intersection array of the distance-regular graph $\Gamma$.

A distance-regular graph with diameter 2 is called a strongly regular graph. We say that a strongly regular graph $\Gamma$ has parameters $(\nu, k, \lambda, \mu)$, if $\nu = |V(\Gamma)|$, $k$ is its valency, $\lambda := a_1$, and $\mu := c_2$.

If a graph $\Gamma$ is distance-regular then, for all integers $h, i, j$ (0 $\leq$ $h, i, j$ $\leq$ $D$), and all vertices $x, y \in V(\Gamma)$ with $d(x, y)$ = $h$, the number

\[ p^h_{ij} := |\{z \in V(\Gamma) \mid d(x, z) = i, d(y, z) = j\}| \]
does not depend on the choice of $x, y$. The numbers $p_{ij}^h$ are called the intersection numbers of $\Gamma$.

Note that $c_i = p_{i+1}^i$, $a_i = p_{i}^i$, and $b_i = p_{1i}^i$.

For each integer $i$ ($0 \leq i \leq D$), the $i$th distance matrix $A_i$ of $\Gamma$ has rows and columns indexed by the vertex of $\Gamma$, and, for any $x, y \in V(\Gamma)$,

$$(A_i)_{x,y} = \begin{cases} 1 & \text{if } d(x, y) = i, \\ 0 & \text{if } d(x, y) \neq i. \end{cases}$$

Then $A := A_1$ is just the adjacency matrix of $\Gamma$, $A_0 = I$, $A_i^T = A_i$ ($0 \leq i \leq D$), and

$$A_i A_j = \sum_{h=0}^{D} p_{ij}^h A_h \quad (0 \leq i, j \leq D),$$

in particular,

$$A_1 A_i = b_{i-1} A_{i-1} + a_i A_i + c_{i+1} A_{i+1} \quad (1 \leq i \leq D-1),$$

$$A_1 A_D = b_{D-1} A_{D-1} + a_D A_D,$$

and this implies that $A_i = p_i(A_1)$ for certain polynomial $p_i$ of degree $i$.

The Bose-Mesner algebra $\mathcal{M}$ of $\Gamma$ is a matrix algebra generated by $A_1$ over $\mathbb{C}$. It follows that $\mathcal{M}$ has dimension $D + 1$, and it is spanned by the set of matrices $A_0 = I, A_1, \ldots, A_D$, which form a basis of $\mathcal{M}$.

Since the algebra $\mathcal{M}$ is semi-simple and commutative, $\mathcal{M}$ also has a basis of pairwise orthogonal idempotents $E_0 := \frac{1}{|V(\Gamma)|} J, E_1, \ldots, E_D$ (the so-called primitive idempotents of $\mathcal{M}$):

$$E_i E_j = \delta_{ij} E_i \quad (0 \leq i, j \leq D),$$

$$E_i = E_i^T \quad (0 \leq i, j \leq D),$$

$$E_0 + E_1 + \ldots + E_D = I,$$

where $J$ is the all ones matrix.

In fact, $E_j$ ($0 \leq j \leq D$) is the matrix representing orthogonal projection onto the eigenspace of $A_1$ corresponding to some eigenvalue of $\Gamma$. In other words, one can write

$$A_1 = \sum_{j=0}^{D} \theta_j E_j,$$

where $\theta_j$ ($0 \leq j \leq D$) are the real and pairwise distinct scalars, known as the eigenvalues of $\Gamma$. We say that the eigenvalues are in natural order if $b_0 = \theta_0 > \theta_1 > \ldots > \theta_D$. We denote $\hat{\theta}_i = -1 - \frac{b_i}{\theta_i}$ for $i \in \{1, D\}$.

The Bose-Mesner algebra $\mathcal{M}$ is also closed under entrywise (Hadamard or Schur) matrix multiplication, denoted by $\circ$. Then the matrices $A_0, A_1, \ldots, A_D$ are the primitive idempotents of $\mathcal{M}$ with respect to $\circ$, i.e., $A_i \circ A_j = \delta_{ij} A_i$, and $\sum_{i=0}^{D} A_i = J$. This implies that

$$E_i \circ E_j = \sum_{h=0}^{D} q_{ij}^h E_h \quad (0 \leq i, j \leq D)$$
holds for some real numbers $q_{ij}^{h}$, known as the Krein parameters of $\Gamma$.

Let $\Gamma$ be a distance-regular graph, and $E$ be a primitive idempotent of its Bose-Mesner algebra. The graph $\Gamma$ is called $Q$-polynomial (with respect to $E$) if there exist real numbers $c_{i}^{*}, a_{i}^{*}, b_{i-1}^{*}$ $(1 \leq i \leq D)$ and an ordering of primitive idempotents such that $E_{0} = \frac{1}{|V(\Gamma)|}J$ and $E_{1} = E$, and

$$E_{1} \circ E_{i} = b_{i-1}^{*}E_{i-1} + a_{i}^{*}E_{i} + c_{i+1}^{*}E_{i+1} \quad (1 \leq i \leq D - 1),$$

$$E_{1} \circ E_{D} = b_{D-1}^{*}E_{D-1} + a_{D}^{*}E_{D}.$$  

Note that a $Q$-polynomial ordering of the eigenvalues/idempotents does not have to be the natural ordering.

Further, the dual eigenvalues of $\Gamma$ associated with $E$ are the real scalars $\theta_{i}^{*} (0 \leq i \leq D)$ defined by

$$E = \frac{1}{|V(\Gamma)|} \sum_{i=0}^{D} \theta_{i}^{*}A_{i}.$$  

We say that a distance-regular graph $\Gamma$ has classical parameters $(D, b, \alpha, \beta)$ if the diameter of $\Gamma$ is $D$, and the intersection numbers of $\Gamma$ satisfy

$$c_{i} = \left[ \begin{array}{c} i \\ 1 \end{array} \right] \left( 1 + \alpha \left[ \begin{array}{c} i - 1 \\ 1 \end{array} \right] \right),$$  

(1)

so that, in particular, $c_{2} = (b + 1)(\alpha + 1),$

$$b_{i} = \left( \left[ \begin{array}{c} D \\ 1 \end{array} \right] - \left[ \begin{array}{c} i \\ 1 \end{array} \right] \right) \left( \beta - \alpha \left[ \begin{array}{c} i \\ 1 \end{array} \right] \right),$$  

(2)

where

$$\left[ \begin{array}{c} j \\ 1 \end{array} \right] := 1 + b + b^{2} + \ldots + b^{j-1}.$$  

The following important fact about $Q$-polynomial distance-regular graphs was proven in [7].

**Result 2.1** Let $\Gamma$ be a $Q$-polynomial distance-regular graph with diameter $D \geq 3$. Then, for any $i = 2, \ldots, D - 1$, there exists a polynomial $T_{i}$ of degree 4 such that, for any vertex $x \in V(\Gamma)$ and any non-principal eigenvalue $\eta$ of the local graph of $x$, $T_{i}(\eta) \geq 0$ holds. The polynomials $T_{i}$, $i = 2, \ldots, D - 1$, differ only in a scalar multiple.

We call these polynomials the Terwilliger polynomials of $\Gamma$. The existence of these polynomials was established in [7]. In [4], the polynomial $T_{2}$ was calculated explicitly.
**Result 2.2** Suppose that $\Gamma$ has classical parameters $(D, b, \alpha, \beta)$. Then the Terwilliger polynomial $T_2(\lambda)$ of $\Gamma$ is

$$T_2(\lambda) = \frac{-b_2}{\alpha + 1} \left( -\lambda^2 + \alpha \left[ \frac{D}{1} \right] + \beta - \alpha - 1 - (\alpha + 1)(b + 1) + \beta \left[ \frac{D}{1} \right] - (\alpha + 1)(b + 1) \right) \times \left( \lambda^2 + \lambda(2 - \alpha b) - \alpha b + 1 \right) - b_2^2 (\lambda + 1)^2. \quad (\text{3})$$

Furthermore, the roots of $T_2(\lambda)$ are

$$\beta - \alpha - 1, -1, -b - 1, \alpha b \frac{b^{D-1} - 1}{b - 1} - 1.$$

Note that the bilinear forms graph $Bil_q(d \times n)$, $n \geq d$, has classical parameters $(D, b, \alpha, \beta) = (d, q, q - 1, q^n - 1)$. In particular, if $\Gamma$ is a distance-regular graph with the same intersection array as $Bil_q(d \times d)$, $d \geq 3$, then, for any vertex $x \in V(\Gamma)$ and any non-principal eigenvalue $\eta$ of the local graph of $x$, one has:

$$\eta \in [-q - 1, -1] \text{ or } \eta = q^n - q - 1, \quad (\text{4})$$

3 Main result

In this section, we suppose that $\Gamma$ is a distance-regular graph with the same intersection array as $Bil_2(d \times d)$, $d \geq 3$.

**Proposition 3.1** The local graph of any vertex $x$ of $\Gamma$ is the $(2^d - 3) \times (2^d - 3)$-grid.

**Proof:** By (4), for $q = 2$, a local non-principal eigenvalue $\eta$ at any vertex $x \in \Gamma$ satisfies:

$$\eta \in [-3, -1] \text{ or } \eta = 2^d - 3.$$

**Claim 3.2** $\Gamma_1(x)$ has only integral eigenvalues, i.e., $-3, -2, -1, \text{ or } 2^d - 3$.

**Proof:** Recall that the eigenvalues of a graph are algebraic integers, and their product is an integer. Let $\eta_1, \ldots, \eta_s$ be all irrational eigenvalues of $\Gamma_1(x)$. Then $\eta_i \in (-3, -1)$ and $\Pi_{i=1}^s \eta_i$ is an integer, and thus $\Pi_{i=1}^s (\eta_i + 2)$ is an integer. Now $\eta_i \in (-3, -1) \Rightarrow |\eta_i + 2| < 1 \Rightarrow \Pi_{i=1}^s (\eta_i + 2) = 0$. The claim is proved.

**Claim 3.3** $\Gamma_1(x)$ has spectrum $2(2^n - 2)^1$, $(2^n - 3)^2(2^n - 2)$, $(-2)^{(2^n - 1)^2}$. 


Proof: Recall the following basic fact from algebraic graph theory. Let $\theta_0^{m_0}, \theta_1^{m_1}, \ldots, \theta_s^{m_s}$ be the spectrum of a regular (with valency $k$) graph on $v$ vertices, and $A$ be its adjacency matrix. Then:

$$\sum_{i=0}^{s} m_i = v, \quad tr(A) = \sum_{i=0}^{s} m_i \theta_i = 0, \quad tr(A^2) = \sum_{i=0}^{s} m_i \theta_i^2 = vk,$$

(5)

where we may put $\theta_0 = k$ and, moreover, $m_0 = 1$ if the graph is connected.

Apply this fact to $\Gamma_1(x)$. In our notation:

$$b_0 = v = (2^n - 1)^2, \quad \theta_0 = k = a_1 = 2(2^n - 2),$$

$$\theta_1 = 2^n - 3, \quad \theta_2 = -1, \quad \theta_3 = -2, \quad \theta_4 = -3,$$

and $m_1, m_2, m_3, m_4$ are unknown multiplicities of $\theta_1, \theta_2, \theta_3, \theta_4$, respectively, while $m_0 = 1$ (as $\Gamma_1(x)$ is connected).

Then (5) gives a system of (three) linear equations with respect to (four) unknowns $m_1, \ldots, m_4$. One can show that this system has the only non-negative integral solution:

$$m_1 = 2(2^n - 2), \quad m_2 = 0, \quad m_3 = (2^n - 1)^2, \quad m_4 = 0,$$

which shows the claim.

We now see that $\Gamma_1(x)$ is a regular graph with exactly 3 distinct eigenvalues. This yields that $\Gamma_1(x)$ is a strongly regular graph with smallest eigenvalue $-2$. It now easily follows from Seidel's classification of strongly regular graphs with smallest eigenvalue $-2$, see [9], that $\Gamma_1(x)$ is a $(2^d - 3) \times (2^d - 3)$-grid.

**Lemma 3.4** For every pair of vertices $x, y \in \Gamma$ with $d(x, y) = 2$, the induced subgraph $\Gamma(x) \cap \Gamma(y)$ is a 6-gon.

**Proof:** The lemma easily follows from Proposition 3.1 and the fact that $c_2 = 6$.

We now see that $\Gamma$ has the same local graphs as $Bil_2(d \times d)$.

Let $\mathcal{H}$ denote the bilinear forms graph $Bil_2(d \times d)$. For vertices $x \in \mathcal{H}, x \in \Gamma$, an isomorphism $\varphi : x^+ \to x^+$ is called extendable if there is a bijection $\varphi' : x^+ \cup \mathcal{H}_2(x) \to x^+ \cup \Gamma_2(x)$, mapping edges to edges, such that $\varphi'|_{x^+} = \varphi$ (in this case $\varphi'$ is called the extension of $\varphi$). We say that $\Gamma$ has distinct $\mu$-graphs if $\Gamma(x, y) = \Gamma(x, z)$ for $y, z \in \Gamma_2(x)$ implies $y = z$. This property yields that the extension $\varphi'$ above is unique.

A graph $\Delta$ is called triangulable if every cycle in it can be decomposed into a product of triangles (see [6, Section 6]).

For the following result, see [6, Theorem 7.1].
Result 3.5 Assume:

(1) $\Gamma$ has distinct $\mu$-graphs.

(2) There exist a vertex $x$ of $H$ and a vertex $x$ of $\Gamma$, and an extendable isomorphism $\varphi : x^\perp \rightarrow x^\perp$.

(3) If $x, x$ are vertices of $H, \Gamma$, respectively, $\varphi : x^\perp \rightarrow x^\perp$ is an extendable isomorphism, $\varphi'$ is its extension, and $w \in H(x)$, then $\varphi'|_w : w^\perp \rightarrow \varphi(w)^\perp$ is extendable.

(4) $H$ is triangulable.

Then $\Gamma$ is covered by $H$.

Indeed, since $\Gamma$ and $H$ have the same intersection arrays, Result 3.5 implies that $\Gamma \cong H$.

It is not difficult to see that $\Gamma$ satisfies Conditions (1) and (4) of Result 3.5.

Let $\Gamma(x) := \{w_{ij}\}_{i,j}$, and, as usually, for distinct pairs $(i, j)$ and $(i', j')$, $w_{ij} \sim w_{i'j'}$ holds if and only if $i = i'$ or $j = j'$. Denote by $L_i$ the maximal clique of $\Gamma(x)$ that contains the vertices $w_{ij}$ for all $j$, and by $L_j^T$ the maximal clique of $\Gamma(x)$ that contains the vertices $w_{ij}$ for all $i$. For a vertex $x \in \Gamma$, $x^\perp$ denotes $\{x\} \cup \Gamma(x)$.

Without loss of generality, we may assume that there is a vertex $z \in \Gamma_2(x)$ such that $\Gamma(x, z) \subset L_1 \cup L_2 \cup L_3$. Define a subgraph $\Sigma$ induced in $\Gamma$ by the vertex subset

$$\{x\} \cup L_1 \cup L_2 \cup L_3 \cup \{z' \in \Gamma_2(x) \mid \Gamma(x, z') \subset L_1 \cup L_2 \cup L_3\},$$

so that $\Sigma(x) = L_1 \cup L_2 \cup L_3$.

In order to show that $\Gamma$ satisfies Conditions (2) and (3) of Result 3.5, one has to show the following.

Lemma 3.6 $\Sigma$ is isomorphic to $\text{Bil}_2(2, d)$.

The main result of this work is the following theorem.

Theorem 3.7 The bilinear forms graphs $\text{Bil}_2(d, d)$, $d \geq 3$, are uniquely determined by their intersection arrays.

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