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Clifford theory for association schemes

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1 Introduction

Association schemes are regarded as generalizations of finite groups. So it is natural to consider the generalization to association schemes of the theory of representation of finite groups.

Let K be an algebraically closed field. Let G be a finite group, N a normal subgroup of G. The usual Clifford theory for finite groups shows that

(CF1) the restriction of an irreducible KG-module to KN is a direct sum of G-conjugates of an irreducible KN-module L with the same multiplicities;

(CF2) there exists a natural bijection between the set of irreducible KG-modules over L and the set of KT-modules over L, where T is the stabilizer of L in G;

(CF3) and there exists a natural bijection between the set of irreducible KT-modules over L and the set of irreducible modules of a generalized group algebra of T/N.

We will generalize them to association schemes. But we only consider module over the complex number field $\mathbb{C}$.

2 Adjacency algebras of association schemes

We fix some notations for association schemes.

Let (X, S) be an association scheme. We denote by σ_s the adjacency matrix of $s \in S$. The intersection number is denoted by p_{st}^u for $s, t, u \in S$, namely $\sigma_s\sigma_t = \sum_{u\in S} p_{st}^u\sigma_u$. The valency is denoted by n_s for $s \in S$. An elements in the quotient scheme $S/\!/T$ is denoted by s^T.

2.1 Generalized adjacency algebras

In this section we define generalized adjacency algebras based on a definition of generalized group algebra. Details for factor sets and generalized group algebra (ring) are available in the literature [6, Chapter 2, Section 7].

Let G be a group and let K be a field. We say that $\alpha : G \times G \to K^\times$ is a *factor set* if it satisfies the following condition:

$$\alpha(xy, z)\alpha(x, y) = \alpha(y, z)\alpha(x, yz) \text{ for all } x, y, z \in G.$$

Note that in general we can consider the action of G on K, but to simplify our arguments, we suppose that the action is trivial. Two factor sets α and β are *cohomologous* if there exists a map $\gamma : G \to K^\times$ such that

$$\alpha(x, y) = \beta(x, y)\gamma(x)\gamma(y)\gamma(xy)^{-1}$$

and we write $\alpha \sim \beta$ in this case. The relation $\sim$ is an equivalence relation on the set of factor sets. A factor set α is said to be *normalized* if $\alpha(x, 1) = \alpha(1, x) = 1$ for all $x \in G$. For a normalized factor set α, $\alpha(x, x^{-1}) = \alpha(x^{-1}, x)$ also holds. For an arbitrary factor set α, there exists a normalized factor set β such that $\beta \sim \alpha$.

Let (X, S) be an association scheme and let T be a strongly normal closed subset of S. Then the quotient $S/\!/T$ can be regarded as a finite group. Let $\alpha : S/\!/T \times S/\!/T \to K^\times$ be a factor set. We define a K-algebra $K^{(\alpha)}S = \bigoplus_{u \in S} K\sigma_u^{(\alpha)}$ with formal basis $\{\sigma_u^{(\alpha)} \mid u \in S\}$ and multiplication

$$\sigma_u^{(\alpha)}\sigma_v^{(\alpha)} = \sum_{w \in S} p_{uv}^w \alpha(u^T, v^T)\sigma_w^{(\alpha)}.$$

The algebra $K^{(\alpha)}S$ is called the *generalized adjacency algebra* of (X, S) over K with factor set α. If the strongly normal closed subset T is trivial, then the scheme is thin and the generalized adjacency algebra is just a generalized group algebra.

2.2 Graded modules and simple modules

Let K be a field. Let (X, S) be a scheme and T a strongly normal closed subset of S. Then $S/\!/T$ is thin and we can regard it as a finite group. Then

$$KS = \bigoplus_{s^T \in S/\!/T} K(TsT)$$

is an $S/\!/T$-graded K-algebra, where $K(TsT) = \bigoplus_{u \in TsT} K\sigma_u$. Obviously $(KS)_{1^T} = KT$. We can apply Dade ' s theory for KS, but we restrict our attention to the case $K = \mathbb{C}$.

Theorem 2.1. [4, Theorem 3.6] For any simple $\mathbb{C}T$-module L and $s \in S$, $L \otimes \mathbb{C}(TsT)$ is a simple $\mathbb{C}T$-module or 0.

For any simple $\mathbb{C}T$-module L, the set of $S/\!/T$-conjugates is $\{L \otimes \mathbb{C}(TsT) | s \in S, L \otimes \mathbb{C}(TsT) = 0\}$. We remark that there exist examples such that L and L' are $S/\!/T$-conjugate simple $\mathbb{C}T$-modules but their dimensions are different.

3 Clifford Theory

First we define some notations. Let A a finite-dimensional K-algebra and let B be a subalgebra of A. For a right B-module L, the induction $L \otimes_B A$ of L to A is denoted by $L \uparrow^A$. For a right A-module M, we write $M \downarrow_B$ if M is considered as a B-module. We denote by $\mathrm{IRR}(A)$ the complete set of representatives of the isomorphism classes of simple A-modules. Suppose that both A and B are semisimple. For a simple B-module L, we define $\mathrm{IRR}(A \mid L) = \{M \in \mathrm{IRR}(A) \mid \mathrm{Hom}_A(L \uparrow^A, M) \neq 0\}$.

Let (X, S) be an association scheme and let T be a closed subset of S. For a right $\mathbb{C}T$-module L and a right $\mathbb{C}S$-module M, we write $L \uparrow^S$ and $M \downarrow_T$ instead of $L \uparrow^{\mathbb{C}S}$ and $M \downarrow_{\mathbb{C}T}$, respectively.

In the rest of this section, we fix a scheme (X, S) and its strongly normal closed subset T.

Let $M \in \mathrm{IRR}(\mathbb{C}S)$. Then $M \in \mathrm{IRR}(\mathbb{C}S|L)$ for some $L \in \mathrm{IRR}(\mathbb{C}T)$. Since M is a direct summand of $L \uparrow^S$, any simple submodule of $M \downarrow_T$ is an $S/\!/T$-conjugate of L. If L and L' are $S/\!/T$-conjugate, then $L \uparrow^S \cong L' \uparrow^S$ as $\mathbb{C}S$-modules. So

$$\dim_{\mathbb{C}} \mathrm{Hom}\, \mathbb{C}T(L, M \downarrow_T) = \dim_{\mathbb{C}} \mathrm{Hom}\, \mathbb{C}T(L', M \downarrow_T).$$

This shows the following theorem.

Theorem 3.1. [4, Theorem 4.1] Let $M \in \mathrm{IRR}(\mathbb{C}S)$. There exists $L \in \mathrm{IRR}(\mathbb{C}T)$ such that $M \in \mathrm{IRR}(\mathbb{C}S \mid L)$. Then there exists a positive integer e such that

$$M \downarrow_T \cong e \left(\bigoplus_{L' \in C} L' \right),$$

where $C = \{L \otimes \mathbb{C}(TsT) \mid s \in S, L \otimes \mathbb{C}(TsT) \neq 0\}$.

Fix a simple $\mathbb{C}T$-module L. Put $U/\!/T$ the stabilizer of L in $S/\!/T$. Then

$$\bigoplus_{s^T \in S/\!/T} L \otimes \mathbb{C}(TsT) = L \otimes_{\mathbb{C}T} \mathbb{C}S \supset L \otimes_{\mathbb{C}T} \mathbb{C}U = \bigoplus_{u^T \in U/\!/T} L \otimes \mathbb{C}(TuT)$$

and, by Theorem 2.1,

$$\bigoplus_{u^T \in U/\!/T} L \otimes \mathbb{C}(TuT) \cong n_{U/\!/T} L$$

as a $\mathbb{C}T$-module. So $\dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}U}(L \uparrow^U, L \uparrow^U) = \dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}T}(L, L \uparrow^U \downarrow_T) = n_{U/\!/T}$. On the other hand, by the Frobenius reciprocity, we have

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}S}(L \uparrow^S, L \uparrow^S) = \dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}T}(L, L \uparrow^S \downarrow_T) = n_{U/\!/T}.$$

So $\dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}S}(L\uparrow^S, L\uparrow^S) = \dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}U}(L\uparrow^U, L\uparrow^U)$. Let $L\uparrow^U \cong \bigoplus_i m_i M_i$ be the irreducible decomposition of $L\uparrow^U$, with the property that $M_i \cong M_j$ if and only if $i = j$. Then

$$\begin{aligned}\dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}U}(L\uparrow^U, L\uparrow^U) &= \dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}U}(\bigoplus_i m_i M_i, \bigoplus_i m_i M_i)\\ &\leq \dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}S}(\bigoplus_i m_i M_i\uparrow^S, \bigoplus_i m_i M_i\uparrow^S)\\ &= \dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}S}(L\uparrow^S, L\uparrow^S)\end{aligned}$$

This means that $\dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}S}(M_i\uparrow^S, M_i\uparrow^S) = 1$ and $M_i\uparrow^S$ is a simple $\mathbb{C}S$-module for every i. Also $M_i\uparrow^S \cong M_j\uparrow^S$ if and only if $i = j$. Obviously $M_i \in \mathrm{IRR}(\mathbb{C}U|L)$ and $M_i\uparrow^S \in \mathrm{IRR}(\mathbb{C}S|L)$.

Conversely, let $N \in \mathrm{IRR}(\mathbb{C}S|L)$. Then N is a direct summand of $L\uparrow^S$. So there exists some M_i such that N is a direct summand of $M_i\uparrow^S$. Since $M_i\uparrow^S$ is simple, such M_i is uniquely determined. This shows the following theorem.

Theorem 3.2. [4, Theorem 4.2] Fix a simple $\mathbb{C}T$-module L. Put $U/\!/T$ the stabilizer of L in $S/\!/T$. Then there exists a bijection $\tau : \mathrm{IRR}(\mathbb{C}U|L) \to \mathrm{IRR}(\mathbb{C}S|L)$ such that $\tau(M) = M\uparrow^S$ and $\tau^{-1}(N)$ is the unique direct summand of $N\downarrow_U$ contained in $\mathrm{IRR}(\mathbb{C}U|L)$.

We consider $\mathrm{End}_{\mathbb{C}U}(L\uparrow^U)$. For $u^T \in U/\!/T$, we define $\rho_{u^T} \in \mathrm{End}_{\mathbb{C}U}(L\uparrow^U)$ by $(\rho_{u^T}(\ell))_{v^T} = \ell_{u^T v^T}$. Then $\mathrm{End}_{\mathbb{C}U}(L\uparrow^U) = \bigoplus_{u^T \in U/\!/T} \mathbb{C}\rho_{u^T}$ and this is a $U/\!/T$-graded algebra ([3, Section 4]). The multiplication is $\rho_{u^T}\rho_{v^T} = \alpha(u^T, v^T)\rho_{u^T v^T}$ and this defines a factor set α. Now $\mathrm{End}_{\mathbb{C}U}(L\uparrow^U) \cong \mathbb{C}^{(\alpha)}(U/\!/T)$ is a generalized group algebra with factor set α.

Proposition 3.3. [5, Theorem 3.1] Under the above assumptions, the irreducible $\mathbb{C}T$-module L is extensible to a $\mathbb{C}^{(\alpha^{-1})}U$-module ($\mathbb{C}^{(\alpha^{-1})}U$ is the generalized adjacency algebra with factor set α^{-1}). The action is given by $\ell\sigma_u^{(\alpha^{-1})} = \rho_{(u^T)^{-1}}(\ell\sigma_u)$ for $\ell \in L$ and $u \in U$.

We denote by $\widetilde{L}$ the extension of L to $\mathbb{C}^{(\alpha^{-1})}U$. Since L is a simple $\mathbb{C}T$-module, $\widetilde{L}$ is a simple $\mathbb{C}^{(\alpha^{-1})}U$-module.

If M is an irreducible $\mathbb{C}^{(\alpha)}(U/\!/T)$-module, then $\widetilde{L}\otimes_{\mathbb{C}} M$ is an irreducible $\mathbb{C}U$-module and is in $\mathrm{IRR}(\mathbb{C}U \mid L)$. So we can define a map $\mu : \mathrm{IRR}(\mathbb{C}^{(\alpha)}(U/\!/T)) \to \mathrm{IRR}(\mathbb{C}U \mid L)$ by

$$\mu(M) = \widetilde{L} \otimes M.$$

Then μ is a bijection. This shows the following theorem.

Theorem 3.4. [5, Theorem 3.6] Let (X, S) be an association scheme, let T be a strongly normal closed subset, and let L be an irreducible $\mathbb{C}T$-module. Let $U/\!/T$ be the stabilizer of L in $S/\!/T$. Then L is extensible to a $\mathbb{C}^{(\alpha^{-1})}U$-module $\widetilde{L}$ and the map $\mu : \mathrm{IRR}(\mathbb{C}^{(\alpha)}(U/\!/T)) \to \mathrm{IRR}(\mathbb{C}U \mid L)$ defined by $\mu(M) = \widetilde{L} \otimes_{\mathbb{C}} M$ is a bijection.

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