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Clifford theory of characters in Brauer induction

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This is joint work with **Britta Späth** [4]. In representation theory of finite groups Clifford theory plays a very important role. Here we shall discuss *extendibility* of ordinary characters of a normal subgroup N of a finite group G, by using a subgroup $G[b]$ which is a normal subgroup of G_b, where b is a p-block of N, G_b is the set of all elements in G stabilizing b by the conjugation action, and p is a prime number. The group $G[b]$ is defined by E.C. Dade in his very distinguished paper [1] of early 1970's. Actually $G[b]$ has remarkably nice properties.

The notation used here in this small note is standard. Throughout this note we assume that G is a finite group, N is its normal subgroup, and b is a p-block of N. We denote by $\mathrm{Irr}(N)$ and $\mathrm{IBr}(N)$, respectively, the set of all irreducible ordinary and Brauer characters of N. Then, we denote by $\mathrm{Irr}(b)$ and $\mathrm{IBr}(b)$, respectively, those characters belonging to b. For a subgroup H of G and a p-block B' of H, $(B')^G$ means the block induction of B' to G if it is defined. A triple $(\mathcal{K}, \mathcal{O}, k)$ is so-called a p-modular system, which is big enough for all finitely many finite groups which we are looking at, including G. Namely, $\mathcal{O}$ is a complete descrete valuation ring, $\mathcal{K}$ is the quotient field of $\mathcal{O}$, $\mathcal{K}$ and $\mathcal{O}$ have characteristic zero, and k is the residue field $\mathcal{O}/\mathrm{rad}(\mathcal{O})$ of $\mathcal{O}$ such that k has characteristic p. We mean by "big enough" above that $\mathcal{K}$ and k are both splitting fields for the finite groups mentioned above. We denote by 1_b the block idempotent of b which is a block algebra of kN (sometimes of $\mathcal{O}N$). We write $\mathrm{Bl}(G)$ and $\mathrm{Bl}(G|b)$ for the set of all p-blocks of G and for the set of all p-blocks of G coverting b, respectively. When $\chi \in \mathrm{Irr}(N)$ and $\phi \in \mathrm{IBr}(N)$, we denote by $\mathrm{bl}(\chi)$ and $\mathrm{bl}(\phi)$, respectively, the p-block of N to which χ and ϕ belong. For $\phi \in \mathrm{IBr}(N)$, we denote by $\mathrm{IBr}(G|\phi)$ the set of all characters $\psi \in \mathrm{IBr}(G)$ such that ϕ is an irreducible constituent of $\psi{\downarrow}_N$, see [8, p.155]. For the notation and terminology we shall not explain precisely, see the books of [9].

Let us keep the notation G, N and b as above throughout. Then, the group $G[b]$ is defined by [1] as follows:

$$G[b] := \{g \in G \,|\, (1_b C_{g^{-1}})(1_b C_g) = 1_b C_1\}$$

where $C_g := C_{\mathcal{O}G}(N) \cap \mathcal{O}Ng \subseteq \mathcal{O}G$ for each $g \in G$. For a p-block B of G we denote by λ_B the central function (central character) $\lambda_B : Z(kG) \to k$ associated to B, see [8, p.48]. When $g \in G$, we denote by $\mathrm{cc}_G(g)$ the conjugacy class of G which contains g, and we define $(\mathrm{cc}_G(g))^+ := \sum_{g \in \mathrm{cc}_G(g)} g \in kG$. Then, we have had several characterizations of $G[b]$. Namely,

Proposition. We have the following three kinds of characterizations of the group $G[b]$.

(i) (see [5]) $G[b] = \{g \in G_b \,|\, \exists u_g \in b^\times$ such that $g^{-1}\beta g = {u_g}^{-1}\beta u_g$ for any $\beta \in b\}$

(ii) (see [3]) $G[b] = \{g \in G_b \,|\, b \otimes_{\mathcal{O}} g \cong b$ as $\mathcal{O}[N \times N]$-modules$\}$.

(iii) (see [6]) $G[b] = \{g \in G_b \,|\, \exists y \in gN,\ \exists B' \in \mathrm{Bl}(\langle N, g\rangle)$ such that $\lambda_{B'}\Big((\mathrm{cc}_{\langle N,g\rangle}(y))^+\Big) \neq 0\}$.

The following three theorems are our main results in this note.

First, we obtain a sort of generalization of the Theorem of Harris-Knörr [2].

Theorem A. Let G be a finite group, and let $N \lhd G$, $H \leq G$ and $M := N \cap H$. Let $b' \in \mathrm{Bl}(M)$ be a block of M that has a defect group D with $C_G(D) \subseteq H$. For $b := (b')^N$ the map from $\mathrm{Bl}(H \mid b')$ to $\mathrm{Bl}(G \mid b)$ given by $B' \mapsto (B')^G$ is well-defined and surjective.

Remark. There is an exmaple where the above map in **Theorem A** is not injective, see [4].

Theorem B. Let b' be a block of M that has a defect group D with $C_G(D) \subseteq H$. Assume further that $G = G[b]$ for $b := (b')^N$. Then for every $\phi \in \mathrm{IBr}(b)$ and every $\phi' \in \mathrm{IBr}(b')$ there is a bijection

$$\Lambda : \mathrm{IBr}(G \mid \phi) \longrightarrow \mathrm{IBr}(H \mid \phi'),$$

such that $\mathrm{bl}(\Lambda(\rho))^G = \mathrm{bl}(\rho)$ for every $\rho \in \mathrm{IBr}(G \mid \phi)$. Further $\rho \in \mathrm{IBr}(G)$ is an extension of ϕ if and only if $\Lambda(\rho)$ is an extension of ϕ'.

Theorem C. Let G be a finite group, and let $N \lhd G$, $H \leq G$ and $M := N \cap H$. Let $b' \in \mathrm{Bl}(M)$ be a block of M with defect group D such that $C_G(D) \subseteq H$, and let $b := (b')^N$. Assume further that $G = G[b]$.

(i) (Ordinary characters)

(1) If $\chi' \in \mathrm{Irr}(b')$ extends to a character $\widetilde{\chi}' \in \mathrm{Irr}(H)$, then there exists a character $\chi \in \mathrm{Irr}(b)$ of height zero which extends to a character $\widetilde{\chi} \in \mathrm{Irr}(G)$ and which satisfies

$$(*) \qquad \mathrm{bl}\Big((\widetilde{\chi}')\downarrow_{J\cap H}\Big)^J = \mathrm{bl}(\widetilde{\chi}\downarrow_J) \text{ for every } J \text{ with } N \leq J \leq G.$$

(2) If $\chi \in \mathrm{Irr}(b)$ extends to a character $\widetilde{\chi} \in \mathrm{Irr}(G)$, then there exists a character $\chi' \in \mathrm{Irr}(b')$ of height zero which extends to a character $\widetilde{\chi}' \in \mathrm{Irr}(H)$ and which satisfies $(*)$.

(ii) (Sylow p-subgroups)

(1) If $\chi' \in \mathrm{Irr}(b')$ extends to a character $\widetilde{\chi}' \in \mathrm{Irr}(H)$ and if $\chi \in \mathrm{Irr}(b)$ extends to a subgroup J_0 of G with $N \leq J_0 \leq G$ and $J_0/N \in \mathrm{Syl}_p(G/N)$, then χ extends to a character $\widetilde{\chi} \in \mathrm{Irr}(G)$ which satisfies $(*)$.

(2) If $\chi \in \mathrm{Irr}(b)$ extends to a character $\widetilde{\chi} \in \mathrm{Irr}(G)$ and if $\chi' \in \mathrm{Irr}(b')$ extends to $J_0 \cap H$ for a subgroup J_0 of G with $N \leq J_0 \leq G$ and $J_0/N \in \mathrm{Syl}_p(G/N)$, then χ' extends to a character $\widetilde{\chi}' \in \mathrm{Irr}(H)$ which satisfies $(*)$.

(iii) (Brauer characters)

(1) If $\phi' \in \mathrm{IBr}(b')$ extends to a character $\widetilde{\phi}' \in \mathrm{IBr}(H)$, then any $\phi \in \mathrm{IBr}(b)$ extends to a character $\widetilde{\phi} \in \mathrm{IBr}(G)$ which satisfies

$$(**) \qquad \mathrm{bl}\Big((\widehat{\phi}')\downarrow_{J\cap H}\Big)^J = \mathrm{bl}(\widetilde{\phi}\downarrow_J) \text{ for every } J \text{ with } N \leq J \leq G.$$

(2) If $\phi \in \mathrm{IBr}(b)$ extends to a character $\widetilde{\phi} \in \mathrm{IBr}(G)$, then any $\phi' \in \mathrm{IBr}(b')$ extends to a character $\widetilde{\phi}' \in \mathrm{IBr}(H)$ which satisfies $(**)$.

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