数理解析研究所講究録
第 1926 巻 2014 年 25-34

On transversal designs and their automorphism groups

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In this talk we consider automorphism groups SCTs of transversal designs acting regularly on the set of point classes and determine the relations among SCTs, RDSs and λ-planar functions.

1 Transversal Designs and Difference Matrices

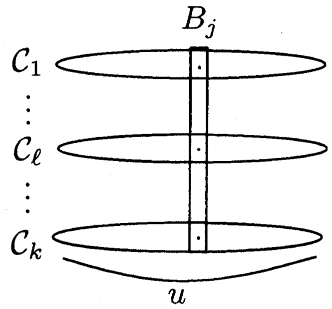
Definition 1.1. A *transversal design* $\mathrm{TD}_\lambda(k,u)$ $(u>1)$ is an incidence structure $\mathcal{D}=(\mathbb{P},\mathbb{B})$, where

(i) $\mathbb{P}$ is a set of uk points partitioned into k classes $\mathcal{C}_1,\cdots,\mathcal{C}_k$ (called *point classes*), each of size u,

(ii) $\mathbb{B}$ is a collection of k-subsets of $\mathbb{P}$ (called blocks),

(iii) Any two distinct points in the same point class are incident with no blocks and any two points in distinct point classes are incident with exactly λ blocks.

By definition, $|\mathbb{P}|=uk$, $|\mathbb{B}|=u^2\lambda$ and every block B_j of $\mathbb{B}$ intersects in each point class $\mathcal{C}_\ell$ $(1\le \ell\le k)$ in exactly one point.



$|\mathbb{P}|=uk$

$|\mathbb{B}|=u^2\lambda$

Example 1.2. Set $F=GF(q)$. Then the following is a $\mathrm{TD}_1(q,q)$.
$\mathbb{P}=F\times F,\ \mathbb{B}=\{y=ax+b\mid a,b\in F\},\ \mathfrak{C}=\{\mathcal{C}_i:=\{i\}\times F\mid i\in F\}.$

Transversal designs and their automorphism groups

Let $\mathcal{D} = (\mathbb{P}, \mathbb{B})$ be a $\mathrm{TD}_\lambda(k, u)$ with k point classes $\mathcal{C}_1, \cdots, \mathcal{C}_k$ and let U be a subgroup of $\mathrm{Aut}(\mathcal{D})$ acting regularly on each $\mathcal{C}_i$. Choose $p_i \in \mathcal{C}_i (1 \le i \le k)$ and let $B_1 U, \cdots B_{u\lambda} U$ be the U-orbits on $\mathbb{B}$. Then a $k \times u\lambda$ matrix $\begin{bmatrix} d_{1,1} & \cdots & d_{1,u\lambda} \\ \vdots & & \vdots \\ d_{k,1} & \cdots & d_{k,u\lambda} \end{bmatrix}$ defined by $p_i d_{ij} \in B_j$ $(d_{ij} \in U)$ has the following property.

$$d_{i,1} d_{\ell,1}^{-1} + \cdots + d_{i,u\lambda} d_{\ell,u\lambda}^{-1} = \lambda U \quad (\in \mathbb{Z}[U]), \quad \forall i \neq \ell$$

Difference matrices

Definition 1.3. Let U be a group of order u and $k, \lambda \in \mathbb{N}$

A $k \times u\lambda$ matrix $\begin{bmatrix} d_{1,1} & \cdots & d_{1,u\lambda} \\ \vdots & & \vdots \\ d_{k,1} & \cdots & d_{k,u\lambda} \end{bmatrix}$ $(d_{ij} \in U)$ is called a (u, k, λ)-**difference matrix** over U (a (U, k, λ)-**DM**) if

$$d_{i,1} d_{\ell,1}^{-1} + \cdots + d_{i,u\lambda} d_{\ell,u\lambda}^{-1} = \lambda U \in \mathbb{Z}[U] \quad (\forall i \neq \ell)$$

Example 1.4. The following is a $(3, 3, 1)$-DM over $(\mathbb{Z}_3, +)$.

$$M = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

Transversal designs obtained from difference matrices

Definition 1.5. Let $D = [d_{ij}]$ be a (u, k, λ)-difference matrix over a group U of order u. A transversal design $\mathrm{TD}_\lambda(k, u)$ $\mathcal{D}_D(\mathbb{P}, \mathbb{B})$ is obtained from D in the following way:
$\mathbb{P} = \{1, \cdots, k\} \times U$
$\mathbb{B} = \{\{(1, d_{1,j} g), (2, d_{2,j} g), \cdots, (k, d_{k,j} g)\} \mid 1 \le j \le u\lambda, \ g \in U\}$
We note that $\{1\} \times U, \cdots, \{k\} \times U$ is the point classes of $(\mathbb{P}, \mathbb{B})$.

Example 1.6. The following is a $\mathrm{TD}_1(3, 3)$ obtained from M in Example 1.4.
$\mathbb{P} = \{1, 2, 3\} \times \mathbb{Z}_3$,

$$\mathbb{B} = \left\{ \begin{Bmatrix} (1,0) \\ (2,0) \\ (3,0) \end{Bmatrix}, \begin{Bmatrix} (1,1) \\ (2,1) \\ (3,1) \end{Bmatrix}, \begin{Bmatrix} (1,2) \\ (2,2) \\ (3,2) \end{Bmatrix}, \begin{Bmatrix} (1,0) \\ (2,1) \\ (3,2) \end{Bmatrix}, \begin{Bmatrix} (1,1) \\ (2,2) \\ (3,0) \end{Bmatrix}, \begin{Bmatrix} (1,2) \\ (2,0) \\ (3,1) \end{Bmatrix}, \begin{Bmatrix} (1,0) \\ (2,2) \\ (3,1) \end{Bmatrix}, \begin{Bmatrix} (1,1) \\ (2,0) \\ (3,2) \end{Bmatrix}, \begin{Bmatrix} (1,2) \\ (2,1) \\ (3,0) \end{Bmatrix} \right\},$$

$\mathfrak{C}$ (the point classes) : $\{1\} \times \mathbb{Z}_3, \ \{2\} \times \mathbb{Z}_3, \ \{3\} \times \mathbb{Z}_3$.

Difference matrices and orthogonal arrays

Let $U = \{g_1, \cdots, g_u\}$ be a group of order u. A $k \times u\lambda$ (U, k, λ)-DM $D = [d_{ij}]$ is said to be *normalized* if each entry in its first row and column is equal to the identity of U.

Remark 1.7. Let notations be as mentioned above. Assume $[d_{ij}]$ is normalized. Then $(Dg_1, Dg_2, \cdots, Dg_u)$ is an $\mathrm{OA}_\lambda(k, u)$ ([13]) with entries from U. Denote by $d_i = (d_{i1}, \cdots, d_{i\,u\lambda})$ the i-th row of $[d_{ij}]$. If $\lambda = 1$, then the followimg is a set of $k-1$ mutually orthogonal Latin squares.

$$\begin{bmatrix} d_2g_1 \\ \vdots \\ d_2g_u \end{bmatrix}, \begin{bmatrix} d_3g_1 \\ \vdots \\ d_3g_u \end{bmatrix}, \cdots, \begin{bmatrix} d_kg_1 \\ \vdots \\ d_kg_u \end{bmatrix}$$

The following results on difference matrices are well known.

Result 1.8. (D. Jungnickel ([6])) If there exist a (u, k, λ)-DM then $k \leq u\lambda$.

The above result says that the $\mathrm{TD}_\lambda(k, u)$ obtained from a (u, k, λ)-DM must satisfy $k \leq u\lambda$. However, in general, the following holds.

Result 1.9. (Drake-Jungnickel [7]) If there exists a $\mathrm{TD}_\lambda(k, u)$, then
$(*)$ $k \leq (u^2\lambda - 1)/(u-1)$.

Example 1.10. Examples are known satisfying the equality in $(*)$ ([13] Proposition I.7.10). For example, there actually exist a $\mathrm{TD}_2(7, 2)$ and a $\mathrm{TD}_3(11, 2)$.

Given $u > 0$ and $\lambda > 0$, the number of rows k of a (u, k, λ)-DM over a group U of order u depends on the group of U.

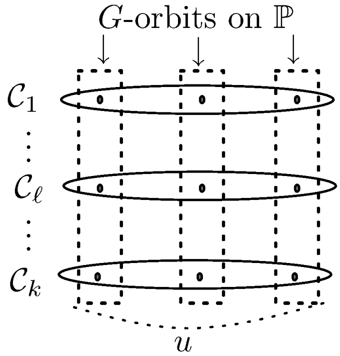
Result 1.11. (D. A. Drake [3]) Let U be any group of even order u with a cyclic Sylow 2-subgroup. If M is a (u, k, λ)-DM over U with $2 \nmid \lambda$, then $k \leq 2$.

For example, it is well known that no $(2, 2n, n)$-DM (i. e. Hadamard matrix) exists for any odd integer $n > 1$. In general, if $2 \nmid \lambda$, there exists no $(2, k, \lambda)$-DM for $k \geq 3$.

In what follows we use a notation $I_m = \{1, 2, \cdots, m\}$ for positive integer m.

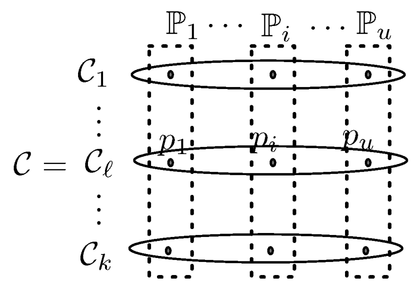
2 SCT groups

Definition 2.1. Let $\mathcal{D}(\mathbb{P},\mathbb{B})$ be a transversal design $\mathrm{TD}_\lambda(k,u)$ with the set of point classes $\mathfrak{C} = \{\mathcal{C}_i \mid i \in I_k\}$, where $|\mathbb{P}| = uk$, $|\mathbb{B}| = u^2\lambda$ and $|\mathcal{C}_i| = u, i \in I_k$. Let G be an automorphism group of $\mathcal{D}$. We say G is *class-transitive* if G is transitive on $\mathfrak{C}$. If G is a class-transitive group of order k and acts semi-regularly on $\mathbb{B}$, we say G is an *SCT*(u,k,λ) *group*. We note that G is semiregular on $\mathbb{P}$.



In the rest of this article we use the following notations.

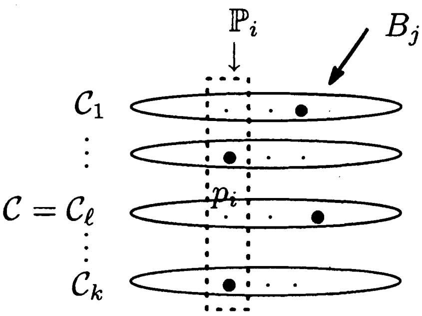
Notation 2.2. Let $\mathcal{D}(\mathbb{P},\mathbb{B})$ be a transversal design $\mathrm{TD}_\lambda(k,u)$, where $|\mathbb{P}| = uk$ and $|\mathbb{B}| = u^2\lambda$ with the set of point classes $\{\mathcal{C}_1,\cdots,\mathcal{C}_k\}$. We fix a point class $\mathcal{C}(\in \{\mathcal{C}_1,\cdots,\mathcal{C}_k\})$ of $\mathcal{D}(\mathbb{P},\mathbb{B})$. Assume a group G $(\leq \mathrm{Aut}(\mathcal{D}))$ is an $\mathrm{SCT}(u,k,\lambda)$ group of $\mathcal{D}$. Let $\mathbb{P}_1,\mathbb{P}_2,\cdots,\mathbb{P}_u$ be the G-orbits on $\mathbb{P}$ $(|\mathbb{P}|/|G| = u)$ and set $\{p_i\} = \mathbb{P}_i \cap \mathcal{C}$ for each $i \in I_u$. Moreover, let $\mathbb{B}_1,\mathbb{B}_2,\cdots,\mathbb{B}_r$ be the G-orbits on $\mathbb{B}$, where $r = |\mathbb{B}|/|G|$, and choose blocks $B_1 \in \mathbb{B}_1$, $B_2 \in \mathbb{B}_2,\cdots$ and $B_r \in \mathbb{B}_r$.



A matrix obtained from an SCT group of $\mathrm{TD}_\lambda(k,u)$

Hypothesis 2.3. Under Notation 2.2, we define a $u \times r$ matrix $M = [D_{ij}]$ $(D_{ij} \subset G)$ over G of order k in the following manner.

$$D_{ij} = \{g \in G \mid {p_i}^g \in B_j\}, \quad i \in I_u, \quad j \in I_r,$$



$$M = \begin{bmatrix} D_{11} & D_{12} & \cdots & D_{1r} \\ \vdots & \cdots & \cdots & \vdots \\ D_{u1} & D_{u2} & \cdots & D_{ur} \end{bmatrix}$$

Theorem 2.4. *Under Hypothesis 2.3, we have*

(i) $\sum_{i \in I_u} |D_{ij}| = k \quad \forall j \in I_r$ *and*

(ii) $\sum_{j \in I_r} D_{ij} D_{\ell j}{}^{(-1)} = \begin{cases} u\lambda + \lambda(G-1) & \textit{if } i = \ell, \\ \lambda(G-1) & \textit{otherwise.} \end{cases}$

We define SCT matrices.

Definition 2.5. Let G be a group of order k and $M = [D_{ij}]$ a $u \times r$ matrix over $\mathbb{Z}[G]$, where $D_{ij} \subset G$ for $i \in I_u, j \in I_r$. We say M is an *SCT*(u, k, λ) *matrix* over G if the following conditions are satisfied.

(i) $\sum_{i \in I_u} |D_{ij}| = k \quad \forall j \in I_r$

(ii) $\sum_{j \in I_r} D_{ij} D_{\ell j}{}^{(-1)} = \begin{cases} u\lambda + \lambda(G-1) & \text{if } i = \ell, \\ \lambda(G-1) & \text{otherwise.} \end{cases}$

Example 2.6. The following is an SCT$(2, 5, 5)$ over $\mathbb{Z}_5 = \langle a \rangle$.

$$\begin{bmatrix} 1 & 1+a & 1+a+a^3 & 1+a+a^2+a^3 \\ a+a^2+a^3+a^4 & a^2+a^3+a^4 & a^2+a^4 & a^4 \end{bmatrix}$$

We define an incidence structure corresponding to an SCT(u, k, λ) matrix over a group G in the following manner.

Definition 2.7. Let $M = [D_{ij}]$ be a $u \times r$ SCT(u, k, λ) matrix over a group G of order k. We define an incidence structure $\mathcal{D}_M = (\mathbb{P}, \mathbb{B})$ in the following manner.

$$\mathbb{P} = \{1, 2, \cdots, u\} \times G, \quad \mathbb{B} = \{B_{j,g} \mid j \in I_r,\ g \in G\}$$

where $B_{j,g} = (B_j)g$ and $B_j = (1, D_{1j}) \cup (2, D_{2j}) \cup \cdots \cup (u, D_{uj})$ $(\subset \mathbb{P})$.

The converse of Theorem 2.4 is true, as shown below.

Theorem 2.8. *Let M be an SCT(u, k, λ) matrix over a group $G = \{g_1, \cdots, g_k\}$ of order k and $\mathcal{D}_M = (\mathbb{P}, \mathbb{B})$ the incidence structure defined in Definition 2.7. Then the following holds.*

(i) *$\mathcal{D}_M$ is a $TD_\lambda(k, u)$ with the point classes*
$\mathcal{C}_1 = I_u \times \{g_1\}, \ \cdots, \ \mathcal{C}_k = I_u \times \{g_k\}$,

(ii) *G acts on $\mathcal{D}_M$ as an SCT(u, k, λ) group under the action*
$(i, w)g = (i, wg)$ *for* $i \in \{1, \cdots, u\}$ *and* $w, g \in G$.

We now give a result on SCT$(2, k, \lambda)$ matrices with $k = \lambda$

Proposition 2.9. *Let G be an group of order λ and let D_1, D_2, D_3, D_4 be subsets of G satisfying*

$(*)$ $D_1 D_1{}^{(-1)} + D_2 D_2{}^{(-1)} + D_3 D_3{}^{(-1)} + D_4 D_4{}^{(-1)} = \lambda + \lambda G$

Then the following is a SCT$(2, \lambda, \lambda)$ matrix over G, from which we obtain a class transitive $TD_\lambda(\lambda, 2)$:

$$M = \begin{bmatrix} D_1 & D_2 & D_3 & D_4 \\ G - D_1 & G - D_2 & G - D_3 & G - D_4 \end{bmatrix}$$

Using some difference sets we can give $\mathrm{SCT}(2,\lambda,\lambda)$ matrices.

Proposition 2.10. Let G be a group of order $v(:=4m^2)$ and D_i a (v,k_i,λ_i) difference set (DS) of order n_i $(:= k_i-\lambda_i)$ in G for $i\in\{1,2,3,4\}$. If $4m^2 = \sum\lambda_i = \sum n_i$, then $\{D_1,\cdots,D_4\}$ satisfies the condition $(*)$ and we obtain a $\mathrm{TD}_v(v,2)$ admitting G as a $\mathrm{SCT}(2,v,v)$ group.

For example, if we choose $D_1,\cdots,D_4$ as $(4m^2, 2m^2\pm m, m^2\pm m)$ DSs (Hadamard DSs), then the condition is satisfied.

Remark 2.11. For each odd integer $n>1$, there exists a $(4n^4, 2n^4\pm n^2, n^4\pm n^2)$-difference set (an Hadamard difference set of order n^4) in an abelian group of order $4n^4$ (Haemer-Xiang[10]). From this we obtain an $\mathrm{SCT}(2,4n^4,4n^4)$ group acting on a $\mathrm{TD}_{4n^4}(4n^4,2)$ applying Proposition 2.10.

Example 2.12. By computer search we can verify that there exists an $\mathrm{SCT}(2,q,q)$ matrix for $q\in\{3,5,9,11,13,17,19\}$. From this we have a $\mathrm{TD}_q(q,2)$. We note that this is unable to obtain from difference matrices applying Drake's result. For example, the following is a $\mathrm{SCT}(2,19,19)$ matrix over $\mathbb{Z}_{19}=\langle a\rangle$.

$$\begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} \\ G-D_{11} & G-D_{12} & G-D_{13} & G-D_{14}\end{bmatrix}, \text{ where}$$

$D_{11} = 1+a+a^2+a^6+a^{13}+a^{14}$,
$D_{12} = 1+a+a^2+a^3+a^4+a^5+a^6+a^9+a^{10}+a^{13}$,
$D_{13} = 1+a+a^2+a^4+a^5+a^8+a^{10}+a^{11}+a^{13}+a^{15}$, and
$D_{14} = 1+a+a^2+a^4+a^5+a^7+a^9+a^{11}+a^{12}+a^{14}+a^{15}+a^{17}$.

We also have the following result on $\mathrm{SCT}(2,k,\lambda)$ matrices with $k=2\lambda$.

Proposition 2.13. *Let G be a group of order $4m^2$. If subsets A and B of G satisfies $(*)$ $AA^{(-1)}+BB^{(-1)} = 4m^2+2m^2(G-1)$, then $\begin{bmatrix} A & B \\ G-A & G-B\end{bmatrix}$ is an $SCT(2,4m^2,2m^2)$ matrix over G.*

Example 2.14. (i) Let G be a group of order $4m^2$ and let C and D be any $(4m^2,2m^2-m,m^2-m)$ and $(4m^2,2m^2+m,m^2+m)$ difference sets of G, respectively. Then we can verify that $CC^{(-1)}+DD^{(-1)} = 4m^2+2m^2(G-1)$ and so by Proposition above we obtain an $\mathrm{SCT}(2,4m^2,2m^2)$ matrix $\begin{bmatrix} C & D \\ G-C & G-D\end{bmatrix}$ over G. From this we have a $\mathrm{TD}_{2m^2}(4m^2,2)$ admitting G as an $\mathrm{SCT}(2,4m^2,2m^2)$ automorphism group of order $4m^2$.
(ii) There are exactly 14 groups of order 16. Nine of them have $(16,6,2)$-difference sets and so have $\mathrm{SCT}(2,16,8)$ matrices by Proposition 2.13. On the other hand, five groups $\mathbb{Z}_{16}$, $\mathbb{Z}_2\times\mathbb{Z}_8$, $\mathbb{Z}_4\times\mathbb{Z}_4$, $\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_4$ and D_{16} have no difference sets. However, we can verify that each of these contains subsets A and B satisfying the condition $(*)$ of Proposition 2.13. Hence there exists an $\mathrm{SCT}(2,16,8)$ matrix over any group of order 16.

3 Spreads, SCT matrices and λ-planar functions

Definition 3.1. Let G be a group of order n^2. A set of subgroups $\{H_1, \cdots, H_{n+1}\}$ of G is called a **spread** of G if

(1) $|H_1| = \cdots |H_{n+1}| = n$ and

(2) $G = H_i H_j \quad (1 \leq \forall i \neq \forall j \leq n+1)$.

Remark 3.2. $G^* = H_1{}^* \cup H_2{}^* \cup \cdots \cup H_{n+1}{}^*$ is a disjoint union.

By Theorems 4.4.9 and 4.9.14 of [15] we can show the following. A shorter proof was communicated to the author by N. Chigira [14].

Lemma 3.3. Let G be a group of order n^2. If there exists a spread in G, Then G is an elementary abelian p-group for a prime p.

Example 3.4. Set $G = (V(2,q), +)$. Then the set of 1-dimensional $GF(q)$-subspaces $H_1, \cdots, H_{q+1}$ of $V(2,q)$ is a spread of G.

We can construct SCT$(p^m, q^2, q^2/p^m)$ matrices using a spread of an elementary abelian p-group of order q^2.

Proposition 3.5. *Let q be a power of a prime p and $G \simeq E_{q^2}$. For a spread $\mathcal{S} = \{H_1, \cdots, H_{q+1}\}$ of G, set $r = q/p^m$ $(1 < p^m \leq q)$ and $A_i = H^*_{ir+1} + H^*_{ir+2} + \cdots + H^*_{(i+1)r}$ $(0 \leq i \leq p^m - 2)$, $A_{p^m-1} = H^*_{(p^m-1)r+1} + H^*_{(p^m-1)r+2} + \cdots + H^*_{p^m \cdot r} + H^*_{p^m \cdot r+1} + 1$.*

Let $[n_{ij}]$ be any Latin square of order p^m with entries from $\{0, 1, \cdots, p^m - 1\}$. Then the following is a SCT$(p^m, q^2, q^2/p^m)$ matrix, which gives a $TD_{q^2/p^m}(q^2, p^m)$.

$$\begin{bmatrix} A_{n_{1,1}} & A_{n_{1,2}} & \cdots & A_{n_{1,p^m}} \\ A_{n_{2,1}} & A_{n_{2,2}} & \cdots & A_{n_{2,p^m}} \\ \vdots & \vdots & \vdots & \vdots \\ A_{n_{p^m,1}} & \cdots & A_{n_{p^m,p^m-1}} & A_{n_{p^m,p^m}} \end{bmatrix}$$

Definition 3.6. Let $\mathcal{G}$ be a group of order $u^2\lambda$ and $U (\lhd \mathcal{G})$ its normal subgroup of order u. A $u\lambda$-subset D of $\mathcal{G}$ is called a $(u\lambda, u, u\lambda, \lambda)$-relative difference set (RDS) relative to U if $DD^{(-1)} = u\lambda + \lambda(\mathcal{G} - U)$. The subgroup U is called a **forbidden subgroup**. We note that from U we obtain a $(u, u\lambda, \lambda)$-difference matrix over U.

Remark 3.7. Denote by $\pi(n)$ the set of primes dividing an integer $n > 1$. In the known examples $\mathcal{G}$ satisfies $\pi(|\mathcal{G}|) \in \{\{p\}, \{3,7\}, \{2,p\}\}$ for a prime p ([1],[4],[5],[8],[12]) and U is a p-group. Moreover, in most cases U is abelian.

We shall consider a relation between RDSs and SCT$(u, u\lambda, \lambda)$ matrices by generalizing the notion of planar functions.

Theorem 3.8. *Let G be a group of order $u\lambda$ and U a group of order u. Let D_y $(y \in U)$ be subsets of G. If a $u \times u$ matrix $D = [D_{yz^{-1}}]_{y,z\in U}$ over $\mathbb{Z}[G]$ whose rows and columns are indexed by the elements of U is a SCT$(u, u\lambda, \lambda)$ matrix, then the following holds.*

(i) $G = \sum_{y\in U} D_y$ *(the disjoint union of u subsets D_y).*

(ii) *A function $f : G \longrightarrow U$ defined by $f(D_y) = y$ $(y \in U)$ satisfies the following:*

$$(\star) \quad \#\{x \in G \mid f(ax)f(x)^{-1} = b\} = \lambda \quad (\forall a \in G \setminus \{1\},\ \forall b \in U)$$

Definition 3.9. Let G and U be groups. We call a function $f : G \longrightarrow U$ a **λ-planar function** if f satisfies $(\star)$.

Remark 3.10. (i) A 1-planar function is just a planar function in the usual sense (A. Pott [11]).

(ii) We can show $|G| = |U|\lambda$ by counting the number of pairs $(x, f(tx)f(x)^{-1})$ with $x \in G$ in two ways.

Proof of Theorem 3.8

As D is an SCT$(u, u\lambda, \lambda)$ matrix over G, we have $\sum_{z\in U} D_{a_1 z^{-1}} D_{a_2 z^{-1}}{}^{(-1)} = \sum_{z\in U} D_{a_1 a_2^{-1}(a_2 z^{-1})} D_{a_2 z^{-1}}{}^{(-1)}$. Hence,

$$(\star) \qquad \sum_{y\in U} D_{by} D_y{}^{(-1)} = \begin{cases} u\lambda + \lambda(G-1) & \text{if } b = 1, \\ \lambda(G-1) & \text{otherwise.} \end{cases}$$

Then, by $(\star)$, we have $\sum_{y\in G} |D_y| = u\lambda$ and $D_y \cap D_z = \phi$ $(y \neq z)$ by putting $b = 1$ and $b \neq 1$, respectively. Thus we have (i).

Let $a \in G \setminus \{1\}$ and $b \in G$ and consider the equation $f(ax)f(x)^{-1} = b$. Set $y = f(x)$. Then $f(ax) = by$. Hence,

$$f(x) = y,\ f(ax) = by \iff x \in D_y,\ ax \in D_{by}\,.$$

By $(\star)$, there exist exactly λ distinct elements $(t_i, x_i) \in D_{by_i} \times D_{y_i}$ such that $a = t_i x_i{}^{-1}$ for $i \in \{1, \cdots, \lambda\}$. As $t_i = ax_i$, $f(t_i) = by_i$ and $f(x_i) = y_i$, we have $f(ax_i)f(x_i)^{-1} = b$ and so (ii) holds. $\square$

We now show that relations among λ-planar functions, SCTs, and RDSs.

Theorem 3.11. *Let G be a group of order $u\lambda$ and U a group of order u. If $f : G \longrightarrow U$ is a λ-planar function, then the following holds.*

(i) *A $u \times u$ matrix $D = [D_{y,z}]$ defined by $D_{y,z} = f^{-1}(yz^{-1})$ $(y, z \in U)$ is an SCT$(u, u\lambda, \lambda)$ matrix.*

(ii) *A subset $D = \{(x, f(x)) \mid x \in G\}$ of $\mathcal{G} := G \times U$ is a $(u\lambda, u, u\lambda, \lambda)$ relative difference set in a group $\mathcal{G}$ relative to U.*

Proof. (i) Fix $a_1, a_2 \in U$ and let $y \in U$. Then, for any $t \in G$,
$t \in D_{a_1,y}D_{a_2,y}{}^{(-1)} \Longleftrightarrow t = x_1x_2{}^{-1},\ \exists x_1 \in D_{a_1,y},\ \exists x_2 \in D_{a_2,y}$
$\Longleftrightarrow x_1 = tx_2,\ f(tx_2) = a_1y^{-1},\ f(x_2) = a_2y^{-1},\ \exists x_2 \in D_{a_2,y}$
$\Longleftrightarrow t = x_1x_2{}^{-1},\ f(tx_2)f(x_2)^{-1} = a_1a_2{}^{-1},\ \exists x_2 \in D_{a_2,y}$. Thus,

$$\sum_{y\in U} D_{a_1,y}D_{a_2,y}{}^{(-1)} = \begin{cases} |G| + \lambda(G-1) & \text{if } a_1 = a_2, \\ \lambda(G-1) & \text{otherwise.} \end{cases}$$

(ii) $(t,b) \in (x_1, f(x_1))(x_2, f(x_2))^{-1},\ \exists x_1, x_2 \in G$
$\Longleftrightarrow t = x_1x_2{}^{-1},\ f(x_1)f(x_2)^{-1} = b,\ \exists x_1, x_2 \in G$
$\Longleftrightarrow f(tx_2)f(x_2)^{-1} = b,\ x_1 = tx_2,\ \exists x_2 \in G.$ □

Two Groups G, U corresponding to a λ-planar function f

Assume there exists a λ-planar function from G to U. Many examples are known where $|G|$ is not a power of a prime ([1],[4],[5],[8],[12]).
These satisfy $\pi(|G|) \in \{\{3,7\}, \{2,p\}\}$.
However, every known example of U is a p-group for a prime p and in the most cases U is abelian. What is the possible group theoretic structure of G or U ?
When $\lambda = 1$, the following result is known.

Result 3.12. (Blokhuis-Jungnickel-Schmidt [9]) Let G and H be abelian groups of order n. If there exists a 1-planar function from G to H, then $n = p^e$ for a prime p and the p-rank of $G \times H$ is at least $e + 1$.

We now construct a λ-planar function with λ a prime power.

Theorem 3.13. *Let p be a prime and U any group of order p^m. Let G be an elementary abelian p-group of order p^{2n} with $n \geq m$. Then there exists a p^{2n-m}-planar function from G to U.*

Proof . Let $G, q, p^m, H_i (i \in I_{q+1})$ be as in Proposition 3.5 and consider an SCT(p^m, p^{2n}, p^{2n-m}) with $q = p^n$. Let U be *any* group of order p^m $(\leq q)$ and $\cup_{y\in U} T_y$ a partition of the spread $\{H_1, \cdots, H_{q+1}\}$ such that $|T_1| = r + 1$ and $|T_y| = r$ $(y \in U^*)$, where $r = q/p^m$. Let D_y be the set of non-identity elements of T_y for $y \in U^*$ and D_1 the set of elements of T_1. Then a matrix $L = [z_{y_1,y_2}]$ defined by $z_{y_1,y_2} = y_1y_2{}^{-1}$ $(y_1, y_2 \in U)$ is a Latin square of order p^m with entries from U. Hence, by Proposition 3.5, $[D_{y_1y_2^{-1}}]_{y_1,y_2\in U}$ is an SCT(p^m, p^{2n}, p^{2n-m}) matrix, which gives a p^{2n-m}-planar function from G to U by Theorem 3.8. □

By Theorems 3.13 and 3.11, we have the following.

Theorem 3.14. *Any p-group can be a forbidden subgroup of an RDS.*

As a corollary we have the following, which gives another proof of de Launey's result on DMs (Corollary 2.8 of [2]).

Corollary 3.15. *There exists a (p^m, p^{2n}, p^{2n-m})-difference matrix over any group of oder p^m whenever $n \geq m$.*

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