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The Gröbner bases for Defining Equations of Simple $K3$ singularities with Indeterminate Exponents

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Abstract

For a certain type singularities, the defining equations are polynomials with parameter coe cients and indeterminate exponents. The forms of Gröbner bases of ideal generated by such polynomials depend on the values of parameters, and those calculations are not easy. In calculation process, we need to classify the conditions of coe cient parameters for the leading term. We are trying to calculate the Gröbner basis for defining equations of simple $K3$ singularities precisely.

1 Simple $K3$ singularities

We de ne the simple $K3$ singularities. The notion of a simple $K3$ singularity was de ned by Ishii and Watanabe [2] as a three-dimensional Gorenstein purely elliptic singularity of $(0,2)$-type, whereas a simple elliptic singularity is two-dimensional purely elliptic singularity of $(0,1)$-type.

De nition 1.1 ([3]) *Let $(X,\ x)$ be a normal isolated singularity. For any positive integer m,*

$$\delta_m(X,\ x)\ =\ \frac{dim_c\Gamma(X-\{x\},\vartheta(mK))}{L^{2/m}(X-\{x\})},$$

where K is the canonical line bundle on $X-\{x\}$, and $L^{2/m}(X-\{x\})$ is the set of all $L^{2/m}$-integrable (at x) holomorphic m-tuple n-forms on $X-\{x\}$.

Then δ_m is nite and does not depend on the choice of a Stein neighborhood on X.

De nition 1.2 ([3]) *A singularity $(X,\ x)$ is said to be purely elliptic if $\delta_m=1$ for every positive integer m.*

When X is a two-dimensional analytic space, purely elliptic singularities are quasi-Gorenstein singularities, i.e., there exists a non-vanishing holomorphic 2-form on $X - \{x\}$.

De nition-Proposition 1.3 ([2]) *A three-dimensional singularity $(X,\ x)$ is a simple $K3$ singularity if the following two equivalent conditions are satisfied:*
(1) *$(X,\ x)$ is Gorenstein purely elliptic of $(0,2)$-type.*
(2) *$(X,\ x)$ is quasi-Gorenstein and the exceptional divisor E is a normal K3 surface for any minimal resolution $\pi\ :\ (\tilde{X},\ E)\ \rightarrow\ (X,\ x)$.*

Simple elliptic singularities and cusp singularities are characterized as two-dimensional purely elliptic singularities of $(0,1)$-type and of $(0,0)$-type, respectively. The notion of a simple $K3$ singularity is de ned as a three-dimensional isolated Gorenstein purely elliptic singularity of (0,2)-type. Let $f\ \in \mathbf{C}[z_0,\ z_1,\ z_2,\ z_3]$ be a polynomial which is nondegenerate with respect to its Newton boundary $\Gamma(f)$ in the sense of [?], and whose zero locus $X\ =\ \{f\ =\ 0\}$ in $\mathbf{C}^4$ has an isolated singularity at the origin $0\ \in\ \mathbf{C}^4$. Then the condition for $(X,\ 0)$ to be a simple $K3$ singularity is given by a property of the Newton boundary $\Gamma(f)$ of f. Next we consider the case where $(X,\ x)$ is a hypersurface singularity de ned by a nondegenerate polynomial $f\ =\ \sum a_\nu z^\nu\ \in\ \mathbf{C}[z_0,\ z_1,\ \dots,\ z_n]$, and $x\ =\ 0\ \in\ \mathbf{C}^{n+1}$. We denote by $\mathbf{R_0}$ the set of all nonnegative real numbers. Recall that the Newton boundary $\Gamma(f)$ of f is the union of the compact faces of $\Gamma_+(f)$, where $\Gamma_+(f)$ is the convex hull of $\bigcup_{a_\nu\neq 0}(\nu\ +\ \mathbf{R_0^{n+1}})$ in $\mathbf{R^{n+1}}$. For any face Δ of $\Gamma_+(f)$, set $f_\Delta\ :=\ \sum_{\nu\in\Delta} a_\nu z^\nu$. We say f to be nondegenerate, if

$$\frac{\partial f_\Delta}{\partial z_0}\ =\ \frac{\partial f_\Delta}{\partial z_1}\ =\ \dots\ =\ \frac{\partial f_\Delta}{\partial z_n}\ =\ 0$$

has no solution in $(\mathbf{C}\ )^{n+1}$ for any face Δ. When f is nondegenerate, the condition for $(X,\ x)$ to be a purely elliptic singularity is given as follows:

Theorem 1.4 ([4])
Let f be a nondegenerate polynomial and suppose X has an isolated singularity at $x = 0 \in \mathbf{C}^{n+1}$.
(1) *$(X,\ x)$ is purely elliptic if and only if $(1,\ 1,\ \dots\ ,\ 1)\ \in\ \Gamma(f)$.*
(2) *Let $n\ =\ 3$ and let Δ_0 be the face of $\Gamma(f)$ containing $(1,\ 1,\ 1,\ 1)$ in the relative interior of Δ_0. Then $(X,\ x)$ is a simple $K3$ singularity if and only if $dim_R\Delta_0\ =\ 3$.*

Thus if f is nondegenerate and de nes a simple $K3$ singularity, then f_{Δ_0} is a quasi-homogeneous polynomial with a uniquely determined weights α , which called the weights of f and denoted $\alpha(f)$. We denote by $\mathbf{Q}_+$ the set of all positive rational numbers. Then $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbf{Q}_+^4$ and $deg_\alpha(\nu) := \sum_{i=1}^{4} \alpha_i \nu_i = 1$ for any $\nu \in \Delta_0$. In particular, $\sum_{i=1}^{4} \alpha_i = 1$, since $(1,1,1,1)$ is always contained in Δ_0.

2 Application of Gröbner Bases

In the elimination theory, one of basic strategy is Elimination Theorem. The calculation of Grobner basis([1]) for such polynomials is not easy. In calculation process, we need to classify conditions of parameters for the leading term. By a study of Comprehensive Grobner bases([5]), the calculation algorithm for a certain type is obtained. The following theorem holds.

Theorem 2.1 ([1]) *Let $I \subset k[x_1, ..., x_n]$ be an ideal and let G be a Gröbner basis of I with respect to lex order where $x_1 > x_2 > ... > x_n$. Then, for every $0 \leq l \leq n$, the set $G_l = G \cap k[x_{l+1}, \ldots, x_n]$ is a Gröbner basis of the lth elimination ideal I_l.*

Let f be a de ning equation, $I := < f, \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} >$. And let G be a Grobner basis of I with respect to *lex* order where $x_1 > x_2 > ... > x_n$. Then, for every $0 \leq l \leq n$, the set

$$G_l = G \cap k[x_{l+1}, \ldots, x_n]$$

is a Grobner basis of the lth elimination ideal I_l. We can obtain the non-degeneracy condition of singularity at the origin from the Grobner basis of the lth elimination ideal I_l. (The degeneracy condition of singularity at the origin means the singularity is non-isolated singularity at the origin.) In the process, we need to classify the conditions of parameters for the leading term. We consider the ideal

$$I = \{f_i(t_1, ..., t_m, x_1, ..., x_n) : 1 \leq i \leq s\}$$

in $k(t_1, ..., t_m)[x_1, ..., x_n]$ and x a monomial order. We thought of $t_1, \ldots, t_m$ as symbolic parameters appearing in the coefficients of $f_1, \ldots, f_s$. By dividing each f_i by its leading coefficient which lies in $k(t_1, \ldots, t_m)$, we assumed that the leading coefficients of the f_i are all equal to 1. Then let $g_1, \ldots, g_s$ be a reduced Grobner basis for I. Thus the leading coefficients of the g_i were also 1.

3 Calculation

We are trying to calculate the Grobner basis of the following de ning equation.

$$x^2 + y^3 + z^7 + \lambda z^6 w^6 + \mu w^{42} + w^{42+k} + xyzw = 0$$

The Groebner basis algorithm can really be applied to input polynomials that contain indeterminates like 'k' in the exponents.

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