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GEOMETRY OF POLAR ACTIONS ON COMPLEX HYPERBOLIC SPACES

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ABSTRACT. Polar actions on complex hyperbolic spaces without singular orbits have been classified by Berndt and Díaz-Ramos. In this paper, we will focus on geometry of their orbits, and mention classifications of the minimal ones and Ricci soliton ones.

1. INTRODUCTION

A submanifold of a complex hyperbolic space $\mathbb{C}H^n$ is said to be *homogeneous* if it is an orbit of an isometric action on $\mathbb{C}H^n$. Note that by an isometric action we always mean an action of a connected closed subgroup of the isometry group. Homogeneous submanifolds of $\mathbb{C}H^n$ have studied very actively and deeply (refer to, for instance, [1, 6, 11, 12, 15] and references therein). They are of particular interest from the viewpoint not only of submanifold geometry of $\mathbb{C}H^n$, but also of studies of homogeneous submanifolds in other symmetric spaces of noncompact type.

In this paper, we consider the case where isometric actions on $\mathbb{C}H^n$ are polar, and study the geometry of the orbits of such actions. An isometric action on a Riemannian manifold M is said to be *polar* if there exists a connected complete submanifold Σ of M such that Σ meets each orbit of the action and is perpendicular to each orbit at every intersection point. Note that such a submanifold Σ is called a *section* of the action, and it is always a totally geodesic submanifold (for instance, see [3, Theorem 3.2.1]). For more details of polar actions, for instance, refer to [2, 4, 5, 9, 13, 14] and references therein.

We moreover restrict to the case where polar actions on $\mathbb{C}H^n$ induce *homogeneous polar foliations*, that is, have no singular orbits. In this case, the actions have been classified by Berndt and Díaz-Ramos in [4]. They have proved that there exist exactly $2n-1$ actions which induce nontrivial homogeneous polar foliations of $\mathbb{C}H^n$ up to orbit equivalence. Recall that a homogeneous foliation of $\mathbb{C}H^n$ is said to be *trivial* if the leaves are points in $\mathbb{C}H^n$ or the leaf coincides with $\mathbb{C}H^n$. Moreover, the author have classified orbits of such polar actions up to isometric congruence, and have given explicit expressions of the orbits in [15]. The subject of this paper is to review the above results on geometry of the orbits, and also to announce our recent work on Ricci solitons (Subsection 3.2).

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Remark 1.1. Our results in this paper include the some known results in the case of cohomogeneity one actions on $\mathbb{C}\mathrm{H}^n$ in [1, 6, 12]. See Remark 3.3 and 3.12 for more details.

This paper is organized as follows. In Section 2, we recall the solvable model of a complex hyperbolic space $\mathbb{C}\mathrm{H}^n$, and review the classifications of homogeneous polar foliations of $\mathbb{C}\mathrm{H}^n$ and the orbits of such actinos. In Section 3, we mention the curvature properties of the orbits in $\mathbb{C}\mathrm{H}^n$. Firstly, we mention the minimality of the orbit and the classification of the minimal orbits, which are obtained in [15]. Secondly, we announce our recent result on homogeneous Ricci soliton submanifolds of $\mathbb{C}\mathrm{H}^n$. In particular, we classify the orbits which are Ricci solitons.

2. Preliminaries

2.1. **The solvable models of complex hyperbolic spaces.** In this subsection, we prepare the solvable models of complex hyperbolic spaces $\mathbb{C}\mathrm{H}^n$ with $n \geq 2$, which we need in the following sections. We refer mainly to [8], [11].

Definition 2.1. We call a triple $(\mathfrak{s}, \langle,\rangle, J)$ the *solvable model* of $\mathbb{C}\mathrm{H}^n$ if

(1) $\mathfrak{s} := \mathrm{span}_{\mathbb{R}}\{A_0, X_1, Y_1, \ldots, X_{n-1}, Y_{n-1}, Z_0\}$ is a Lie algebra whose bracket relations are defined by
$$[A_0, X_i] = (1/2)X_i, \ [A_0, Y_i] = (1/2)Y_i, \ [A_0, Z_0] = Z_0, \ [X_i, Y_i] = Z_0,$$
(2) $\langle,\rangle$ is an inner product on $\mathfrak{s}$ such that the above basis is orthonormal,
(3) J is a complex structure on $\mathfrak{s}$ defined by
$$J(A_0) = Z_0, \ J(Z_0) = -A_0, \ J(X_i) = Y_i, \ J(Y_i) = -X_i.$$

Let S be the simply-connected Lie group with Lie algebra $\mathfrak{s}$. One knows the expression of the complex hyperbolic space as a homogeneous space, that is,
$$\mathbb{C}\mathrm{H}^n = \mathrm{SU}(1,n)/\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(n)).$$
The Lie group S coincides with the solvable part of the Iwasawa decomposition of $\mathrm{SU}(1,n)$. Furthermore, we can naturally identify $\mathbb{C}\mathrm{H}^n$ with the Lie group S. More precisely, we have the following.

Proposition 2.2. *Denote by the same symbols $\langle,\rangle$ and J the induced left-invariant Riemannian metric and the complex structure on S, respectively. Then, $(S, \langle,\rangle, J)$ is holomorphically isometric to $\mathbb{C}\mathrm{H}^n$ with constant holomorphic sectional curvature -1.*

2.2. **Classifications of polar actions on complex hyperbolic spaces and their orbits.** In this subsection, we recall the classification of polar actions on $\mathbb{C}\mathrm{H}^n$ without singular orbits by Berndt and Díaz-Ramos. And then, we recall the classification of the orbits of such actions by the author.

First of all, we introduce Lie subgroups $S_b(\varphi)$ of S, which play essential roles in the study of polar actions on $\mathbb{C}\mathrm{H}^n$ without singular orbits. For $\varphi \in [0, \pi/2]$, let us define
$$\xi_0 := \cos(\varphi)X_1 + \sin(\varphi)A_0.$$
We mean always by $\ominus$ the orthogonal complement with respect to $\langle,\rangle$.

Definition 2.3. For $b \in \{1, \ldots, n\}$ and $\varphi \in [0, \pi/2]$, we define $\mathfrak{s}_b(\varphi)$ as follows:
(1) if $\varphi \in [0, \pi/2)$, then set

$$\mathfrak{s}_b(\varphi) := \mathfrak{s} \ominus \mathrm{span}_{\mathbb{R}}\{\xi_0, X_2, \ldots, X_b\},$$

where $b \in \{1, \ldots, n-1\}$,
(2) if $\varphi = \pi/2$, then set

$$\mathfrak{s}_b(\pi/2) := \mathfrak{s} \ominus \mathrm{span}_{\mathbb{R}}\{X_1, X_2, \ldots, X_b\},$$

where $b \in \{1, \ldots, n\}$.

One can easily check that $\mathfrak{s}_b(\varphi)$ is a Lie subalgebra of $\mathfrak{s}$ of codimension b. Note that $\mathfrak{s}_b(\pi/2)$ is nilpotent, whereas $\mathfrak{s}_b(\varphi)$ is solvable but is not nilpotent for $\varphi \in [0, \pi/2),$.

Denote by $S_b(\varphi)$ the connected Lie subgroup of S with Lie algebra $\mathfrak{s}_b(\varphi)$. Then, one can show that the action of $S_b(\varphi)$ on $\mathbb{C}\mathrm{H}^n$ induces a homogeneous polar foliation of cohomogeneity b, and its section is a totally geodesic real hyperbolic space $\mathbb{R}\mathrm{H}^b$. Furthermore, we have the following classification result.

Theorem 2.4 ([4]). *An isometric action on $\mathbb{C}\mathrm{H}^n$ induces a nontrivial homogeneous polar foliation of $\mathbb{C}\mathrm{H}^n$ if and only if it is orbit equivalent to one of the following:*
(1) *the action of $S_b(0)$, where $b \in \{1, \ldots, n-1\}$,*
(2) *the action of $S_b(\pi/2)$, where $b \in \{1, \ldots, n\}$.*

We next mention the classification of the orbits of polar actions on $\mathbb{C}\mathrm{H}^n$ without singular orbits, up to isometric congruence. Denote by o the origin of $\mathbb{C}\mathrm{H}^n$.

Theorem 2.5 ([15]). *Every orbit of polar actions on $\mathbb{C}\mathrm{H}^n$ without singular orbits is isometrically congruent to one of the following:*
(1) *the orbit $S_b(\varphi).o$, where $b \in \{1, \ldots, n-1\}$ and $\varphi \in [0, \pi/2)$,*
(2) *the orbit $S_b(\pi/2).o$, where $b \in \{1, \ldots, n\}$.*

Remark 2.6. For $p \in \mathbb{C}\mathrm{H}^n$, the orbit $S_b(0).p$ is isometrically congruent to the orbit $S_b(\varphi).o$ for some $\varphi \in [0, \pi/2)$ ([15, Proposition 4.5]). In paticular, we note that φ is explicitly given by

$$\varphi = \arcsin(\tanh(t_0/2)),$$

where t_0 is the distance between the origin o and the orbit $S_b(0).p$. On the other hand, the action of $S_b(\pi/2)$ has congruency of orbits ([15, Theorem 5.1]). Namely, for each $b \in \{1, \ldots, n\}$, every orbit of $S_b(\pi/2)$ is isometrically congruent to the orbit $S_b(\pi/2).o$. Refer to [16] for more details of the congruency of orbits.

3. Curvature properties

In this section, we study the curvature properties of the orbit $S_b(\varphi).o$.

Let us recall that $\mathbb{C}\mathrm{H}^n$ can be identified with the Lie group S. One thus can identify the submanifold $S_b(\varphi).o$ with the Lie subgroup $S_b(\varphi)$ equipped with the induced left-invariant metric. Throughout this section, therefore, we shall express the curvatures in terms of the metric Lie algebra $\mathfrak{s}_b(\varphi)$.

3.1. **The minimality.** In this subsection, we mention the minimality of the orbit $S_b(\varphi).o$ obtained in [15].

Proposition 3.1 ([15]). *The mean curvature vector $\mathcal{H}$ of $S_b(\varphi)$ is given by*

$$\mathcal{H} = (1/2)(2n - b + 1)\sin(\varphi)\xi_0,$$

Hence, the orbit $S_b(\varphi).o$ is minimal if and only if $\varphi = 0$.

Recall that $\varphi = \arcsin(\tanh(t_0/2))$, where t_0 denotes the distance between o and $S_b(0).p$. It hence follows from the monotonicity of this function that $\varphi = 0$ if and only if $t_0 = 0$. Altogether, we have the following result.

Theorem 3.2. *For $\varphi \in [0, \pi/2)$, the action of $S_b(0)$ has the unique minimal orbit $S_b(0).o$, whereas the action of $S_b(\pi/2)$ has no minimal orbits.*

Remark 3.3. In the case of cohomogeneity one, namely, $b = 1$, the actions of $S_1(\pi/2)$ and $S_1(0)$ on $\mathbb{C}H^n$ are well-known. Especially, Theorem 3.2 has been proved in [1] (also refer to [6]).

(1) The action of $S_1(\pi/2)$ induces the so-called solvable foliation, and the orbit $S_1(0).o$ is a unique minimal orbit. In fact, $S_1(0).o$ is known as the homogeneous ruled minimal hypersurface, and the other orbits are equidistant hypersurfaces to $S_1(0).o$.
(2) The action of $S_1(\pi/2)$, which is the nilpotent part of the Iwasawa decomposition of $\mathrm{SU}(1,n)$, induces the so-called horosphere foliation, and every orbit is known as a horosphere of $\mathbb{C}H^n$. In this case, the action of $S_1(\pi/2)$ has congruency of orbits, and has no minimal orbits.

The mean curvature vector is said to be *parallel* if $\nabla^{\perp}_X \mathcal{H} = 0$ holds for any $X \in \mathfrak{s}_b(\varphi)$, where $\nabla^{\perp}$ denotes the normal connection of $S_b(\varphi)$. One can obtain the following by direct calculations.

Proposition 3.4. *The mean curvature vector $\mathcal{H}$ of $S_b(\varphi)$ is always parallel.*

Remark 3.5. We note that the proposition above can be shown by the general theory of polar actions. As we mentioned before, the action of $S_b(\varphi)$ is polar, and the orbit $S_b(\varphi).o$ is a principal orbit of the action. Therefore, it follows from [3, Corollary 3.2.5] that the mean curvature vector field on $S_b(\varphi).o$ is parallel with respect to $\nabla^{\perp}$.

3.2. **Ricci solitons.** In this subsection, we announce our recent result on homogeneous Ricci soliton submanifolds of complex hyperbolic spaces $\mathbb{C}H^n$. In particular, we classify the orbits $S_b(\varphi)$ which are Ricci solitons.

First of all, let us recall the notion of Ricci solitons.

Definition 3.6. A Riemannian manifold (M, g) is called a *Ricci soliton* if its Ricci curvature ric satisfies

$$\mathrm{ric} = cg + \mathcal{L}_X g$$

for some $c \in \mathbb{R}$ and some vector field $X \in \mathfrak{X}(M)$, where $\mathcal{L}_X$ denotes the usual Lie derivative.

It is easy to see that the notion of Ricci solitons is a generalization of Einstein manifolds. We now also recall the notion of algebraic Ricci solitons, which essentially have been introduced by Lauret (see [17, 18]).

Definition 3.7. A metric Lie algebra $(\mathfrak{g}, \langle , \rangle)$ is called an *algebraic Ricci soliton* if its Ricci operator $\mathrm{Ric} : \mathfrak{g} \to \mathfrak{g}$ satisfies

$$\mathrm{Ric} = c \cdot \mathrm{id}_{\mathfrak{g}} + D \tag{3.1}$$

for some $c \in \mathbb{R}$ and some $D \in \mathrm{Der}(\mathfrak{g})$, where $\mathrm{Der}(\mathfrak{g})$ denotes the Lie algebra of derivations on $\mathfrak{g}$.

Remark 3.8. Let (G, g) be a simply-connected Lie group with a left-invariant metric, and $(\mathfrak{g}, \langle , \rangle)$ be the corresponding metric Lie algebra. Then, one knows that (G, g) is a Ricci soliton if $(\mathfrak{g}, \langle , \rangle)$ is an algebraic Ricci soliton. In addition, if G is *completely solvable*, which means that the eigenvalues of any ad_X are all real, then the converse also holds. See [18] for more details.

By direct calculations, one can see that $S_b(\varphi)$ is completely solvable. Therefore, we have only to study whether $\mathfrak{s}_b(\varphi)$ is an algebraic Ricci soliton for our goal.

Firstly, we consider the case of $\mathfrak{s}_b(\pi/2)$, that is, the nilpotent case. It is easy to show that a direct sum as metric Lie algebras of an algebraic Ricci soliton and an abelian Lie algebra is also an algebraic Ricci soliton. Hence, one has the following.

Proposition 3.9. *For each $b \in \{1, \ldots, n\}$, the metric Lie algebra $\mathfrak{s}_b(\pi/2)$ is an algebraic Ricci soliton.*

In the other cases, the structure theorem for algebraic Ricci solitons [18, Theorem. 4.8] yields the following.

Proposition 3.10. *For $b \in \{1, \ldots, n-1\}$ and $\varphi \in [0, \pi/2)$, the metric Lie algbera $\mathfrak{s}_b(\varphi)$ is an algebraic Ricci soliton if and only if $b = n-1$ and $\varphi = 0$.*

Altogether, we have the following result.

Theorem 3.11. *The orbit of $S_b(\varphi).o$ in $\mathbb{C}\mathrm{H}^n$ is a Ricci soliton if and only if*
(1) *$b = n-1$ and $\varphi = 0$, or*
(2) *$\varphi = \pi/2$.*

Remark 3.12. In the case of cohomogeneity one, Theorem 3.11 has been proved in [12]. Note that the proofs in [12], which are based on the explicit formulas for the Ricci operators, are direct and elementary.

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