Finding a Zero Path in $\mathbb{Z}_3$-Labeled Graphs[*]

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Abstract

The parity of the length of paths and cycles is a classical and well-studied topic in graph theory and theoretical computer science. A basic problem concerned with parity is to find an odd or even s–t path in an undirected graph. It can be solved in polynomial time easily, whereas its directed version is NP-hard. In this paper, as a generalization of this problem, we focus on the problem of finding an s–t path in a group-labeled graph, which is a directed graph with a group label on each arc. It is not difficult to see that finding an odd or even path in an undirected graph can be formulated as finding a zero path in a $\mathbb{Z}_2$-labeled graph.

For group-labeled graphs, efficient algorithms for finding non-zero paths or cycles with some conditions have been devised recently. On the other hand, the difficulty of finding a zero path is heavily dependent on the group, e.g., it is NP-complete to determine whether there exists an s–t path of label zero (or another specified label) in a $\mathbb{Z}$-labeled graph, but quite easy in a $\mathbb{Z}_2$-labeled graph. It is in fact known that a zero path in a Γ-labeled graph can be found in polynomial time for any constant-size abelian group Γ by getting help of the graph minor theory.

In this paper, we present a solution to finding an s–t path in a group-labeled graph with two labels forbidden. This also leads to an elementary solution to finding a zero path in a $\mathbb{Z}_3$-labeled graph, which is the first nontrivial case of finding a zero path. This case generalizes the 2-disjoint paths problem in undirected graphs, which also motivates us to consider that setting. More precisely, we provide an elementary polynomial-time algorithm for testing whether there are at most two possible labels of s–t paths in a group-labeled graph or not, and finding s–t paths attaining at least three distinct labels if exist. We also give a necessary and sufficient condition for a group-labeled graph to have exactly two possible labels of s–t paths, and our algorithm is based on this characterization.

1 Introduction

1.1 Background

The parity of the length of paths and cycles in a graph is a classical and well-studied topic in graph theory and theoretical computer science. As the simplest example, one can easily check the bipartiteness of a given undirected graph, i.e., we can determine whether it contains a cycle of odd length or not. This can be done in polynomial time also in the directed case by using the ear decomposition. It is also an important problem to test whether a given directed graph contains a directed cycle of even length or not, which is

known to be equivalent to Pólya's permanent problem [13] (see, e.g., [12]). A polynomial time algorithm for this problem was devised by Robertson, Seymour, and Thomas [14].

In this paper, we focus on paths connecting two specified vertices s and t. It is easy to test whether a given undirected graph contains an s–t path of odd (or even) length or not, whereas the same problem is NP-complete in the directed case [11] (follows from [5]). A natural generalization of this problem is to consider paths of length p modulo q. One can easily see that, when $q = 2$, both of the following problems generalize the problem of finding an odd (or even) s–t path in an undirected graph:

- Finding an s–t path of length p modulo q in an undirected graph.

- Finding an s–t path whose length is NOT p modulo q in an undirected graph, which is equivalent to determining whether all s–t paths are of length p modulo q or not.

Although these two generalizations are similar to each other, they are essentially different in the case of $q \geq 3$. In fact, a linear time algorithm for the second generalization was given by Arkin, Papadimitriou, and Yannakakis [1] for any q, whereas not so much has been known about the first generalization.

Recently, as another generalization of the parity constraints, paths and cycles in a group-labeled graph have been investigated, where a group-labeled graph is a directed graph with each arc labeled by a group element. In a group-labeled graph, the label of a walk is defined as the sum (or the ordered product if the underlying group is non-abelian) of the labels on traversed arcs, where each arc can be traversed in the converse direction and then the label is inversed (see Section 2 for the precise definition). In a similar way to paths of length p modulo q, it is natural to consider the following two problems:

(I) Finding an s–t path of label α in a group-labeled graph for a given element α.

(II) Finding an s–t path whose label is NOT α in a group-labeled graph, which is equivalent to determining whether all s–t paths are of label α or not.

Note that, when we consider Problem (I) or (II), by changing uniformly the labels of the arcs incident to s if necessary, we may assume that α is the identity of the underlying group. Hence, each problem is equivalent to finding a zero path or a non-zero path in a group-labeled graph.

If the underlying group is $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z} = (\{0,1\}, +)$ and the label of each arc is 1, then the label of a path corresponds to the parity of its length because $-1 = 1$ in $\mathbb{Z}_2$. This shows that both of these two problems generalize the problem of finding an odd (or even) s–t path in an undirected graph. We note that, in a $\mathbb{Z}_2$-labeled graph, finding an s–t path of label $\alpha \in \mathbb{Z}_2$ is equivalent to finding an s–t path whose label is not $\alpha + 1 \in \mathbb{Z}_2$, but such equivalence cannot hold for any other nontrivial group.

Problem (II) can be reduced to testing whether the group-labeled graph (precisely, the graph obtained from the original input by adding an arc from s to t with label α and by removing vertices that are not contained in any s–t path) contains a *non-zero* cycle, whose label is not the identity. With this observation, Problem (II) can be easily solved in polynomial time for any underlying group (see, e.g., [18] and Proposition 6). We note that there are several results for packing non-zero paths [2,3,18,20] and non-zero cycles [9,19] with some conditions.

On the other hand, the difficulty of Problem (I) is heavily dependent on the underlying group. When the underlying group is isomorphic to $\mathbb{Z}_2$, since Problems (I) and (II) are equivalent as discussed above, it can be easily solved in polynomial time. When the

underlying group is $\mathbb{Z}$, Problem (I) is NP-complete since the undirected Hamiltonian path problem reduces to this problem by replacing each edge with a pair of two arcs of opposite directions with label 1 and letting $\alpha := n - 1$, where n denotes the number of vertices. Huynh [8] showed the polynomial-time solvability of Problem (I) for any constant-size abelian group, which is deeply dependent on the graph minor theory.

To investigate the gap between Problems (I) and (II), we make a new approach to these problems by generalizing Problem (II) so that multiple labels are forbidden. In this paper, we provide a solution to the case that two labels are forbidden. Our result also leads to an elementary solution to the first nontrivial case of Problem (I), i.e., when the underlying group is isomorphic to $\mathbb{Z}_3 = \mathbb{Z}/3\mathbb{Z} = (\{0, \pm 1\}, +)$.

1.2 Our contribution

Let Γ be an arbitrary group. For a Γ-labeled graph G and two distinct vertices s and t, let $l(G; s, t)$ be the set of all possible labels of s-t paths in G. Our first contribution is to give a characterization of Γ-labeled graphs G with two specified vertices s, t such that $l(G; s, t) = \{\alpha, \beta\}$, where α and β are distinct elements in Γ. Roughly speaking, we show that $l(G; s, t) = \{\alpha, \beta\}$ if and only if G is obtained from planar graphs with some condition by "gluing" them together (see Section 3.2). It is interesting that the planarity, which is a topological condition, appears in the characterization.

Note that there exists an easy characterization of triplets (G, s, t) with $l(G; s, t) = \{\alpha\}$, which is used to solve Problem (II) (see Section 2.2 for details). Hence, our characterization leads to the first nontrivial classification of Γ-labeled graphs in terms of the possible labels of s-t paths, and the classification is complete when $\Gamma \simeq \mathbb{Z}_3$.

We also show an algorithmic result, which is our second contribution. Based on the fact that our characterization can be tested in polynomial time, we present a polynomial-time algorithm for testing whether $|l(G; s, t)| \le 2$ or not and finding at least three s-t paths whose labels are distinct if exist (see Theorem 11). In particular, our algorithm leads to an elementary solution to Problem (I) when $\Gamma \simeq \mathbb{Z}_3$, i.e., we can test whether $\alpha \in l(G; s, t)$ or not, and find an s-t path of label $\alpha \in \Gamma$ if exists.

Note again that our results do not depend on Γ, which can be non-abelian or infinite.

1.3 2-disjoint paths problem

Problem (I) in a $\mathbb{Z}_3$-labeled graph in fact generalizes the 2-disjoint paths problem, which also motivates us to consider the situation when two labels are forbidden. The 2-disjoint paths problem is to determine whether there exist two vertex-disjoint paths such that one connects s_1 and t_1 and the other connects s_2 and t_2 for distinct vertices s_1, s_2, t_1, t_2 in a given undirected graph. We can reduce the 2-disjoint paths problem to Problem (I) in a $\mathbb{Z}_3$-labeled graph as follows: let $s := s_1$ and $t := t_2$, replace every edge in the given graph with an arc with label 0, add one arc from t_1 to s_2 with label 1, and ask whether there exists an s-t path of label 1 or not. If the answer is YES, then there exist desired two vertex-disjoint paths, and otherwise there do not.

The 2-disjoint paths problem can be solved in polynomial time [15–17], and the following theorem characterizes the existence of two disjoint paths.

Theorem 1 (Seymour [16]). *Let $G = (V, E)$ be an undirected graph and $s_1, t_1, s_2, t_2 \in V$ distinct vertices. Then, there exist two vertex-disjoint paths P_i connecting s_i to t_i ($i =*

$1,2)$ *if and only if there is no family of disjoint vertex sets* $X_1, \ldots, X_k \subseteq V \setminus \{s_1, t_1, s_2, t_2\}$ *such that*

1. $N(X_i) \cap X_j = \emptyset$ *for distinct* $i, j \in \{1, \ldots, k\}$,

2. $|N(X_i)| \leq 3$ *for* $i = 1, \ldots, k$, *and*

3. *if* G' *is the graph obtained from* G *by deleting* X_i *and adding new edges joining every pair of distinct vertices in* $N(X_i)$ *for every* i, *then* G' *can be embedded on a plane so that* s_1, s_2, t_1, t_2 *are on the outer boundary of* G' *in this order.*

Our characterization (Theorem 14) for triplets (G, s, t) with $l(G; s, t) = \{\alpha, \beta\}$ is inspired by Theorem 1, and we use this theorem in our proof.

2   Preliminaries

2.1   Terms and notations

Throughout this paper, let Γ be a group (which can be non-abelian or infinite), for which we usually use multiplicative notation with denoting the identity by 1_Γ (also we sometimes use additive notation with denoting the identity by 0, e.g., for $\Gamma \simeq \mathbb{Z}_3$). We assume that the following operations can be done in constant time for any $\alpha, \beta \in \Gamma$: getting the inverse element $\alpha^{-1} \in \Gamma$, computing the product $\alpha\beta \in \Gamma$, and testing the identification $\alpha = \beta$. A directed graph $G = (V, E)$ with a mapping $\psi_G : E \to \Gamma$ (called a *label function*) is called a Γ-*labeled graph*.

2.1.1   Graphs

Let $G = (V, E)$ be a directed graph. A sequence $W = (v_0, e_1, v_1, e_2, v_2, \ldots, e_l, v_l)$ is called a *walk* in G if $v_0, v_1, \ldots, v_l$ are vertices, $e_1, \ldots, e_l$ are arcs, and either $e_i = v_{i-1}v_i \in E$ or $e_i = v_i v_{i-1} \in E$ for each $i = 1, \ldots, l$. A walk W is called a *path* (or, in particular, a v_0-v_l *path*) if $v_0, \ldots, v_l$ are distinct, and a *cycle* if $v_0, \ldots, v_{l-1}$ are distinct and $v_0 = v_l$. We call v_0 and v_l (which may coincide) the *end vertices of* W. Let $\bar{W}$ denote the reversed walk of W, i.e., $\bar{W} = (v_l, e_l, \ldots, v_1, e_1, v_0)$.

Let $X \subseteq V$ be a vertex set. We denote by $\delta_G(X)$ the set of arcs incident to X in G and by $N_G(X)$ the *neighbor* of X in G, i.e., $\delta_G(X) := \{e = xy \in E \mid |\{x, y\} \cap X| = 1\}$ and $N_G(X) := \{y \in V \setminus X \mid \delta_G(X) \cap \delta_G(\{y\}) \neq \emptyset\}$. We often omit the subscript G and denote a singleton $\{x\}$ by its element x if there is no confusion.

Let $G[X] := (X, E(X))$ denote the subgraph of G *induced by* X, where $E(X) := \{e = uv \in E \mid \{u, v\} \subseteq X\}$. We denote by $G - X$ the subgraph of G obtained by removing all vertices in X, i.e., $G - X = G[V \setminus X]$. For an arc set $F \subseteq E$, we also denote by $G - F$ the subgraph of G obtained by removing all arcs in F, i.e., $G - F = (V, E \setminus F)$.

A directed graph $H = (U, F)$ is called *2-connected* if $H - u$ is connected for every $u \in U$. A *2-connected component* of G is a maximal 2-connected induced subgraph $G[X]$ ($X \subseteq V$ with $|X| \geq 2$) of G. It can be efficiently computed (see, e.g., [6]).

Suppose that G is embedded on a plane. We call a unique unbounded face of G the *outer face* of G, and another face an *inner face*. For a face F of G, let $\mathrm{bd}(F)$ denote the cycle obtained by walking the boundary of F in an arbitrary direction from an arbitrary vertex.

2.1.2 Labels

Let $G = (V, E)$ be a Γ-labeled graph with a label function ψ_G, and $W = (v_0, e_1, v_1, \ldots, e_l, v_l)$ a walk in G. The *label* $\psi_G(W)$ of W is defined as the ordered product $\psi_G(e_l, v_l) \cdots \psi_G(e_2, v_2) \cdot \psi_G(e_1, v_1)$, where $\psi_G(e_i, v_i) := \psi_G(e_i)$ if $e_i = v_{i-1}v_i$ and $\psi_G(e_i, v_i) := \psi_G(e_i)^{-1}$ if $e_i = v_i v_{i-1}$. Note that, for the reversed walk $\bar{W}$ of W, we have $\psi_G(\bar{W}) = \psi_G(W)^{-1}$. In particular, since an arc uv with label α and an arc vu with label α^{-1} are equivalent, we identify such two arcs. We say that W is *balanced* (or a *zero walk*) if $\psi_G(W) = 1_\Gamma$ and *unbalanced* (or a *non-zero walk*) otherwise, and also that G is *balanced* if G contains no unbalanced cycle. Note that whether a cycle C is balanced or not does not depend on the choices of the direction and the end vertex, since $\psi_G(\bar{C}) = \psi_G(C)^{-1}$ and $\psi_G(C') = \psi_G(e_1) \cdot \psi_G(C) \cdot \psi_G(e_1)^{-1}$ for $C' := (v_1, e_2, v_2, \ldots, e_l, v_l = v_0, e_1, v_1)$. For any cycle, we consider only whether it is balanced or not in this paper, and hence we can choose the direction and the end vertex arbitrarily.

For two distinct vertices $s, t \in V$, let $l(G; s, t)$ be the set of all possible labels of s–t paths in G. If $l(G; s, t) = \{\alpha\}$ for some $\alpha \in \Gamma$, we also denote the element α itself by $l(G; s, t)$. Without loss of generality, we may assume that there is no vertex $v \in V$ such that v is not contained in any s–t path, since such a vertex does not make an effect on $l(G; s, t)$. To consider only such graphs, let $\mathcal{D}$ denote the set of all triplets (G, s, t) of a Γ-labeled graph G with two specified vertices $s, t \in V(G)$ in which every vertex is contained in some s–t path. The following lemma guarantees that one can efficiently compute a subgraph G' of G such that $(G', s, t) \in \mathcal{D}$ and $l(G'; s, t) = l(G; s, t)$.

Lemma 2. *For a connected Γ-labeled graph $G = (V, E)$ and distinct vertices $s, t \in V$, one can test whether $(G, s, t) \in \mathcal{D}$ or not in polynomial time. Moreover, one can compute, in polynomial time, the set X of vertices which are not contained in any s–t path in G.*

2.2 Finding a non-zero path

In this section, we show that a non-zero s–t path can be found (i.e., Problem (II) can be solved) efficiently by using well-known properties of Γ-labeled graphs. The following techniques are often utilized in dealing with Γ-labeled graphs (see, e.g., [2, 3, 18]).

Definition 3 (Shifting). Let $G = (V, E)$ be a Γ-labeled graph. For a vertex $v \in V$ and an element $\alpha \in \Gamma$, *shifting* (*a label function* ψ_G) *by* α *at* v means the following operation: update ψ_G to ψ_G' defined as, for each $e \in E$,

$$\psi_G'(e) := \begin{cases} \psi_G(e) \cdot \alpha^{-1} & (e \in \delta_G(v) \text{ leaves } v) \\ \alpha \cdot \psi_G(e) & (e \in \delta_G(v) \text{ enters } v) \\ \psi_G(e) & (\text{otherwise}). \end{cases}$$

Shifting at $v \in V$ does not change the label of any walk whose end vertices are not v, and neither that of any cycle C whose end vertex is v up to conjugate, i.e., $\psi_G'(C) = \alpha \cdot \psi_G(C) \cdot \alpha^{-1}$. Furthermore, when we apply shifting multiple times, the order of applications does not make any effect on the resulting label function. We say that two Γ-labeled graphs G_1 and G_2 are (s, t)-*equivalent* if there exists a sequence of labels $\alpha_v \in \Gamma$ ($v \in V \setminus \{s, t\}$) such that G_2 is obtained from G_1 by shifting by α_v at each $v \in V \setminus \{s, t\}$ (and then G_1 is obtained from G_2 by shifting by α_v^{-1} at each v). Note that $l(G_1; s, t) = l(G_2; s, t)$ if G_1 and G_2 are (s, t)-equivalent.

Lemma 4. *For a connected and balanced* Γ*-labeled graph* $G = (V, E)$ *and distinct vertices* $s, t \in V$, *there exists a* Γ*-labeled graph* G' *which is* (s, t)*-equivalent to* G *such that*

$$\psi_{G'}(e) = \begin{cases} \alpha & (e \in \delta_G(s) \text{ leaves } s), \\ \alpha^{-1} & (e \in \delta_G(s) \text{ enters } s), \\ 1_\Gamma & (\text{otherwise}), \end{cases}$$

for every arc $e \in E(G') = E$ *and for some* $\alpha \in \Gamma$ *(in fact,* $\alpha = l(G; s, t)$*).*

Lemma 5. *For any* $(G, s, t) \in \mathcal{D}$, $|l(G; s, t)| = 1$ *if and only if* G *is balanced.*

Lemmas 2 and 5 lead to the following proposition. It should be remarked that G' in Lemma 4 can be found in $O(|E|)$ time by one breadth first search and $|V| - 1$ shiftings.

Proposition 6. *Let* $G = (V, E)$ *be a* Γ*-labeled graph with a label function* ψ_G *and two specified vertices* $s, t \in V$. *Then, for any* $\alpha \in \Gamma$, *one can test whether* $l(G; s, t) \subseteq \{\alpha\}$ *or not in polynomial time. Furthermore, if* $l(G; s, t) \not\subseteq \{\alpha\}$, *then one can find an* s–t *path* P *with* $\psi_G(P) \neq \alpha$ *in polynomial time.*

2.3 New operations

For our characterization of triplets $(G, s, t) \in \mathcal{D}$ with $|l(G; s, t)| = 2$, we introduce several new operations which do not make any effect on $l(G; s, t)$. Let $(G = (V, E), s, t) \in \mathcal{D}$.

Definition 7 (3-contraction). For a vertex set $X \subseteq V \setminus \{s, t\}$ such that $|N_G(X)| = 3$ and $G_X := G[X \cup N_G(X)] - E(N_G(X))$ is connected and balanced, the *3-contraction* of X is the following operation (see Fig. 1):

- remove all vertices in X, and
- add an arc from x to y with label $l(G_X; x, y)$ (which consists of a single element by Lemma 4) for each pair of $x, y \in N_G(X)$ if there is no such arc.

The resulting graph is denoted by $G/_3 X$.

For vertex sets $X, Y, Z \subseteq V$, we say that X *separates* Y *and* Z *in* G if every two vertices $y \in Y \setminus X$ and $z \in Z \setminus X$ are in different connected components of $G - X$.

Definition 8 (2-contraction). For a vertex set $X \subseteq V$ with $x \in X$ and $y \in V \setminus X$ such that

- $G[X]$ is a balanced 2-connected component of $G - y$ (hence, $|X| \geq 2$), and
- $\{x, y\}$ separates X and $\{s, t\}$,

the *2-contraction* of X is the following operation (see Fig. 2):

- apply shifting by $l(G[X]; v, x)$ at each vertex $v \in X - x$, so that the label of every arc in $G[X]$ becomes 1_Γ (which can be checked similarly to Lemma 4),
- merge all vertices in X into a single vertex, which we refer also as x, and
- identify the parallel arcs with the same label.

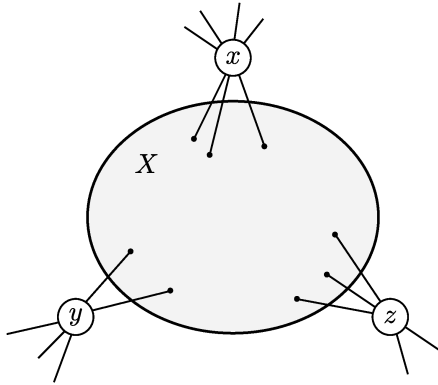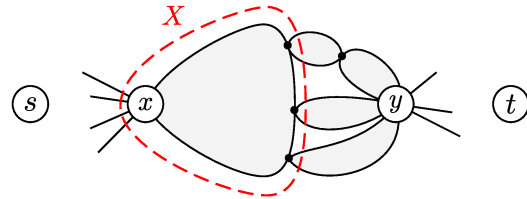The resulting graph is denoted by $G/_2 X$.

Figure 1: 3-contraction.

Figure 2: 2-contraction.

Remark. Although the 2-contraction and the 3-contraction are different operations, we use the same term *"contraction"* to refer one of these two operations because of the following correspondence. Let $X \subseteq V \setminus \{s, t\}$ be a vertex set such that $N_G(X) = \{x, y\}$ for some distinct vertices $x, y \in V$ and $G_X := G[X \cup N_G(X)] - E(N_G(X))$ is balanced. Then, by a sequence of the 2-contractions of some $X' \subseteq X + x$ with $x \in X'$ and $y \notin X'$, we can replace the balanced subgraph G_X with a single arc from x to y with label $l(G_X; x, y)$. This sequential operation and the 3-contraction are analogous to the contractions of X_i in Theorem 1.

We call a sequence of contractions a *contraction sequence*. We also say that a vertex set $X \subseteq V$ is *2-contractible* (or *3-contractible*) if the 2-contraction (or 3-contraction) of X can be performed in G. Furthermore, X is said to be *contractible* if X is 2-contractible or 3-contractible.

It should be noted that any contraction does not change $l(G; s, t)$, since each s–t path enters the removed vertex set at most once in both cases (2-contraction and 3-contraction).

Definition 9 (Replacing). For a triplet $(H, x', y') \in \mathcal{D}$ with $V(H) \cap V = \emptyset$ and parallel arcs $e_i \in E$ ($i \in I$) (possibly $|I| = 1$) from $x \in V$ to $y \in V$ with $l(H; x', y') = \{ \psi_G(e_i) \mid i \in I \}$, we say that G' is obtained from G by *replacing* e_i ($i \in I$) *with* (H, x', y') when G' is obtained from the disjoint union of G and H by removing e_i ($i \in I$) and by identifying x and y with x' and y', respectively.

The following operation can be regarded as the inverse operation of the replacing.

Definition 10 (Reduction). For a connected induced subgraph H of G with $3 \le |V(H)| < |V|$ such that $\{x, y\} \subsetneq V(H)$ separates $\{s, t\}$ and $V(H)$, the *reduction* of (H, x, y) is the following operation:

- remove all vertices in $V(H) \setminus \{x, y\}$, and

- add an arc from x to y with label α for each $\alpha \in l(H; x, y)$ if there is no such arc.

The resulting graph is denoted by $G/(H, x, y)$.

It should be remarked again that $l(G; s, t) = l(G'; s, t)$ for a Γ-labeled graph G' obtained from G by any operation shown here, since there exists an s–t path in G of label $\alpha \in \Gamma$ if and only if so does in G'.

3 Main Results

3.1 Algorithmic results

As described in Section 2.2, Problem (II) can be solved efficiently, i.e., one can find a non-zero s–t path in polynomial time (Proposition 6). The following theorem, one of our main results, is the first nontrivial extension of this property, which claims that not only one label but also another can be forbidden simultaneously.

Theorem 11. *Let $G = (V, E)$ be a Γ-labeled graph with a label function ψ_G and two specified vertices $s, t \in V$. Then, for any distinct $\alpha, \beta \in \Gamma$, one can test whether $l(G; s, t) \subseteq \{\alpha, \beta\}$ or not in polynomial time. Furthermore, if $l(G; s, t) \not\subseteq \{\alpha, \beta\}$, then one can find an s–t path P with $\psi_G(P) \notin \{\alpha, \beta\}$ in polynomial time.*

Such an algorithm is constructed based on a characterization of Γ-labeled graphs with exactly two possible labels of s–t paths shown in Section 3.2. Our algorithm is presented later in Section 4. It should be mentioned that this theorem leads to a solution to Problem (I) for $\Gamma \simeq \mathbb{Z}_3$.

Corollary 12. *Let $G = (V, E)$ be a $\mathbb{Z}_3$-labeled graph with a label function ψ_G and two specified vertices $s, t \in V$. Then one can compute $l(G; s, t)$ in polynomial time. Furthermore, for each $\alpha \in l(G; s, t)$, one can find an s–t path P with $\psi_G(P) = \alpha$ in polynomial time.*

3.2 Characterizations

Recall that $\mathcal{D}$ denotes the set of all triplets (G, s, t) such that G is a Γ-labeled graph with $s, t \in V(G)$ in which every vertex is contained in some s–t path. In this section, we provide a complete characterization of triplets $(G, s, t) \in \mathcal{D}$ with $l(G; s, t) = \{\alpha, \beta\}$ for some distinct $\alpha, \beta \in \Gamma$. We consider two cases separately: when $\alpha\beta^{-1} = \beta\alpha^{-1}$ and when $\alpha\beta^{-1} \neq \beta\alpha^{-1}$.

First, we give a characterization in the easier case: when $\alpha\beta^{-1} = \beta\alpha^{-1}$. Note that this case does not appear when $\Gamma \simeq \mathbb{Z}_3$. The following proposition holds analogously to Lemmas 4 and 5, which characterize triplets $(G, s, t) \in \mathcal{D}$ with $|l(G; s, t)| = 1$.

Proposition 13. *Let α and β be distinct elements in Γ with $\alpha\beta^{-1} = \beta\alpha^{-1}$. For any $(G, s, t) \in \mathcal{D}$, $l(G; s, t) = \{\alpha, \beta\}$ if and only if G is not balanced and there exists a Γ-labeled graph G' which is (s, t)-equivalent to G such that*

$$\psi_{G'}(e) = \begin{cases} \alpha \text{ or } \beta & (e \in \delta_G(s) \text{ leaves } s), \\ \alpha^{-1} \text{ or } \beta^{-1} & (e \in \delta_G(s) \text{ enters } s), \\ 1_\Gamma \text{ or } \alpha\beta^{-1} & (\text{otherwise}), \end{cases}$$

for every arc $e \in E(G') = E(G)$.

We next consider the much more difficult case: when $\alpha\beta^{-1} \neq \beta\alpha^{-1}$. The following theorem, one of our main results, completes a characterization of triplets $(G, s, t) \in \mathcal{D}$ with $l(G; s, t) = \{\alpha, \beta\}$ for some distinct $\alpha, \beta \in \Gamma$. The definition of the set $\mathcal{D}_{\alpha,\beta} \subseteq \mathcal{D}$, which appears in the theorem, is shown later through Definitions 15–17 in Section 3.3. In short, $(G, s, t) \in \mathcal{D}_{\alpha,\beta}$ if one can obtain, from G by a contraction sequence, a Γ-labeled graph constructed by "gluing" together planar Γ-labeled graphs with some simple conditions.

Theorem 14. *Let α and β be distinct elements in Γ with $\alpha\beta^{-1} \neq \beta\alpha^{-1}$. For any $(G, s, t) \in \mathcal{D}$, $l(G; s, t) = \{\alpha, \beta\}$ if and only if $(G, s, t) \in \mathcal{D}_{\alpha,\beta}$.*

Recall that $|l(G; s, t)| = 1$ if and only if G is balanced by Lemma 5, which can be easily tested. Hence, these characterizations lead to the first nontrivial classification of Γ-labeled graphs in terms of the number of possible labels of s–t paths, and the classification is also complete when $\Gamma \simeq \mathbb{Z}_3$.

3.3 Definition of $\mathcal{D}_{\alpha,\beta}$

Fix distinct elements $\alpha, \beta \in \Gamma$ with $\alpha\beta^{-1} \neq \beta\alpha^{-1}$. In order to characterize triplets $(G, s, t) \in \mathcal{D}$ with $l(G; s, t) = \{\alpha, \beta\}$, let us define several sets of triplets $(G, s, t) \in \mathcal{D}$ for which it is easy to see that $l(G; s, t) = \{\alpha, \beta\}$. Theorem 14 claims that any triplet $(G, s, t) \in \mathcal{D}$ with $l(G; s, t) = \{\alpha, \beta\}$ is in fact contained in one of them.

Definition 15. For distinct $\alpha, \beta \in \Gamma$, let $\mathcal{D}^0_{\alpha,\beta}$ be the set of all triplets $(G, s, t) \in \mathcal{D}$ such that

- there is no contractible vertex set $X \subseteq V(G)$ (*well-contracted*), and
- G can be embedded on a plane with the face set $\mathcal{F}$ satisfying the following properties:
  - both s and t are on the boundary of the outer face $F_0 \in \mathcal{F}$,
  - one s–t path along the boundary of F_0 is of label α and the other is of β, and
  - there exists a unique inner face F_1 whose boundary is unbalanced, i.e., $\psi_G(\mathrm{bd}(F)) = 1_\Gamma$ for any $F \in \mathcal{F} \setminus \{F_0, F_1\}$.
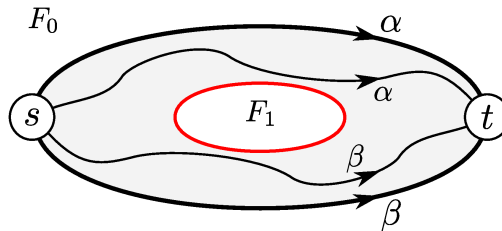

Figure 3: $(G, s, t) \in \mathcal{D}^0_{\alpha,\beta}$.

It is not difficult to see that $l(G; s, t) = \{\alpha, \beta\}$ for any triplet $(G, s, t) \in \mathcal{D}^0_{\alpha,\beta}$ (see Fig. 3).

Definition 16. For distinct $\alpha, \beta \in \Gamma$, we define $\mathcal{D}^1_{\alpha,\beta}$ as the minimal set of triplets $(G, s, t) \in \mathcal{D}$ satisfying the following conditions:

- $\mathcal{D}^0_{\alpha,\beta} \subseteq \mathcal{D}^1_{\alpha,\beta}$, and
- if $(G/(H, x, y), s, t) \in \mathcal{D}^1_{\alpha,\beta}$ (recall that $G/(H, x, y)$ denotes the Γ-labeled graph obtained from G by the reduction of (H, x, y)) for some triplet $(H, x, y) \in \mathcal{D}^0_{\alpha',\beta'}$ with distinct $\alpha', \beta' \in \Gamma$, then $(G, s, t) \in \mathcal{D}^1_{\alpha,\beta}$.

We are now ready to define $\mathcal{D}_{\alpha,\beta}$.

Definition 17. For distinct $\alpha, \beta \in \Gamma$, let $\mathcal{D}_{\alpha,\beta}$ be the set of all triplets $(G, s, t) \in \mathcal{D}$ with the following property: there exists a Γ-labeled graph $\tilde{G}$ obtained from G by a contraction sequence such that $(\tilde{G}, s, t) \in \mathcal{D}^1_{\alpha,\beta}$.

It is also easy to see that $l(G; s, t) = \{\alpha, \beta\}$ for any triplet $(G, s, t) \in \mathcal{D}_{\alpha,\beta}$ since contractions and reductions do not change $l(G; s, t)$. A proof sketch of the non-trivial direction ("only if" part of Theorem 14) is presented later in Section 5.

4 Algorithm

In this section, we present an algorithm to test whether $l(G; s, t) \subseteq \{\alpha, \beta\}$ or not for given distinct $\alpha, \beta \in \Gamma$ and to find an s–t path of label $\gamma \in \Gamma \setminus \{\alpha, \beta\}$ if $l(G; s, t) \subseteq \{\alpha, \beta\}$, in a given $\mathbb{Z}_3$-labeled graph $G = (V, E)$ with $s, t \in V$. It should be mentioned that, when $\Gamma \simeq \mathbb{Z}_3$, such an algorithm can compute $l(G; s, t)$ itself and find an s–t path of label α for each $\alpha \in l(G; s, t)$. Without loss of generality, we assume that G does not have parallel arcs with the same label.

For the simple description, we separate our algorithm into two parts: to test whether $|l(G; s, t)| \leq 2$ or not and return at most two s–t paths which attain all labels in $l(G; s, t)$ when $|l(G; s, t)| \leq 2$, and to find three s–t paths whose labels are distinct when it has turned out that $|l(G; s, t)| \geq 3$ (note that it is easy to find two s–t paths whose labels are distinct when $|l(G; s, t)| \geq 2$).

We first present an algorithm to test whether $|l(G; s, t)| \leq 2$ or not and return at most two s–t paths which attain all labels in $l(G; s, t)$ when $|l(G; s, t)| \leq 2$. It should be noted again that this algorithm can compute $l(G; s, t)$ itself when $\Gamma \simeq \mathbb{Z}_3$. Throughout this algorithm, let $G' = (V', E')$ denote a temporary graph currently considered.

TESTTWOLABELS(G, s, t)

Input A Γ-labeled graph $G = (V, E)$ and distinct vertices $s, t \in V$.

Output The set $l(G; s, t)$ of all possible labels of s–t paths in G with those which attain the labels if $|l(G; s, t)| \leq 2$, and $|l(G; s, t)| \geq 3$ otherwise.

Step 0. Compute the set X of vertices which are not contained in any s–t path by Lemma 2. If $X = V$, then halt with returning $\emptyset$ since there is no s–t path in G. Otherwise, set $G' \leftarrow G - X$. Note that $(G', s, t) \in \mathcal{D}$ and $l(G'; s, t) = l(G; s, t)$.

Step 1. Test whether G' is balanced or not by Lemma 4 (takeing an arbitrary spanning tree, and apply shifting along it). If G' is balanced, then halt with returning the label of an arbitrary s–t path in G with the path. Otherwise, by using an unbalanced cycle, obtain two s–t paths in G whose labels are distinct, say $\alpha, \beta \in \Gamma$. In the following steps, we check whether $l(G'; s, t) = \{\alpha, \beta\}$ or not. To make G' 2-connected, add to G' an arc from s to t with label α if there is no such arc.

Step 2. While G' is not 3-connected (and $|V'| \geq 3$), do the following procedure. Let $x, y \in V'$ be distinct vertices such that $G' - \{x, y\}$ is not connected. Let X be the vertex set of a connected component of $G' - \{x, y\}$ that contains none of s and t (such X exists, since s and t are adjacent in G'), and $Y := X \cup \{x, y\} \subsetneq V$. Test whether $|l(G'[Y]; x, y)| \leq 2$ or not recursively by TESTTWOLABELS$(G'[Y], x, y)$. Update $G' \leftarrow G'/(G'[Y], x, y)$ (reduction) if $|l(G'[Y]; x, y)| \leq 2$, and return $|l(G; s, t)| \geq 3$ otherwise.

Step 3. While there exists a 3-contractible vertex set $X \subseteq V' \setminus \{s, t\}$, update $G' \leftarrow G'/_3 X$ (3-contraction).

Step 4. Test whether $(G', s, t) \in \mathcal{D}_{\alpha, \beta}^0$ or not, which can be done in polynomial time by testing the planarity of G' (e.g., by [7]) and checking the embedding conditions[1] in Definition 15. Return $\{\alpha, \beta\}$ with the s–t paths in G whose labels are α and β which have been obtained in Step 1 if $(G', s, t) \in \mathcal{D}_{\alpha, \beta}^0$, and $|l(G; s, t)| \geq 3$ otherwise.

Next, we show an algorithm to find three s–t paths whose labels are distinct when it has turned out that $|l(G; s, t)| \geq 3$. Also it should be noted again that this algorithm finds three s–t paths which attain all labels when $\Gamma \simeq \mathbb{Z}_3$.

FINDTHREEPATHS(G, s, t)

Input A Γ-labeled graph $G = (V, E)$ and distinct vertices $s, t \in V$ with $|l(G; s, t)| \geq 3$.

Output Three s–t paths in G whose labels are distinct.

Step 0. If $V = \{s, t\}$, then halt with returning three s–t paths each of which consists of a single arc $st \in E$. Note that E consists of at least three parallel arcs st with distinct labels.

Step 1. For each $s' \in N_G(s) - t$, by TESTTWOLABELS$(G - s, s', t)$, test whether $|l(G - s; s', t)| \leq 2$ or not.

Step 2. If $|l(G - s; s', t)| \leq 2$ for all $s' \in N_G(s) - t$, then we have already obtained s'–t paths which attain all labels in $l(G - s; s', t)$. Choose three s–t paths whose labels are distinct among the s–t paths obtained by extending such s'–t paths using an arc (possibly parallel arcs) $ss' \in E$ for each $s' \in N_G(s) - t$ and the s–t paths each of which consists of a single arc $st \in E$, and halt with returning them.

Step 3. Otherwise, for at least one $\tilde{s} \in N_G(s) - t$, we obtained $|l(G - s; \tilde{s}, t)| \geq 3$. Then, find three $\tilde{s}$–t paths whose labels are distinct by FINDTHREEPATHS$(G - s, \tilde{s}, t)$. Extend the three $\tilde{s}$–t paths using an arc $s\tilde{s} \in E$, and return the extended s–t paths.

5 Proof Sketch of Necessity of Theorem 14

In this section, we describe an outline of our proof of the necessity of Theorem 14.

To derive a contradiction, assume that there exist distinct $\alpha, \beta \in \Gamma$ and a triplet $(G, s, t) \in \mathcal{D}$ such that $l(G; s, t) = \{\alpha, \beta\}$ and $(G, s, t) \notin \mathcal{D}_{\alpha, \beta}$. We choose such $\alpha, \beta \in \Gamma$ and $(G, s, t) \in \mathcal{D}$ so that G is as small as possible.

Fix an arbitrary arc e_0 in G leaving s, and consider the graph $G' := G - e_0$. By using the minimality of G, we can show that $(G', s, t) \in \mathcal{D}_{\alpha, \beta}$. Furthermore, we can see that a triplet $(\tilde{G}, s, t) \in \mathcal{D}_{\alpha, \beta}^0$ is obtained from $(G', s, t) \in \mathcal{D}_{\alpha, \beta}$ by applying a contraction at most once.

By the definition of $\mathcal{D}_{\alpha, \beta}^0$, $\tilde{G}$ can be embedded on a plane satisfying the conditions in Definition 15. By using the fact that G is obtained from $\tilde{G}$ by expanding a vertex set and adding e_0, we try to extend the planar embedding of $\tilde{G}$ to G. Then, we have one of the following cases.

- Such an extension is possible, i.e., G can be embedded on a plane satisfying the conditions in Definition 15. This contradicts that $(G, s, t) \notin \mathcal{D}_{\alpha, \beta}$.

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[1]The face set is almost unique due to the 3-connectivity (see, e.g., [4, Chapter 4])

- $\tilde{G}$ contains a contractible vertex set, which contradicts that $(\tilde{G}, s, t) \in \mathcal{D}^0_{\alpha,\beta}$.

- G contains a contractible vertex set or a 2-cut $\{x, y\} \subsetneq V$ (i.e., $G - \{x, y\}$ is not connected for some distinct vertices $x, y \in V$), which contradicts that G is a minimal counterexample.

- We can construct an s–t path of label $\gamma \in \Gamma \setminus \{\alpha, \beta\}$ in G by using e_0 and some arcs in G', which contradicts that $l(G; s, t) = \{\alpha, \beta\}$.

In each case, we have a contradiction, which completes the proof of the necessity of Theorem 14. We note that Theorem 1 plays an important role in this case analysis.

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