数理解析研究所講究録
第 1933 巻 2015 年 32-43



# Optimal policy for two-tier pension system*

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**Abstract**

Growing aging population with the low fertility has brought a severe picture to maintain the pension scheme in near future. The financial viability of public pension requires the reserve should be positive to pay the benefit in the demographic and economical environment change subject to maintain the certain level of the income replacement ratio. The two-tier public pension has a scheme which consists of constant and wage-proportional benefit and premium. The policy depends only four variables of premium and benefit for two schemes but the difficulty exits in the long time decision for life span and the economic equality of various cohorts in the uncertain future environment.

Assuming the price of market asset and the average wage follow stochastic processes, we maximize the net present value of pension for the cohort. To guarantee the pension fund viability, we obtain conditions by the martingale method of the optimal consumption and investment theory. For the pension population change, we consider the condition for various cohorts to equalized the net present value of cost benefit in the two-tier pension system.

**Keywords:** two-tier public pension, portfolio risk management, population cohort

## 1 Introduction

Japanese government has started two-tier pension system since 1986. The first-tier is called National Pension (NP) where all residents aged 20-59 contribute a constant premium compulsorily. The second-tier is called wage-Proportional Pension (PP) where the employer pays the half of the premium. The benefit of pension also are separated as 2-tier; one is called Basic pension which are equally payed to all including dependent spouses of wage-proportional pensioners, who did not pay explicitly any contribution. The benefit of wage-proportional is payed on the basis of qualifying years and the total amount of contribution.

The aging society with the low birth rate makes worse the balance of pension account in the near future with the prolonged deflationary economy in Japan. Japanese government has announced every five years the actuarial valuation of pension plan for 100 years, as seen [5] and [6]. It reviewed the long term financial viability under significant changes in the demographic and economical environment. In 2014, the new actuarial valuation of pension plan has been reviewed in policy alternatives, so called, Macro-economic slide adjustment where premium is increased and benefit is decreased in order to keep the reserve to a certain viable level for 100 years. The demographic scenario is seen in Table 1 where life expectancy is improved and fertility decreases in about 40 years. Economic scenarios for the simulation are set for the inflation rate, the rate of

*This research was supported in part by the Grant-in-Aid for Scientific Research (No. 25350461) of the Japan Society for the Promotion of Science in 2013–2015.

| 2014 projection | 2060 | 2012 data |
|---|---|---|
| Fertility | 1.35 | 1.41 |
| Male life expectancy | 84.19 | 79.94 |
| Female lefe expectancy | 90.93 | 86.414 |

Table 1: Population projections

wage change, and the rate of return of investment as Table 2. The previous actuarial valuation is fixed inflation as 1 % as government policy. However, the new valuation assumes it from 2.0% which is the recent target number of Bank of Japan, to 0.6% of near-deflation scenario. In the bottom of Table 2, we see the stopping year of Macro-economic slide adjustment which means premium increase and benefit cut.[†] After the stopping year, benefits follow inflation adjustment. In the good economic performance cases except scenario H, the premium and benefit adjustment finish within 30 years. In the scenario H the reserve is used up in National pension. How to avoid this situation of National pension is the main purpose of this paper.

The objective of the Macro-economic slide is to ensure the 100 years viability of public pension. However, the macro-economic slide policy is not complete guarantee of 100 years viability. It will increase the number of younger generation to refuse premium contribution due to the distrust to pension system. However, the public pension has a huge reserve, namely 130 trillion yen and also the half amount of benefit in fundamental pension is received as government subsidy. Our goal of paper is to examine the conditions of parameters for the viability of public pension.

The structure of paper is as follows. In section 2 we explain the Japanese two-tier pension system and describe the mathematical notations. In section 3 the cohort model of population is introduced in order to include the criteria of generation equality. Section 4 is the viability conditions for National pension. Last section is some concluding remarks.

## 2 Two-tier public pension

In the pension contract, policy holders pay premium from age $\omega_1$ to age $\omega_2$ and they receive benefit from age $\omega_2 + 1$ to the end age $\omega_3$. The two-tier pension premium and benefit are depicted in Figure 1. The first-tier of pension is compulsory and the premium and benefit are fixed constants for all policy holders of National pension (NP). Let the constant pension premium

[†]The policy forces the premium rate $a_t$ increases up to the fixed year $T_a$ as follows,

$$a_t = \begin{cases} a_{t-1}(1+m_a), & t < T_a \\ a_{max}, & t \geq T_a \end{cases}$$

The policy forces to cut benefit rate $b_t$ up to stoping time $\tau_p$ in order to keep the reserve level;

$$b_t = \begin{cases} b_{t-1}(1-m_b), & 0 < t < \tau_p \\ b_{min}, & \tau_p \leq t \leq T \end{cases}$$

| Valuation year | 2014(%) | | | | 2009(%) | | |
|---|---|---|---|---|---|---|---|
| Scenario | A | C | E | H | High | Mid | Low |
| inflation rate | 2.00 | 1.60 | 1.20 | 0.60 | 1.00 | 1.00 | 1.00 |
| wage change | 2.30 | 1.80 | 1.30 | 0.70 | 1.90 | 1.50 | 1.10 |
| rate of return | 3.40 | 3.20 | 3.00 | 1.70 | 3.20 | 3.10 | 2.90 |
| End year of Macro-economic slide adjustment | | | | | | | |
| National pension | 2044 | 2043 | 2043 | Default | 2037 | 2038 | 2043 |
| Proportional | 2017 | 2018 | 2020 | 2051 | 2018 | 2019 | 2028 |

Table 2: Economic scenarios

be $\alpha_t^0$ and the population of contributors be $\xi_t^0$. The NP contribution equals $\alpha_t^0\xi_t^0$. The second-tier pension is called as Proportional pension (PP), where the premium is wage proportional and it is payed by employees and the employer; Let the wage proportional premium ratio be $a_t$, the average wage be $H_t^0$ and the numbers of contributors be $\xi_t^1$. The premium equals to $a_t H_t^0 \xi_t^1$. The total of two premiums at time $t$ is the sum as follows;

$$u_t = \alpha_t^0\xi_t^0 + a_t H_t^0 \xi_t^1$$

The benefits are also depicted in the negative part of Figure 1. The box area below zero is called fundamental pension benefit which is the same value for all pensioners. The total of all pensioners is $\xi_t^2 + \xi_t^3 + \xi^4$, where $\xi_t^2$ is the population of NP pensioners at $t$. Let $\alpha_t^1$ be the benefit amount per pensioner and then the benefit of fundamental pension equals $\alpha_t^1(\xi_t^2 + \xi_t^3 + \xi_t^4)$.

Let proportional pension benefit ratio be $b_t$ and let $H_t^1$ be the average of premium paid at $t$ before age $\omega_2$. Let $\xi_t^3$ be the population of PP pensioners and $\xi_t^4$ be the population of spouse of PP pensioners. Total benefit is the following sum;

$$s_t = \alpha_t^1(\xi_t^2 + \xi_t^3 + \xi_t^4) + b_t H_t^1 \xi_t^3$$

Numbers of contributors and beneficiaries in National pension at $t$ are given by

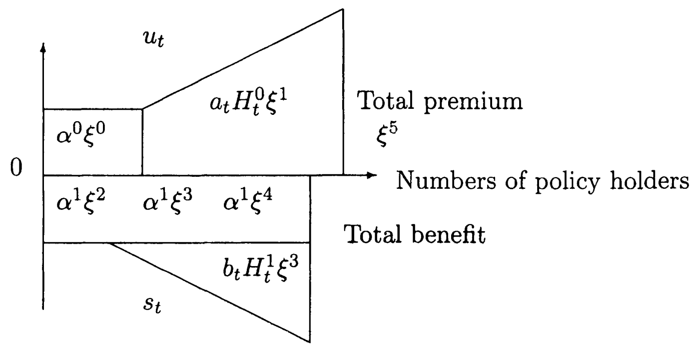


Figure 1: Premium and benefit balance in two-tier Pension

$$\xi_t^0 = \sum_{y=\omega_1}^{\omega_2} p_n(t,y), \quad \xi_t^2 = \sum_{y=\omega_2}^{\omega_3} p_n(t,y).$$

Numbers of contributors and beneficiaries in Proportional pension at $t$ are also given by

$$\xi_t^1 = \sum_{y=\omega_1}^{\omega_2} p_p(t,y), \quad \xi_t^3 = \sum_{y=\omega_2}^{\omega_3} p_p(t,y)$$

where $p(t,y)$ is the numbers of population at time $t$ of age $y$. Let $l_t^p$ be the rate of Proportional pension participation and $l_t^0$ be no policy holder rate. Let $p_n(t,y) = (1-l_t^p-l_t^0)p(t,y)$ be numbers of the National pension policyholders at $t$ of age $y$. The numbers of the Proportional pension policyholders is $p_p(t,y) = l_t^p p(t,y)$. The period of premium payment is defined as $L = \omega_2 - \omega_1$. The first tier balance is defined as follows;

$$q_t^n(\alpha^0, \alpha^1) := \alpha_t^0 \xi_t^0 - \alpha_t^1 \xi_t^2$$

The second tier balance is defined as

$$q_t^p(a,b,\alpha) := u_t^p - s_t^p = H_t^0 a_t \xi_t^1 - (b_t H_t^1 \xi_t^3 + \alpha_t^1 \xi_t^3 + \alpha_t^1 \xi_t^4)$$

Thus we define the balance of 2-tier pension as

$$q_t(a,b,\alpha) := u_t - s_t = q_t^n(\alpha^0,\alpha^1) + q_t^p(a,b,\alpha).$$

We define the total asset value of pension as follows,

$$W_t = R_t + q_t(a,b,\alpha) + \beta_t(\xi_t^2 + \xi_t^3 + \xi_t^4), \tag{2.1}$$

where the government subsidy to each pensioner is $\beta_t$.
Let $R_t$ is the reserve at $t$ which is invested to a risky asset $A_t$ The stochastic processes of rate of return of risky asset price $A_t$ is assumed as the following process;

$$dr_t := \frac{dA_t}{A_t} = \mu_t^r dt + \sigma_t^r dB_t^r$$

where $B_t$ is a Brownian motion. For discretization let the increment be $\Delta r_t = \Delta A_t / A_t$. Wage process $H_t$ is assumed similarly,

$$dx_t := \frac{dH_t}{H_t} = \mu_t^x dt + \sigma_x^r dB_t^x$$

The values of National pension and Wage-proportional pension are defined as follows,

$$\begin{aligned} W_t^n &= R_t^n + q_t^n(\alpha^0,\alpha^1) + \beta_t \xi_t^2 && (2.2) \\ W_t^p &= R_t^n + q_t^p(a,b,\alpha) + \beta_t \xi_t^3 + \beta_t \xi_t^4 && (2.3) \end{aligned}$$

Clearly $W_t = W_t^n + W_t^p$.

## 2.1 Economic criteria for public pension

The objectives of public pension are redistribution of income to elders and self help for old age. The performance of the objective in the redistribution of income to elders is measured by "Income replacement ratio " which should be more than some constant e.g.50%. The definition of income replacement for wage proportional pension is average ratio benefit over income after premium as,

$$\frac{H_t^1 b_t + \alpha_t^1}{H_t^0(1-a_t)} > 0.50 \tag{2.4}$$

The condition of self help for old age is economic rationality which mean that the present value of pension should be positive. Let $\delta_t$ be discount function of time $t$. The Expected value of discount value to join National pension should be positive as,

$$E[-\sum_{t=\omega_1}^{\omega_2} \alpha_t^0 \delta_t + \sum_{t=\omega_2}^{\omega_3} \alpha_t^1 \delta_t] > 0 \tag{2.5}$$

The Expected value of discount value to join Wage-proportional pension should be

$$E[-\sum_{t=\omega_1}^{\omega^2} a_t H_t^0 \delta_t + \sum_{t=\omega_2}^{\omega_3} (b_t H_t^1 + \alpha_t^1)\delta_t] > 0 \tag{2.6}$$

## 2.2 Viability condition for two-tier pension

We consider the pension policy $\{a_t, b_t, \alpha_t^0, \alpha_t^1, \beta_t\}$ which should satisfy the following conditions; The first is that the benefit of Basic pension should be more than social welfare benefit, which is supported by government expenditure. If the condition is not satisfied, it causes hesitation to pay the premium of National pension. Let $\hat{\beta}$ be the social welfare benefit , the benefit of National pension should satisfies the constraint;

$$\alpha_t^1 > \hat{\beta} \tag{2.7}$$

We define the viability for each pension of two-tier system;
(i) For National pension account, the wealth is greater than the benefit payment;

$$W_t^n \geq \alpha_t^1 \xi_t^2 \tag{2.8}$$

(ii) For Proportional pension account, similarly,

$$W_t^p \geq \alpha_t^1(\xi_t^3 + \xi_t^4) + b_t H_t^0 \xi_t^3 \tag{2.9}$$

(iii) For the total account of public pension, the sum of two-tier should be satisfied as,

$$W_t \geq s_t \tag{2.10}$$

### 2.3 PAYG scheme and default of pension

The objective of actuarial valuation is to prevent the shortage of reserve for benefit. But in the case when reserve is exhausted, the pension system moves to PAYG scheme. Then the all policy variables are decided by the following conditions;

(i) For National pension, from $\alpha_t^0\xi_t^0 - \alpha_t^1\xi_t^2 + \beta_t\xi_t^2 = 0$, the benefit is proportional for population ratio of contributors and pensioners;

$$\alpha_t^1 = \alpha_t^0\frac{\xi_t^0}{\xi_t^2} + \beta_t$$

The benefit should be larger than the social welfare from the condition (2.7), then the premium should be larger than the product of population ratio; $\frac{\xi_t^0}{\xi_t^2}$ and the difference of fiscal burden between the social welfare and pension subsidy; $\hat{\beta} - \beta_t$.

$$\alpha_t^0 > \frac{\xi_t^2}{\xi_t^0}(\hat{\beta} - \beta_t)$$

(ii) The balance of PAYG for the Proportional pension account should be positive;

$$a_tH_t^0\xi_t^1 - \alpha_t^1(\xi_t^3 + \xi_t^4) - b_tH_t^1\xi_t^3 + \beta_t(\xi_t^3 + \xi_t^4) \geq 0$$

Then the benefit ratio satisfies;

$$b_t \leq a_t\frac{H_t^0\xi_t^1}{H_t^1\xi_t^3} - \frac{\alpha_t^0 - \beta_t}{H_t^1}\frac{\xi_t^3 + \xi_t^4}{\xi_t^3}.$$

In the case where the income substitution rate is 0.5, $\frac{b_tH_t^1+\alpha_t^1}{(1-a_t)H_t^0} \geq 1/2$, then the rate of benefit satisfies the following condition.

$$b_t \geq \frac{1}{2}(1-a_t)\frac{H_t^0}{H_t^1} - \frac{\alpha_t^1}{H_t^1}$$

## 3 Cohort and population dynamics

The population is a function of age $y$ and time $t$ which satisfied McKendrick-von Foerster PDE

$$\frac{\partial p(t,y)}{\partial t} = -\frac{\partial p(t,y)}{\partial y} - \mu(t,y)p(t,y)$$

where $\mu(t,y)$ is the death rate of age $y$ at time $t$,. Let $k := t-y$ , $v(k,y) = p(t,y)$ , then $v(k,y)$ implies the $k$-cohort population age $y$. The PDE is simplified as

$$dv(k,y) = -\mu(k,y)v(k,y)dy.$$

Thus $k$-cohort population can be obtained in the discrete modeling as follows,

$$v(k,y) = v(k,0)\exp(-\sum_{s=0}^{y}\mu(k,s)) \tag{3.1}$$

The boundary condition is for given $m(k,y)$ which is the fertility rate of k-cohort and age $y$.

$$v(k,0) = \sum_{y=0}^{\omega_3} m(k-y,y)v(k-y,y)$$

The explicit solution has been given in Feller[2], however the numerical simulation uses the estimated value $\mu(k,y)$ in discrete time.

### 3.1 The three groups of cohorts

There exists three groups of cohort. At first we define Planning groups of cohort whose all premium and benefit are determined within the planning period $t \in (0, T)$ Secondly we define Future group of cohort whose premium and benefit cannot determined within the end of planning year $T$. Finally we define Existing group of cohort whose premium and benefit are determined in before the planning time 0. The value cannot be control to attain the optimality.

In Figure 2 we depict three groups in age-axis and time-axis. The age starts from 0 and $\omega_1$ of starting premium payment. The time starts from 0 where 0-th cohort starts premium payment. The planning period is assumed to be $t \in (0, T)$. Let define $T_l := T - (\omega_3 - \omega_1)$ . 0-cohort starts paying of premium and $T_l$-cohort will finish all benefit witch is seen in highest diagonal line in Figure 2.

Planning cohort group is marked as $\mathcal{N}$ where the cohort k satisfies $0 \leq k \leq T_l$. In the group of cohorts, all policy variables $a_t, b_t$ are determined in panning period 0 to $T$. Future cohort group is marked as $\mathcal{F}$ where the cohort k satisfies $T_l + 1 \leq k \leq T$. For the period $t = y + k \geq T$ $\tilde{a}_t, \tilde{b}_t$ is beyond of the planning period. Existing cohort group is marked as $\mathcal{P}$ where the cohort k satisfies $-\omega_3 \leq k \leq -1$. The past fixed parameters $a_t^c, b_t^c$ is effective for the group.

### 3.2 The net present value of pension for cohort groups

The economic rationality of pension has been discussed in section 1.3. The same condition should be satisfied for each group of cohorts. The net present value of planning cohort$(0 \leq k \leq T_l)$ is defined as follows,‡

$$c_n(k) := f(a_t, b_t, \alpha_t^0, \alpha_t^1) \tag{3.2}$$

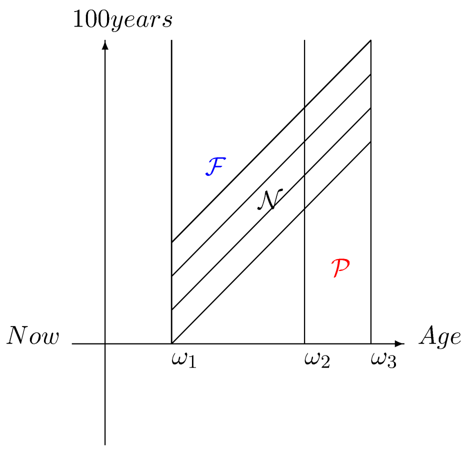


Figure 2: Cohorts and 100 year pension plan(Lexis diagram)

‡ $H_t = H_t^0 = H_t^1$

where

$$\begin{aligned} f(a_t, b_t, \alpha_t^0, \alpha_t^1) &:= -\sum_{y=\omega_1}^{\omega_2-1} (\alpha_{y+k}^0 v(k,y)\delta_{y+k} + a_{y+k} H_{y+k}^* v(k,y)) \\ &+ \sum_{y=\omega_2}^{\omega_3} (\alpha_{y+k}^1 v(k,y)\delta_{y+k} + b_{y+k} H_{y+k}^* v(k,y)) \end{aligned}$$

and $H_t^* = H_t \delta_t$.

The net present value of future cohort ($T_l < k < T$) is defined by denoting the following notations; The policy variables which exists out of plan period are denoted as $\tilde{a}_{y+k}, \tilde{b}_{y+k}, \tilde{\alpha}_{y+k}^0, \tilde{\alpha}_{y+k}^1$ for $k + y > T$.

Define $\hat{a}_t := \begin{cases} a_t, t < T \\ \tilde{a}_t, t \geq T \end{cases}$, $\hat{b}_t := \begin{cases} b_t, t < T \\ \tilde{b}_t, t \geq T \end{cases}$, $\hat{\alpha}_t^i = \begin{cases} \alpha_t^i, t < T \\ \tilde{\alpha}_t^i, t \geq T \end{cases}$,

Then the net present value of future cohort is defined as follows,

$$c_f(k) = f(\hat{a}_t, \hat{b}_t, \hat{\alpha}_t^0, \hat{\alpha}_t^1) \tag{3.3}$$

The net present value of existing cohort ($-\omega_3 \leq k \leq -1$) is defined by denoting the following notations; The policy variables which existed already are the following past parameters; $a_c, b_c, \alpha_c^0, \alpha_c^1$: Then we define

$\hat{a}_t := \begin{cases} a_t, t > 0 \\ a_c, t \leq 0 \end{cases}$, $\hat{b}_t := \begin{cases} b_t, t > 0 \\ b_c, t \leq 0 \end{cases}$, $\hat{\alpha}_t^i = \begin{cases} \alpha_t^i, t > 0 \\ \alpha_c^i, t \leq 0 \end{cases}$

Then the net present value of future cohort is defined as follows,

$$c_p(k) = f(\hat{a}_t, \hat{b}_t, \hat{\alpha}_t^0, \hat{\alpha}_t^1) \tag{3.4}$$

The past policy variables $a_c, b_c, \alpha_c^0, \alpha_c^1$ are constants. And the future policy variables $\tilde{a}_{y+k}, \tilde{b}_{y+k}, \tilde{\alpha}_{y+k}^0, \tilde{\alpha}_{y+k}^1$ also assumed to be constant and not policy variables.

The economic condition for cohort groups is the positivity of expected net present value under risk neutral measure $Q$. For all cohort which belongs to Planning group the expected value should be positive as

$$E^Q[c_n(k)|\mathcal{F}_k] > 0, 0 \leq k \leq T_l \tag{3.5}$$

For all cohort which belongs to Future groupthe expected value should be positive as

$$E^Q[c_f(k)|\mathcal{F}_k] > 0, T_l < k < T \tag{3.6}$$

For all cohort which belongs to Existing group the expected value should be positive as

$$E^Q[c_p(k)|\mathcal{F}_k] > 0, -\omega_3 \leq k \leq -1 \tag{3.7}$$

## 3.3 Evaluation of objective performance

We consider two evaluation approach for measure the goodness of policy selection. The first is focus on only the planning cohorts, because all parameter are indigenous and it is less influential from past policy decision.

Let $U(\cdot)$ be the utility function for planning groups. The objective function are considered discounted sum of the utility function over all cohorts of the planning group as;

$$\max_{a_t, b_t, \alpha, \beta} E[\sum_{k=0}^{T_l} \delta_k U(c_n(k))] \tag{3.8}$$

The second which we consider is to minimize the variance of present value of pension for cohorts. It may be an equity policy for generations. Let $m$ be the average of net present values for all cohorts as,

$$m = \frac{1}{\omega_3 + T} \sum_{k=-\omega_3}^{T} c(k)$$

We choose the policy $\{a_t, b_t, \alpha, \beta\}_{t=1,\cdots,T}$ in order to minimize the variance of net present values,

$$\min_{a_t, b_t, \alpha, \beta} \sum_{k=-\omega_3}^{T} (c(k) - m)^2 \tag{3.9}$$

# 4 Viability conditions to policy variables

The economic factors and population are stochastic processes. The former changes with a significant volatility but the later changes slowly and steadily. The critical problem for pension is fear of default, however there is no default if we adopt the PAYG policy but it will worsen the income replacement ratio.

The viability of pension is defined in (2.8),(2.9) and (2.10) which implies the positivity of reserve after the payment of benefits for stochastic change in economic factors. We consider the problem similar to the optimal investment and consumption problem, which has been formulated in [3]. In this paper we reformulate for discretized conditions. This viability condition reduces the numbers of free variables to numerical calculation.

## 4.1 Stochastic process of asset value of pensions

### 4.1.1 The process for National pension

The dynamics of $W_t^n$ is from (2.2),

$$\Delta W_t^n = R_t^n \Delta r_t + \Delta q_t^n(a, b, \alpha) + \beta_t \Delta \xi_t^2 \tag{4.1}$$

The wealth process $W_{t+1}^n = W_t^n + \Delta W_t^n$ is determined by new policy parameters of premium and benefit $\alpha_{t+1}^0, \alpha_{t+1}^1$. From this self financing condition the wealth at $t+1$ satisfies as follows,

$$\begin{aligned} W_{t+1}^n &= R_t^n(1 + \Delta r_t) + \alpha_t^0(\xi_t^0 + \Delta \xi_t^0) - \alpha_t^1(\xi_t^2 + \Delta \xi_t^2) + \beta_t(\xi_t^2 + \Delta \xi_t^2) \\ &= R_{t+1} + \alpha_{t+1}^0 \xi_{t+1}^0 - \alpha_{t+1}^1 \xi_{t+1}^2 + \beta_{t+1}^n \xi_{t+1}^2 \end{aligned}$$

### 4.1.2 The process for wage-proportional pension

The dynamics of value of proportional pension follows

$$dW_t^p = R_t^p dr_t + dq_t^p(a,b,\alpha) + \beta_t(d\xi_t^3 + d\xi_t^4) \tag{4.2}$$

where $dq_t^p = a_t(dH_t^0\xi_t^1 + H_t^0 d\xi_t^1) - (b_t H_t^1 d\xi_t^3 + \alpha_t^1 d\xi_t^3 + \alpha_t^1 d\xi_t^4)$.

From $dH_t^0\xi_t^1 + H_t^0 d\xi_t^1 = H_t^0(dx_t + d\xi_t^1)$ where $dH_t^0/H_t^0 = dx_t$.

$$q_{t+dt}^p(a,b,\alpha) \quad = \quad a_t H_t^0(1 + d\xi_t^1 + dx_t) - (b_t H_t^1(1 + d\xi_t^3) + \alpha_t^1(\xi_t^3 + d\xi_t^3 + \xi_t^4 + d\xi_t^4))$$

New policy for premium rate and benefit rates $a_{t+dt}, b_{t+dt}$ becomes from self financing condition;

$$W_{t+dt}^p = R_{t+dt}^p + q_{t+dt}^p(a,b,\alpha) + \beta_{t+dt}(\xi_{t+dt}^3 + \xi_{t+dt}^4)$$

### 4.1.3 The total wealth of two-tier pension

The total wealth of two-tier pension is the sum of the values as,

$$W_t = W_t^p + W_t^n$$

where the sum of balances of the NP and PP is

$$q_t(a,b,\alpha) := \alpha_t^0\xi_t^0 + H_t^0 a_t\xi_t^1 - (b_t H_t^1\xi_t^3 + \alpha_t^1(\xi_t^2 + \xi_t^3 + \xi_t^4))$$

and the government subsidy $\beta_t(\xi_t^2 + \xi_t^3 + \xi_t^4)$.
The dynamics of wealth process for total pension follows as

$$dW_t = R_t dr_t + dq_t(a,b,\alpha) + \beta_t d(\xi_t^2 + \xi_t^3 + \xi_t^4)$$

The new policy $\beta_{t+dt}$ of self financing condition satisfies,

$$W_{t+dt} \quad = \quad R_{t+dt} + q_{t+dt}(a,b,\alpha) + \beta_{t+dt}(\xi_{t+dt}^2 + \xi_{t+dt}^3 + \xi_{t+dt}^4)$$

## 4.2 Viability condition for National pension

From (4.1) the wealth change of National pension satisfies,

$$\begin{aligned}\Delta W_t^n &= R_t^n\Delta r_t + \Delta q_t^n(a,b,\alpha) + \beta_t\Delta\xi_t^2 \\ &= \frac{R_t^n}{A_t}\Delta A_t + \alpha_t^0\Delta\xi_t^0 - (\alpha_t^1 - \beta_t)\Delta\xi_t^2\end{aligned}$$

We take the discounted values which are denoted with $*$ as $W_t^{n*} := W_t^n\delta_t, A_t^* := A_t\delta_t, \beta_t^* = \beta_t\delta_t$. Denote the amount of market asset be $\delta_t = e^{-\sum_{i=0}^t r_i^f}$, where $r_i^f$ is the risk free rate and $\phi_t = \frac{R_t^n}{A_t}$.

$$\Delta W_t^{n*} = \phi_t\Delta A_t^* + \delta_t(\alpha_t^0\Delta\xi_t^0 - (\alpha_t^1 - \beta_t)\Delta\xi_t^2)$$

Then the wealth could divide martingale part $M_t$ and the others $N_t$ as,

$$W_t^{n*} = \underbrace{W_0^n + \sum_{s=0}^{t-1}\phi_s\Delta A_s^*}_{M_t} + \underbrace{\sum_{s=0}^{t-1}(\alpha_s^0\Delta\xi_s^0 - (\alpha_s^1 - \beta_s)\Delta\xi_s^2)\delta_s}_{N_t}$$

Under a risk neutral probability $Q$, $M_t$ is $Q$-martingale. From $M_t = E_t^Q[M_T] = E_t^Q[W_T^* - N_T]$ we obtain as

$$W_t^{n*} = M_t + N_t = E_t^Q[W_T^{n*} - N_T + N_t]$$

Therefore, by substituting $N_t$ the following equation is derived;

$$W_t^{n*} = E_t^Q[W_T^{n*} - \sum_{s=t}^{T-1}(\alpha_s^0 \Delta\xi_s^0 - (\alpha_s^1 - \beta_s)\Delta\xi_s^2)\delta_s]$$

where $W_0^n = R_0^n$.
The terminal condition satisfies

$$W_T^n = s_T^n = (\alpha_T^1 - \beta_T)\xi_T^2. \tag{4.3}$$

In oder to guarantee $W_{T-1}^n \geq 0$, it is necessary to have the following condition;

$$W_{T-1}^n = E_{T-1}^Q[s_T^n - (\alpha_{T-1}^0 \Delta\xi_{T-1}^0 - (\alpha_{T-1}^1 - \beta_{T-1})\Delta\xi_{T-1}^2)] \geq 0$$

Because $\alpha^0, \alpha^1$ is predictable and the assumption of deterministic population change, at time $T-1$ it is simplified as follows,

$$W_{T-1}^n = (\alpha_T^1 - \beta_T)\xi_T^2 + (\alpha_{T-1}^1 - \beta_{T-1})\Delta\xi_{T-1}^2 - \alpha_{T-1}^0 \Delta\xi_{T-1}^0 \geq 0$$

Similarly we obtain the following;

$$W_{T-2}^n = E_{T-2}^Q[W_{T-1}^n] + (\alpha_{T-2}^1 - \beta_{T-2})\Delta\xi_{T-2}^2 - \alpha_{T-2}^0 \Delta\xi_{T-2}^0 \geq 0$$

$$\cdots$$

$$W_1^n = E_1^Q[W_2^n] + (\alpha_2^1 - \beta_2)\Delta\xi_2^2 - \alpha_2^0 \Delta\xi_2^0 \geq 0$$

$$W_0^n = E^Q[W_1^n] + (\alpha_1^1 - \beta_1)\Delta\xi_1^2 - \alpha_1^0 \Delta\xi_1^0 \geq 0$$

where $W_0^n = R_0^n$. Consequently,

$$E^Q[W_1^n] = R_0^n - (\alpha_1^1 - \beta_1)\Delta\xi_1^2 + \alpha_1^0 \Delta\xi_1^0 \geq 0$$

for $j \in (2, T-1)$ using the chain rule of conditional expectation;

$$E^Q[W_j^n] = R_0^n - \sum_{i=1}^{j}\{(\alpha_i^1 - \beta_i)\Delta\xi_i^2 - \alpha_i^0 \Delta\xi_i^0\} \geq 0$$

and for $T$;

$$E^Q[W_T^n] = R_0^n - \sum_{i=1}^{T-1}\{(\alpha_i^1 - \beta_i)\Delta\xi_i^2 - \alpha_i^0 \Delta\xi_i^0\} - (\alpha_T^1 - \beta_T)\xi_T^2 \geq 0$$

The positivity conditions of value of pension are the constraints where the positivity of initial reserved minus the change of value due to the numbers of policy holders.

## 5 Concluding remarks

The optimal selection of premium and benefit are depending very long time schedule. In this paper we try to find them without direct parameter assumption as Macro-economic slide policy which uses exponential decreasing assumption [5]. The strong assumption enable to calculate numerically but it cannot give the viability of pension. In order to obtain the viable premium and benefit, we need rather simpler modeling than this two-tier pension system.

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