数理解析研究所講究録
第 1934 巻 2015 年 152-157

ON EISENSTEIN SERIES IN THE KOHNEN PLUS SPACE FOR HILBERT MODULAR FORMS

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ABSTRACT. This short article introduces a generalization for the so-called Cohen Eisenstein series to the case of general Hilbert modular forms of half-integral weight. We will recall the definition of the Cohen Eisenstein series and the Kohnen plus space and give some very basic properties. Then we define the Kohnen plus space for general Hilbert modular forms, which was initially given by Hiraga and Ikeda, and also give some analogues of the results from Kohnen. After that, we state the two main theorems of this article, where the first one gives the generalized Cohen Eisenstein series and the second one gives a property of the structure of generalized Kohnen plus space. Finally, we sketch how we construct the Eisenstein series and how we prove the second theorem.

1. BACKGROUND

First we recall the definition of Cohen Eisenstein series.

For any half-integer $k \in \frac{1}{2}\mathbb{Z}$, $M_{k+1/2}(\Gamma, \chi)$ and $S_{k+1/2}(\Gamma, \chi)$ denote the space of modular forms and cusp forms of weight k and character χ for the congruence subgroup Γ of $SL_2(\mathbb{Z})$. If $\chi = 1$, we may simply write $M_k(\Gamma)$ and $S_k(\Gamma)$.

Theorem 1 (Cohen, 1975, [1]). *Let $r \geq 2$ be an integer. There is a modular form $\mathcal{H}_r \in M_{r+1/2}(\Gamma_0(4))$ of weight $r + 1/2$ which is defined by*

$$\mathcal{H}_r(z) = \zeta(1-2r)$$
$$+ \sum_{\substack{N \geq 0 \\ (-1)^r N \equiv 0,1 (\mathrm{mod} 4)}} \left(L(1-r, \chi_{D_{(-1)^r N}}) \sum_{d \mid f_{(-1)^r N}} \mu(d) \chi_{D_{(-1)^r N}d}(d) d^{r-1} \sigma_{2r-1}(f/d) \right) q^N,$$

where for any integer n, D_n is the discriminant of $\mathbb{Q}(\sqrt{n})/\mathbb{Q}$, f_n is the positive integer such that $n = f_n^2 D_n$, and, as usual, $q = \exp(2\pi i z)$.

Cohen used his modular forms to give some applications. For example, he gave the "generalized class number relations", which state that for integer $D \equiv 0$ or 1 (mod 4) such that $(-1)^{r-1}D = |D|$, we have

$$\sum_{N \geq 0} \left(\sum_s H\left(r, \frac{4N - s^2}{|D|}\right) \right) q^N \in M_{r+1}(\Gamma_0(D), \chi_D),$$

where $H(r,x)$ is the x-th Fourier coefficient of H_r if $x \in \mathbb{Z}_{\geq 0}$ or 0 otherwise.

Inspired by these modular forms, in 1980, Kohnen [3] defined the plus spaces $M^+_{r+1/2}(\Gamma_0(4))$ and $S^+_{r+1/2}(\Gamma_0(4))$, which are subspaces of $M_{r+1/2}(\Gamma_0(4))$ and $S_{r+1/2}(\Gamma_0(4))$ characterized by the Fourier coefficients of the modular forms in them.

Definition 1 (Kohnen, 1980, [3]). *The Kohnen plus spaces are defined by*

$$M^+_{r+1/2}(\Gamma_0(4)) = \left\{ f \in M_{r+1/2}(\Gamma_0(4)) \,\middle|\, f(z) = \sum_{(-1)^r N \equiv 0,1 (\mathrm{mod} 4)} a(N) q^N \right\},$$

$$S^+_{r+1/2}(\Gamma_0(4)) = M^+_{r+1/2}(\Gamma_0(4)) \cap S_{r+1/2}(\Gamma_0(4)).$$

So the modular form H_r given by Cohen is in $M^+_{r+1/2}(\Gamma_0(4))$ but not in $S^+_{r+1/2}(\Gamma_0(4))$. Some properties of the plus space are also showed by Kohnen.

Theorem 2 (Kohnen, 1980 ,[3]). *The following statements hold.*

(1) *Let U_4 and W_4 be operators on $S_{r+1}(\Gamma_0(4))$ such that $(U_4 f)(z) = \frac{1}{4}\sum_{i=0}^{3} f(\frac{z+i}{4})$ and $(W_4 f)(z) = (-2\sqrt{-1}z)^{-4-1/2} f(-\frac{1}{4z})$. Then $S^+_{r+1/2}(\Gamma_0(4))$ is the eigenspace of $W_4 U_4$ with respect to eigenvalue $(-1)^{r(r+1)/2} 2^r$.*
(2) *$S^+_{r+1/2}(\Gamma_0(4))$ has a basis consisting of Hecke eigenforms over $\mathbb{C}$.*
(3) *$dim_{\mathbb{C}} S^+_{r+1/2}(\Gamma_0(4)) = dim_{\mathbb{C}} S_{2r}(SL_2(\mathbb{Z}))$.*

Now we introduce the generalization of Kohnen plus space for general Hilbert modular forms of half-integral weight. Let F be a totally real number field of degree n over $\mathbb{Q}$ and $\mathfrak{o}$ and $\mathfrak{d}$ be its ring of integers and different over $\mathbb{Q}$. We denote $\iota_1, ..., \iota_n$ the n embeddings of F to $\mathbb{R}$.

Definition 2. *For any $\xi \in F$, we say $\xi = \square\,(\mathrm{mod}\, 4)$ if there is $x \in \mathfrak{o}$ such that $\xi - x^2 \in 4\mathfrak{o}$.*

We define the congruence subgroup Γ by

$$\Gamma = \Gamma[\mathfrak{d}^{-1}, 4\mathfrak{d}] = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(F) \,\middle|\, a, d \in \mathfrak{o}, b \in \mathfrak{d}^{-1}, c \in 4\mathfrak{d} \right\}.$$

Let κ be an integer. Denote $M_{\kappa+1/2}(\Gamma)$ and $S_{\kappa+1/2}(\Gamma)$ the spaces of Hilbert modular forms and cusp forms of the parallel weight $\kappa + 1/2$ for Γ. We only consider the case of parallel weight. For any $\xi \in F$ and $z = (z_1, z_2, ..., z_n) \in \mathfrak{h}^n$, let $q^\xi = \exp(2\pi i \sum_{i=1}^{n} \iota_i(\xi) z_i)$ for simplicity. Now the generalized Kohnen plus space for Hilbert modular forms are defined as follow.

Definition 3 (Hiraga and Ikeda, 2013, [2]). *The generalized Kohnen plus spaces are defined by*

$$M^+_{\kappa+1/2}(\Gamma) = \left\{ f \in M_{\kappa+1/2}(\Gamma) \;\middle|\; f(z) = \sum_{(-1)^\kappa \xi \equiv \square \,(\mathrm{mod}\, 4)} a(\xi) q^\xi \right\},$$

$$S^+_{\kappa+1/2}(\Gamma) = M^+_{\kappa+1/2}(\Gamma) \cap S_{\kappa+1/2}(\Gamma).$$

So it is easy to see that the definition from Hiraga and Ikeda coincides with which given by Kohnen for the case $F = \mathbb{Q}$ and $\kappa \geq 2$. Hiraga and Ikeda also gave proper analogues of Kohnen's results for generalized Kohnen plus spaces.

Theorem 3 (Hiraga and Ikeda, 2013, [2]). *For $\kappa \geq 2$, the following statements hold.*

(1) *$M^+_{\kappa+1/2}(\Gamma)$ is the fixed subspace of some idempotent operator E^K on $M^+_{\kappa+1/2}(\Gamma)$. If $F = \mathbb{Q}$, $E^K = (\alpha_1 - \alpha_2)^{-1}(W_4 U_4 - \alpha_2)$ where $\alpha = (-1)^{\kappa(\kappa+1)/2} 2^\kappa$ and $\alpha_2 = -2^{-1}\alpha_1$. This also holds for $\kappa = 1$.*

(2) *$S^+_{\kappa+1/2}(\Gamma)$ has a basis consisting of Hecke eigenforms over $\mathbb{C}$.*

(3) *$dim_{\mathbb{C}} S^+_{\kappa+1/2}(\Gamma) = dim_{\mathbb{C}} \mathcal{A}^{\mathrm{CUSP}}_{2\kappa}(PGL_2(F)\backslash PGL(\mathbb{A}_F)/\mathcal{K}_0)$ where $\mathbb{A}_F$ is the adele ring of F, $\mathcal{K}_0 = \prod_{v<\infty} PGL_2(\mathfrak{o}_v)$ and $\mathcal{A}^{\mathrm{CUSP}}_{2\kappa}$ is the space of cuspidal automorphic forms come from $S_{2\kappa}(SL_2(\mathfrak{o}))$, the space of cuspidal Hilbert modular forms of weight 2κ.*

So as mentioned before, the main result of this article is to give some Hilbert modular form in the generalized Kohnen plus space which are corresponding to the one given by Cohen. In fact, there are h such modular forms where h is the class number of F.

2. Main Theorems

Throughout this section, we use the same notations as given in the last section. We denote Cl_F and h the ideal class group and class number of F. For any $\xi \in F$, let $\mathfrak{D}_\xi$ and χ_ξ be the relative discriminant and quadratic character of $F(\sqrt{\xi})/F$ and $\mathfrak{F}_\xi$ be the integral ideal such that $\mathfrak{F}_\xi^2 \mathfrak{D}_\xi = (\xi)$, the principal ideal generated by ξ.

Main Theorem 1. *We set $\kappa \geq 1$ and $\kappa \neq 1$ if $F = \mathbb{Q}$. Let χ' be a Hecke character on Cl_F. Define the function $G_{\chi'}$ on $\mathfrak{h}^n$ by*

$$G_{\kappa,\chi'}(z) = L_F(1-2\kappa, \overline{\chi'^2}) + \sum_{\substack{(-1)^\kappa \xi \equiv \square \,(\mathrm{mod}\, 4) \\ \xi \succ 0}} \chi'(\mathfrak{D}_{(-1)^\kappa \xi}) L_F(1-\kappa, \overline{\chi_{(-1)^\kappa \xi} \chi'}) c((-1)^\kappa \xi) q^\xi$$

where for any $\xi \in F$,

$$c(\xi) = \sum_{\mathfrak{a} \mid \mathfrak{F}_\xi} \mu(\mathfrak{a}) \chi_\xi(\mathfrak{a}) \chi'(\mathfrak{a}) N_{F/\mathbb{Q}}(\mathfrak{a})^{\kappa-1} \sigma_{2\kappa-1,\chi'^2}(\mathfrak{F}_\xi \mathfrak{a}^{-1})$$

and $\xi \succ 0$ *means* ξ *is totally positive. Here in the summation* $\mathfrak{a}$ *runs over all integral ideals dividing* $\mathfrak{F}_\xi$, μ *is the Möbius function for ideals and*

$$\sigma_{k,\chi}(\mathfrak{I}) = \sum_{\mathfrak{b} \mid \mathfrak{I}} N_{F/\mathbb{Q}}(\mathfrak{b})^k \chi(\mathfrak{b})$$

for integer k, *ideal character* χ *and integral ideal* $\mathfrak{I}$. *Then* $G_{\kappa,\chi'} \in M^+_{\kappa+1/2}(\Gamma)$. *Moreover,* $G_{\chi'}$ *is a Hecke eigenform.*

So there are h such modular forms. We call them Eisenstein series in the generalized Kohnen plus space for Hilbert modular forms. It is easily to see that $G_{\kappa,1}$ is the Cohen Eisenstein series if $F = \mathbb{Q}$. In fact, we can get Eisenstein series of weight $\frac{3}{2}$ if $F \neq \mathbb{Q}$ while if $F = \mathbb{Q}$ there is a non-holomorphic function which transforms like a modular form of weight $\frac{3}{2}$ under $SL_2(\mathbb{Z})$. The next main theorem is a corollary of the first main theorem.

Main Theorem 2. *The space* $M^+_{\kappa+1/2}$ *is spanned by the cusp forms and the* h *Eisenstein series given above. That is,*

$$M^+_{\kappa+1/2}(\Gamma) = S^+_{\kappa+1/2}(\Gamma) \bigoplus \oplus_{i=1}^h \mathbb{C} \cdot G_{\kappa,\chi_i'}$$

where $\chi_1', \chi_2', ..., \chi_h'$ *are the* h *distinct characters on* Cl_F.

Combining the second main theorem with the results given by Hiraga and Ikeda, we get that $M^+_{\kappa+1/2}(\Gamma)$ contains a basis consisting of Hecke modular forms.

3. Sketch of The Proofs

We give a brief on how the generalized Cohen Eisenstein series are constructed. Denote the metaplectic double covering of SL_2 by $\widetilde{SL_2}$. The multiplication of the double covering is with respect to Kubota's 2-cocycle. Then for any subset $S \subset SL_2$, denote $\widetilde{S}$ the inverse image of S in $\widetilde{SL_2}$. Let $\mathcal{A}_{\kappa+1/2}$ be the space of automorphic forms come from $M_{\kappa+1/2}(\Gamma)$. Then the idempotent E^K mentioned above can be considered as an operator on $\mathcal{A}_{\kappa+1/2}$ and decomposes as $E^K = \prod_{v \leq \infty} E_v^K$. Let v be a finite place of F. If $\mathfrak{s}_v$ is a complex number, let $\tilde{I}(\mathfrak{s}_v)$ be the space of genuine function f on $\widetilde{SL_2(F_v)}$ induced by the map

$$\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \to \frac{\alpha_v(1)}{\alpha_v(a)} |a|_v^{\mathfrak{s}_v+1},$$

where $\alpha(\star)$ is the Weil constant. Then E_v^K is an idempotent operator on $\widetilde{SL_2(F_v)}$. It is shown in [2] that the fixed subspace of E_v^K is a subspace of one dimension spanned by some function $f^+_{K,v}$. Now take a Hecke character χ' on Cl_F. Then $\chi_v'(x) = |x|^{s_v}$ for any $x \in F_v^\times$ and some $s_v \in \mathbb{C}$.

Now if $\kappa \geq 2$, define the function $f_{\chi'}$ on $\widetilde{SL_2(\mathbb{A}_F)}$ by

$$f_{\chi'} = \prod_{v<\infty} f^+_{K,v} \prod_{v\,|\,\infty} \tilde{j}(\star_v, \sqrt{-1})^{-2\kappa-1}$$

where $f^+_{K,v} \in \tilde{I}(\kappa - 1/2 + s_v)$ and $\tilde{j}$ is the unique factor of automorphy on $\widetilde{SL_2(\mathbb{R})} \times \mathfrak{h}$ such that $\tilde{j}^2$ is the usual factor of automorphy on $SL_2(\mathbb{R}) \times \mathfrak{h}$. One can show that $f_{\chi'}$ is invariant under the left transformation of upper triangular matrix. Denote B the subgroup of $SL_2(F)$ consisting of all upper triangular matrices. Then we define

$$\mathbb{E}_{\kappa,\chi'}(g) = \sum_{\gamma \in B\backslash SL_2(F)} f_{\chi'}(\gamma g) \in \mathcal{A}_{\kappa+1/2}$$

for $g \in \widetilde{SL_2(\mathbb{A}_F)}$. Here $SL_2(F)$ is considered as a subgroup of $\widetilde{SL_2(\mathbb{A}_F)}$. Take the corresponding Hilbert modular form $E_{\kappa,\chi'}$ of $\mathbb{E}_{\kappa,\chi'}$. After calculating the Fourier coefficients of $E_{\kappa,\chi'}$ and some normalizing, we get $G_{\kappa,\chi'}$.

Now let $\kappa = 1$ and $F \neq \mathbb{Q}$. For the convergent issue, we define

$$f_{\chi',\epsilon} = \prod_{v<\infty} f^+_{K,v} \prod_{v\,|\,\infty} \left(\tilde{j}(\star_v, \sqrt{-1})^{-3} |\tilde{j}(\star_v, \sqrt{-1})|^{-2\epsilon}\right)$$

where $f^+_{K,v} \in \tilde{I}(1/2 + s_v + \epsilon)$ and

$$\mathbb{E}_{\kappa,\chi',\epsilon}(g) = \sum_{\gamma \in B\backslash SL_2(F)} f_{\chi',\epsilon}(\gamma g).$$

For any $g \in \widetilde{SL_2(\mathbb{A}_F)}$, this series converges for $\Re(\epsilon)$ large and has a analytic continuation to $\epsilon = 0$. Taking $\mathbb{E}_{\kappa,\chi',0}$ and repeating the same process as above, we get $G_{1,\chi'}$.

Note that if $\kappa = 1$ and $F = \mathbb{Q}$, in the calculation of Fourier coefficients we will get non-vanishing non-holomorphic terms. So finally it turns out that we get a function which transforms like a modular form of weight 3/2 but not regular.

For the sketch of proof of the second main theorem, we give three easy lemmas which are used in the proof. From the three lemmas, the theorem immediately follows.

Lemma 1. *Let*

$$P = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \in SL_2(\mathbb{A}_F) \,\middle|\, a \in F^\times, b \in \mathbb{A}_F \right\}$$

and

$$\Xi = \prod_{v<\infty} \Gamma[(4\mathfrak{d}_v)^{-1}, 4\mathfrak{d}_v].$$

Then the order of the space of double cosets

$$\tilde{P}\backslash\widetilde{SL_2(\mathbb{A}_F)}/\Xi\prod_{v|\infty}SL_2(F_v)$$

is h, the class number of F.

We take a system of representatives $\mathfrak{m}_1, \mathfrak{m}_2, ..., \mathfrak{m}_h$ for the double cosets.

Lemma 2. *Let $a^0_{\chi'}$ be the constant of the Fourier expansion for $\mathbb{E}_{\kappa,\chi'}$, then we have*

$$\det\left(a^0_{\chi'_i}(\mathfrak{m}_j)_{1\leq i,j\leq h}\right) \neq 0.$$

Note that the nature of the matrix above is in fact the table of characters on Cl_F.

Lemma 3. *For any automorphic form $\Phi \in \mathcal{A}_{\kappa+1/2}$ which is invariant under E^K, the the constant term of the Fourier expansion for Φ is determined by its values on $\mathfrak{m}_1, \mathfrak{m}_2, ..., \mathfrak{m}_h$.*

References

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