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Automorphic pairs of distributions and its application to explicit constructions of Maass forms

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1 Automorphic pairs of distributions

Let $\lambda \in \mathbb{C}$, $\varepsilon = 0, 1$. We define the "automorphic factor" $J_{\lambda,\varepsilon}(x)$ on $\mathbb{R}^\times$ by $J_{\lambda,\varepsilon}(x) = \mathrm{sgn}(x)^\varepsilon \cdot |x|^{-2\lambda}$. For $f_0 \in C_0^\infty(\mathbb{R}^\times)$, we put

$$f_\infty(x) = J_{\lambda,\varepsilon}(x) f_0\left(-\frac{1}{x}\right) \qquad (x \neq 0). \tag{1}$$

Let $\boldsymbol{a} = \{a(n)\}_{n\in\mathbb{Z}}, \boldsymbol{b} = \{b(n)\}_{n\in\mathbb{Z}}$ be sequences of complex numbers of polynomial growth, and $N \geq 1$ is a natural number. Consider the mappings $T_0, T_\infty : C_0^\infty(\mathbb{R}^\times) \to \mathbb{C}$ defined by

$$T_0(\varphi) = \sum_{n=-\infty}^{\infty} a(n)(\mathcal{F}\varphi)(n), \quad T_\infty(\varphi) = \sum_{n=-\infty}^{\infty} b(n)(\mathcal{F}\varphi)\left(\frac{n}{N}\right) \qquad (\varphi \in C_0^\infty(\mathbb{R}^\times)),$$

where $(\mathcal{F}\varphi)(t)$ denotes the Fourier transform of φ:

$$(\mathcal{F}\varphi)(t) = \int_{\mathbb{R}} \varphi(x) e^{2\pi i x t} dx.$$

If T_0, T_∞ satisfy the condition

$$T_0(f_0) = T_\infty(f_\infty) \tag{2}$$

for all $f_0 \in C_0^\infty(\mathbb{R}^\times)$, then the pair (T_0, T_∞) is called an *automorphic pair of level* N *with automorphic factor* $J_{\lambda,\varepsilon}(x)$. The relation (2) can be written in a sum formula as

$$\sum_{n=-\infty}^{\infty} a(n)(\mathcal{F}f_0)(n) = \sum_{n=-\infty}^{\infty} b(n)(\mathcal{F}f_\infty)\left(\frac{n}{N}\right). \tag{3}$$

Associated Dirichlet series are defined as follows:

$$\begin{aligned} \xi_\pm(\boldsymbol{a}; s) &= \sum_{n=1}^{\infty} \frac{a(\pm n)}{n^s}, & \xi_\pm(\boldsymbol{b}; s) &= \sum_{n=1}^{\infty} \frac{b(\pm n)}{n^s}, \\ \Xi_\pm(\boldsymbol{a}; s) &= (2\pi)^{-s}\Gamma(s)\xi_\pm(\boldsymbol{a}; s), & \Xi_\pm(\boldsymbol{b}; s) &= (2\pi)^{-s}\Gamma(s)\xi_\pm(\boldsymbol{b}; s). \end{aligned} \tag{4}$$

Then we have

Theorem (T. Suzuki [7]). *The L-functions $\xi_\pm(\boldsymbol{a};s)$ and $\xi_\pm(\boldsymbol{b};s)$ have analytic continuations to meromorphic functions with a finite number of poles, and satisfy the following functional equations:*

$$\gamma(s)\begin{pmatrix} \Xi_+(\boldsymbol{a};s) \\ \Xi_-(\boldsymbol{a};s) \end{pmatrix} = N^{2-2\lambda-s}\cdot\Sigma\cdot\gamma(2-2\lambda-s)\begin{pmatrix} \Xi_+(\boldsymbol{b};2-2\lambda-s) \\ \Xi_-(\boldsymbol{b};2-2\lambda-s) \end{pmatrix},$$

where

$$\gamma(s) = \begin{pmatrix} e^{\pi s\sqrt{-1}/2} & e^{-\pi s\sqrt{-1}/2} \\ e^{-\pi s\sqrt{-1}/2} & e^{\pi s\sqrt{-1}/2} \end{pmatrix}, \qquad \Sigma = \begin{pmatrix} 0 & (-1)^\varepsilon \\ 1 & 0 \end{pmatrix}. \tag{5}$$

Example. Let $\operatorname{Re}\lambda > 1/2$, $\varepsilon = 0$, $N = 1$. We have

$$\zeta(2\lambda-1)\cdot(\mathcal{F}f_0)(0)+\sum_{n\neq 0}\sigma_{1-2\lambda}(|n|)(\mathcal{F}f_0)(n) = \zeta(2\lambda-1)\cdot(\mathcal{F}f_\infty)(0)+\sum_{n\neq 0}\sigma_{1-2\lambda}(|n|)(\mathcal{F}f_\infty)(n),$$

where $\sigma_a(n) := \sum_{0<d|n} d^a$. This equality is proved by using the Fourier expansion of the distribution E_λ defined by

$$E_\lambda(f_0) = \frac{1}{2}\sum_{m,n\neq 0}|m|^{-2\lambda}f_0\left(\frac{n}{m}\right). \qquad \text{("Eisenstein distribution")} \tag{6}$$

2 Principal series representations of $G = SL_2(\mathbb{R})$.

We introduce the following function space:

$$\mathcal{V}^\infty_{\lambda,\varepsilon} = \{f_0 \in C^\infty(\mathbb{R}) \mid f_\infty(x), \text{ defined by (1), can be extended to an element of } C^\infty(\mathbb{R})\}.$$

The action of $G = SL_2(\mathbb{R})$ on $\mathcal{V}^\infty_{\lambda,\varepsilon}$ is defined by

$$(\pi_{\lambda,\varepsilon}(g)f_0)(x) = \begin{cases} J_{\lambda,\varepsilon}(-cx+a)f_0\left(\dfrac{dx-b}{-cx+a}\right) & (\text{if } -cx+a\neq 0) \\ J_{\lambda,\varepsilon}(-dx+b)f_\infty\left(\dfrac{-cx+a}{-dx+b}\right) & (\text{if } -dx+b\neq 0) \end{cases} \tag{7}$$

for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G = SL_2(\mathbb{R})$ and $f_0 \in \mathcal{V}^\infty_{\lambda,\varepsilon}$. To be precise, elements of $\mathcal{V}^\infty_{\lambda,\varepsilon}$ should be regarded as sections of a line bundle over $\mathbb{P}^1(\mathbb{R}) \cong G/P$. We set $f_0(\infty) := f_\infty(0)$. It is known that $\mathcal{V}^\infty_{\lambda,\varepsilon}$ is one of the realizations of the (non-unitary) principal series representations of G.

We define a topology through seminorms on $\mathcal{V}^\infty_{\lambda,\varepsilon}$ given by

$$f_0 \mapsto \sup_{x\in K}\left|\frac{d^N f_0}{dx^N}(x)\right|, \qquad f_0 \mapsto \sup_{x\in K}\left|\frac{d^N f_\infty}{dx^N}(x)\right|,$$

where K is any compact subset of $\mathbb{R}$ and $N \in \mathbb{Z}_{\geq 0}$. We call a continuous linear mapping $T : \mathcal{V}^\infty_{\lambda,\varepsilon} \to \mathbb{C}$ a distribution on $\mathcal{V}^\infty_{\lambda,\varepsilon}$, and denote by $\mathcal{V}^{-\infty}_{\lambda,\varepsilon}$ the space of distributions. For $g \in G$ and $T \in \mathcal{V}^{-\infty}_{\lambda,\varepsilon}$, we define $(\pi_{-\lambda,\varepsilon}(g)T)(f_0) = T(\pi_{\lambda,\varepsilon}(g^{-1})f_0)$.

For a subgroup Γ of $SL_2(\mathbb{Z})$ of finite index, we define

$$\left(\mathcal{V}^{-\infty}_{\lambda,\varepsilon}\right)^\Gamma = \left\{T \in \mathcal{V}^{-\infty}_{\lambda,\varepsilon} \mid \pi_{-\lambda,\varepsilon}(\gamma)T = T \text{ for all } \gamma\in\Gamma\right\}.$$

We call T an *automorphic distribution* after Miller and Schmid [4]. Now we take $T \in (\mathcal{V}_{\lambda,\varepsilon}^{-\infty})^{\Gamma_0(N)}$, where $\Gamma_0(N)$ is the congruence subgroup of level N. Let $\gamma_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\gamma_2 = \begin{pmatrix} 1 & 0 \\ N & 1 \end{pmatrix} \in \Gamma_0(N)$. The invariance of T under γ_1 implies that T has a Fourier expansion as

$$T(f_0) = a(\infty)f_0(\infty) + \sum_{n=-\infty}^{\infty} a(n)(\mathcal{F}f_0)(n),$$

and since

$$(\pi_{\lambda,\varepsilon}(g)f_\infty)(x) = J_{\lambda,\varepsilon}(bx+d)f_\infty\left(\frac{ax+c}{bx+d}\right),$$

the invariance under γ_2 implies that

$$T(f_\infty) = b(\infty)f_\infty(\infty) + \sum_{n=-\infty}^{\infty} b(n)(\mathcal{F}f_\infty)\left(\frac{n}{N}\right).$$

Hence one can construct an automorphic pair of distributions of level N from $T \in (\mathcal{V}_{\lambda,\varepsilon}^{-\infty})^{\Gamma_0(N)}$.

3 Poisson transforms

Note that $G/P \cong \mathbb{P}^1(\mathbb{R})$ is the boundary of $G/K \cong \mathcal{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$. Roughly speaking, we construct automorphic forms on G/K for Γ from Γ-invariant distributions on G/P.

Now we define the Poisson transform after Lewis and Zagier [3], Unterberger [9]. For $z \in \mathcal{H}$, $\lambda \in \mathbb{C}$, $l \in \mathbb{Z}$, we define the Poisson kernel $f_{\lambda,l}(t,z)$ by

$$f_{\lambda,l}(t,z) = \frac{y^\lambda}{|z-t|^{2\lambda}} \cdot \left(\frac{z-t}{|z-t|}\right)^{-l} = \frac{y^\lambda}{|z-t|^{2\lambda-l}(z-t)^l}.$$

When we fix $z \in \mathcal{H}$ (resp. $t \in \mathbb{P}^1(\mathbb{R})$) and regard $f_{\lambda,l}(t,z)$ as a function of t (resp. z), we write $f_{\lambda,l,z}(t)$ (resp. $f_{\lambda,l,t}(z)$).

Lemma. *(1) $f_{\lambda,l,z}$ is an element of $\mathcal{V}_{\lambda,\varepsilon(l)}^{\infty}$, where $\varepsilon(l) = 0(l \equiv 0 \pmod 2), = 1(l \equiv 1 \pmod 2)$.*

(2) For $g \in SL_2(\mathbb{R})$, we have $\left(\pi_{\lambda,\varepsilon(l)}(g)f_{\lambda,l,z}\right)(t) = \left(f_{\lambda,l,t}\big|_l g\right)(z)$, where $\big|_l$ is the slash operator defined by

$$(F|_l g)(z) = \left(\frac{cz+d}{|cz+d|}\right)^{-l} F\left(\frac{az+b}{cz+d}\right) \qquad (z \in \mathcal{H}).$$

(3) $\Delta_l f_{\lambda,l,t}(z) = \lambda(1-\lambda)f_{\lambda,l,t}(z)$, where Δ_l is the Laplace-Beltrami operator defined by

$$\Delta_l = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + ily\frac{\partial}{\partial x}.$$

Definition. For $T \in \mathcal{V}_{\lambda,\varepsilon}^{-\infty}$ and $l \in \mathbb{Z}$ with $\varepsilon(l) = \varepsilon$, we define the *Poisson transform* $\mathcal{P}_{\lambda,l}$ by

$$\mathcal{P}_{\lambda,l}(T)(z) = T(f_{\lambda,l,z}).$$

Definition. Let Γ be a subgroup of $SL_2(\mathbb{Z})$ of finite index, and χ a character of Γ. A function $F : \mathcal{H} \to \mathbb{C}$ is said to be a *Maass form for* Γ *of weight* $l \in \mathbb{Z}$ *with eigenvalue* $\lambda \in \mathbb{C}$ *and character* χ if the following conditions are satisfied:

(1) $(F|_l\gamma)(z) = \chi(\gamma) \cdot F(z)$ for $z \in \mathcal{H}$ and $\gamma \in \Gamma$.
(2) $\Delta_l F = \lambda(1-\lambda)F$
(3) F is slowly increasing at every cusp of Γ.

We denote by $\mathcal{M}_l(\Gamma, \lambda; \chi)$ the space of the Maass forms.

Theorem. *The Poisson transform* $\mathcal{P}_{\lambda,l}$ *defines a map from* $(\mathcal{V}_{\lambda,\varepsilon}^{-\infty})^{\Gamma,\chi}$ *to* $\mathcal{M}_l(\Gamma, \lambda; \chi^{-1})$, *where*

$$(\mathcal{V}_{\lambda,\varepsilon}^{-\infty})^{\Gamma,\chi} = \{T \in \mathcal{V}_{\lambda,\varepsilon}^{-\infty} \mid \pi_{-\lambda,\varepsilon}(\gamma)T = \chi(\gamma)T \text{ for all } \gamma \in \Gamma\}.$$

Example. Let $l \in 2\mathbb{Z}$ and $\operatorname{Re}\lambda > 1/2$. We define the "genuine" Eisenstein distribution by

$$E_\lambda^\circ(f_0) = \frac{1}{2} \sum_{(m,n)\neq(0,0)} |m|^{-2\lambda} f_0\left(\frac{n}{m}\right).$$

Here, for $m = 0$, we put $|m|^{-2\lambda} f_0(\frac{n}{m}) := f_0(\infty) \cdot |n|^{-2\lambda}$. This distribution differs from E_λ defined in (6) by the constant terms. The Poisson transform $\mathcal{P}_{\lambda,l}(E_\lambda^\circ)$ is nothing but the so-called real analytic Eisenstein series

$$E_l(\lambda, z) = \frac{1}{2} \sum_{(m,n)\neq(0,0)} \frac{y^\lambda}{|mz+n|^{2\lambda-l} \cdot (mz+n)^l}.$$

4 A converse theorem for automorphic distributions

Let N be a positive integer, λ a complex number with $\operatorname{Re}(\lambda) > 1/2$ and $2 - 2\lambda \notin \mathbb{Z}_{\leq 0}$. Let $\varepsilon = 0, 1$. Further, let χ be a Dirichlet character of mod$N$ such that $\chi(-1) = (-1)^\varepsilon$. For complex sequences $\boldsymbol{a} = \{a(n)\}_{n\in\mathbb{Z}\setminus\{0\}}, \boldsymbol{b} = \{b(n)\}_{n\in\mathbb{Z}\setminus\{0\}}$ of polynomial growth, we define the Dirichlet series $\xi_\pm(\boldsymbol{a}; s), \xi_\pm(\boldsymbol{b}; s)$ and the completed zeta functions $\Xi_\pm(\boldsymbol{a}; s), \Xi_\pm(\boldsymbol{b}; s)$ by (4).

Let r be an odd prime with $(N, r) = 1$. We take an arbitrary Dirichlet character ψ of mod r and define the twisted zeta functions $\xi_\pm(\boldsymbol{a}, \psi; s), \Xi_\pm(\boldsymbol{a}, \psi; s), \xi_\pm(\boldsymbol{b}, \psi; s), \Xi_\pm(\boldsymbol{b}, \psi; s)$ by

$$\xi_\pm(\boldsymbol{a}, \psi; s) = \sum_{n=1} \frac{a(\pm n)\tau_\psi(\pm n)}{n^s}, \qquad \Xi_\pm(\boldsymbol{a}; \psi, s) = (2\pi)^{-s}\Gamma(s)\xi_\pm(\boldsymbol{a}, \psi; s),$$

$$\xi_\pm(\boldsymbol{b}, \psi; s) = \sum_{n=1} \frac{b(\pm n)\tau_\psi(\pm n)}{n^s}, \qquad \Xi_\pm(\boldsymbol{b}, \psi; s) = (2\pi)^{-s}\Gamma(s)\xi_\pm(\boldsymbol{b}, \psi; s),$$

where $\tau_\psi(n)$ is the Gauss sum defined by

$$\tau_\psi(n) = \sum_{\substack{(m,r)=1 \\ \text{mod } r}} \psi(m) e^{2\pi\sqrt{-1}mn/r}.$$

These twisted zeta functions were first considered by Razar [6]. We assume

[**A1**] $\xi_\pm(\boldsymbol{a}; s), \xi_\pm(\boldsymbol{b}; s)$ converges absolutely for $\operatorname{Re} s > 1$ and have analytic continuations to meromorphic functions of s to $\mathbb{C}$.

[A2] (1) $\Xi_\pm(\boldsymbol{a};s), \Xi_\pm(\boldsymbol{b};s)$ satisfy the functional equation

$$\gamma(s)\begin{pmatrix}\Xi_+(\boldsymbol{a};s)\\ \Xi_-(\boldsymbol{a};s)\end{pmatrix} = N^{2-2\lambda-s}\cdot\Sigma\cdot\gamma(2-2\lambda-s)\begin{pmatrix}\Xi_+(\boldsymbol{b};2-2\lambda-s)\\ \Xi_-(\boldsymbol{b};2-2\lambda-s)\end{pmatrix},$$

where $\gamma(s)$ and Σ are defined by (5).

(2) $\Xi_\pm(\boldsymbol{a},\psi;s), \Xi_\pm(\boldsymbol{b},\psi;s)$ satisfy the functional equation

$$\gamma(s)\begin{pmatrix}\Xi_+(\boldsymbol{a},\psi;s)\\ \Xi_-(\boldsymbol{a},\psi;s)\end{pmatrix} = \overline{\chi(r)}\cdot\overline{\psi(-N)}\cdot r^{2\lambda-2}\cdot(Nr^2)^{2-2\lambda-s}$$
$$\cdot\Sigma\cdot\gamma(2-2\lambda-s)\begin{pmatrix}\Xi_+(\boldsymbol{b},\overline{\psi};2-2\lambda-s)\\ \Xi_-(\boldsymbol{b},\overline{\psi};2-2\lambda-s)\end{pmatrix}.$$

[A3] $\xi_\pm(\boldsymbol{a};s), \xi_\pm(\boldsymbol{b};s), \xi_\pm(\boldsymbol{a},\psi,s), \xi_\pm(\boldsymbol{b},\overline{\psi},s)$ have poles only at $s=1, 2-2\lambda$ of order at most 1, and the residues satisfy the following relations:

$$\operatorname*{Res}_{s=1}\xi_\pm(\boldsymbol{a},\psi;s) = \overline{\chi(r)}\cdot\overline{\psi(-N)}\cdot r^{-2\lambda}\cdot\tau_{\overline{\psi}}(0)\cdot\operatorname*{Res}_{s=1}\xi_\pm(\boldsymbol{a};s),$$
$$\operatorname*{Res}_{s=2-2\lambda}\xi_\pm(\boldsymbol{a},\psi;s) = \overline{\chi(r)}\cdot\overline{\psi(-N)}\cdot r^{2\lambda-2}\cdot\tau_{\overline{\psi}}(0)\cdot\operatorname*{Res}_{s=2-2\lambda}\xi_\pm(\boldsymbol{a};s),$$
$$\operatorname*{Res}_{s=1}\left(\overline{\chi(r)}\cdot\overline{\psi(-N)}\cdot r^{2\lambda}\xi_\pm(\boldsymbol{b},\overline{\psi};s)\right) = \tau_\psi(0)\operatorname*{Res}_{s=1}\xi_\pm(\boldsymbol{b};s),$$
$$\operatorname*{Res}_{s=2-2\lambda}\left(\overline{\chi(r)}\cdot\overline{\psi(-N)}\cdot r^{2-2\lambda}\xi_\pm(\boldsymbol{b},\overline{\psi};s)\right) = \tau_\psi(0)\operatorname*{Res}_{s=2-2\lambda}\xi_\pm(\boldsymbol{b};s).$$

[A4] $\xi_\pm(\boldsymbol{a};s), \xi_\pm(\boldsymbol{a},\psi;s), \xi_\pm(\boldsymbol{b};s), \xi_\pm(\boldsymbol{b},\psi;s)$ have finite order in lacunary vertical strips, i.e., For any $\alpha_1<\alpha_2(\alpha_1,\alpha_2\in\mathbb{R})$, there exists some $\tau_0, K, \rho>0$ such that

$$|\xi_\pm(\boldsymbol{a};\alpha+\sqrt{-1}\tau)|,\ |\xi_\pm(\boldsymbol{a},\psi;\alpha+\sqrt{-1}\tau)| < K\cdot e^{|\tau|^\rho}$$
$$|\xi_\pm(\boldsymbol{b};\alpha+\sqrt{-1}\tau)|,\ |\xi_\pm(\boldsymbol{b},\psi;\alpha+\sqrt{-1}\tau)| < K\cdot e^{|\tau|^\rho}$$

for any $\alpha\in[\alpha_1,\alpha_2]$ and τ with $|\tau|>\tau_0$.

Theorem. *We assume that [A1]–[A4] hold for every (not necessarily primitive) Dirichlet character ψ of* mod r. *We put*

$$a(0) = \left(\frac{2\pi}{N}\right)^{2\lambda-2}\Gamma(2-2\lambda)\left\{e^{\frac{\pi\sqrt{-1}}{2}(2-2\lambda)}\operatorname*{Res}_{s=2-2\lambda}\xi_+(\boldsymbol{b};s) + e^{-\frac{\pi\sqrt{-1}}{2}(2-2\lambda)}\operatorname*{Res}_{s=2-2\lambda}\xi_-(\boldsymbol{b};s)\right\},$$
$$a(\infty) = \frac{N}{2}\left(\operatorname*{Res}_{s=1}\xi_+(\boldsymbol{b};s) + \operatorname*{Res}_{s=1}\xi_-(\boldsymbol{b};s)\right),$$
$$b(0) = (-1)^\varepsilon(2\pi)^{2\lambda-2}\Gamma(2-2\lambda)\left\{e^{\frac{\pi\sqrt{-1}}{2}(2-2\lambda)}\operatorname*{Res}_{s=2-2\lambda}\xi_+(\boldsymbol{a};s) + e^{-\frac{\pi\sqrt{-1}}{2}(2-2\lambda)}\operatorname*{Res}_{s=2-2\lambda}\xi_-(\boldsymbol{a};s)\right\},$$
$$b(\infty) = \frac{(-1)^\varepsilon}{2}\left(\operatorname*{Res}_{s=1}\xi_+(\boldsymbol{a};s) + \operatorname*{Res}_{s=1}\xi_-(\boldsymbol{a};s)\right),$$

and define the linear functionals T_0, T_∞ on $\mathcal{V}^\infty_{\lambda,\varepsilon}$ by

$$T_0(\varphi) = a(\infty)\varphi(\infty) + \sum_{n=-\infty}^{\infty} a(n)(\mathcal{F}\varphi)(n),$$
$$T_\infty(\varphi) = b(\infty)\varphi(\infty) + \sum_{n=-\infty}^{\infty} b(n)(\mathcal{F}\varphi)\left(\frac{n}{N}\right)$$

for $\varphi \in \mathcal{V}^{\infty}_{\lambda,\varepsilon}$. Then, $T_0(f_0) = T_\infty(f_\infty)$ and T_0 is an automorphic distribution for $\Gamma_0(N)$ with character χ.

Corollary. *The Poisson transform $(\mathcal{P}_{\lambda,l}T_0)(z)$ is a Maass form for $\Gamma_0(N)$ of weight l, with eigenvalue λ and character χ^{-1}. Moreover, $(\mathcal{P}_{\lambda,l}T_0)(z)$ has the following Fourier expansion:*

$$(\mathcal{P}_{\lambda,l}T_0)(z) = a(\infty)y^\lambda + a(0)\cdot(-1)^{\frac{l}{2}}\frac{(2\pi)2^{1-2\lambda}\Gamma(2\lambda-1)}{\Gamma(\lambda+\frac{l}{2})\Gamma(\lambda-\frac{l}{2})}y^{1-\lambda}$$

$$+(-1)^{\frac{l}{2}}\pi^\lambda\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}|n|^{\lambda-1}a(n)\frac{W_{\frac{\mathrm{sgn}(n)l}{2},\,\lambda-\frac{1}{2}}(4\pi|n|y)}{\Gamma\left(\lambda+\frac{\mathrm{sgn}(n)l}{2}\right)}e^{2\pi inx}.$$

Remark. (1) *"It is an open question whether or not Weil's argument applies to Maass forms. A key point for Weil is that radially symmetric holomorphic functions are necessarily constant; this is not true in the non-holomorphic case."* (quoted from Gelbart and Miller [2]).

(2) Recently, Diamantis and Goldfeld [1] proved the converse theorem for double Dirichlet series associated with metaplectic Eisenstein series. Their twists of L-functions involve the Gauss sum $\tau_\psi(n)$, not the value $\psi(n)$ of the character ψ. Moreover, it is necessary to include non-primitive Dirichlet characters. We have followed Diamantis-Goldfeld's method.

(3) Diamantis-Goldfeld's result is a converse theorem for vector-valued Dirichlet series, where the dimension of the vector (=the number of Dirichlet series) is equal to the number of cusps of $\Gamma_0(N)$. On the contrary, our argument is rather irrelevant to the discrete subgroup in question.

5 Application to zeta functions associated with quadratic forms

We recall the zeta functions studied by Peter [5], Ueno [8]. Put $V = \mathbb{C}^{m+2}$ and let $Q(x)$ be a non-degenerate integral quadratic form on V of the form

$$Q(x) = x_0x_{m+1} + \sum_{1\le i,j\le m} a_{ij}x_ix_j,$$

where $a_{ij} = a_{ji} \in \frac{1}{2}\mathbb{Z}\,(i \neq j)$ and $a_{ii} \in \mathbb{Z}$. The matrix of Q is given by

$$\begin{pmatrix} 0 & 0 & 1/2 \\ 0 & A & 0 \\ 1/2 & 0 & 0 \end{pmatrix}$$

with $A = (a_{ij})$. We consider the maximal subgroup of $SO(Q)$ of the form

$$P = \left\{ \begin{pmatrix} a & -2a\,{}^tuAh & -aA[u] \\ 0 & h & u \\ 0 & 0 & a^{-1} \end{pmatrix} \;\middle|\; \begin{array}{l} a \in \mathbb{C}^\times \\ h \in SO(A) \\ u \in \mathbb{C}^m \end{array} \right\}.$$

Then the triplet $(P \times GL_1(\mathbb{C}), V)$ is a prehomogeneous vector space.

Let $D = \det(2A)$. For positive integers l, n, we put

$$r(l,n) = \sharp\{v \in \mathbb{Z}^m/(l\mathbb{Z})^m \mid A[v] \equiv n \pmod{l}\},$$
$$r^*(l,n) = \sharp\{v^* \in \mathbb{Z}^m/2lA\mathbb{Z}^m \mid 4^{-1}\cdot|D|A^{-1}[v^*] \equiv n \pmod{|D|l}\}.$$

and define the Dirichlet series $Z(n,w), Z^*(n,w)$ $(n \in \mathbb{Z})$ by

$$Z(n,w) = \sum_{l=1}^{\infty} r(l,n) l^{-w}, \qquad Z^*(n,w) = \sum_{l=1}^{\infty} r^*(l,n) l^{-w}$$

The prehomogeneous zeta functions associated with $(P \times GL_1(\mathbb{C}), V)$ coincide with

$$\zeta_\epsilon(w,s) = \sum_{n=1}^{\infty} Z(\epsilon n, w) n^{-s}, \quad \zeta_\eta^*(w,s) = |D|^s \sum_{n=1}^{\infty} Z^*(\eta n, w) n^{-s} \quad (\epsilon, \eta = \pm).$$

By using the theory of prehomogeneous vector spaces, Ueno proved that $\zeta_\epsilon(w,s)$ and $\zeta_\eta^*(w,s)$ have analytic continuations to meromorphic functions on $\mathbb{C}^2$ and satisfy functional equations.

Theorem. *Assume that m is even and let $D = \det(2A)$. Then, under a suitable adjustment ($w = 2\lambda - 1 + \frac{m}{2}$, etc.), ζ_ϵ and ζ_η^* satisfy the assumption of our converse theorem, and we can construct Maass forms for $\Gamma_0(|D|)$ or $\Gamma_0(4|D|)$ with explicit Fourier coefficients defined by $Z(n,w)$ and $Z^*(n,w)$.*

Remark. When m is odd, it is expected that ζ_ϵ and ζ_η^* correspond to Maass forms of half-integral weight. To include the Maass forms of general weight in our framework, we need to consider the principal series representation of the universal covering group $\widetilde{G}$ of $G = SL_2(\mathbb{R})$.

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