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Lifting from Maass cusp forms for $\Gamma_0(2)$ to cusp forms on GL(2) over a division quaternion algebra

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Abstract

This note is a write-up of our results on an explicit construction of Maass cusp forms on $GL_2(\mathbb{H})$ by lifting from Maass cusp forms for the congruence subgroup $\Gamma_0(2)$, presented by the second named author in the RIMS conference on automorphic forms (2014, Jan. 20–24). We also report our recent results on an explicit description of cuspidal automorphic representations generated by our lifts, which has been obtained after the conference. We have known that our cuspidal representations provide examples of CAP representations, and in fact, counterexamples of the Generalized Ramanujan conjecture. This note also serves as a summary of our recent paper [13].

1 Results

1.1 Lifting to cusp forms on $GL_2(\mathbb{H})$

Let B be the definite quaternion algebra over $\mathbb{Q}$ with discriminant two, which are realized as $B = \mathbb{Q} + \mathbb{Q}i + \mathbb{Q}j + \mathbb{Q}ij$ with $i, j \in B$ characterized by

$$i^2 = j^2 = -1, \quad ij = -ji.$$

By tr and ν we denote the reduced trace and the reduced norm of B respectively. Let $\mathcal{G}$ denote the $\mathbb{Q}$-algebraic group defined by its group of $\mathbb{Q}$-rational points

$$\mathcal{G}(\mathbb{Q}) = GL_2(B).$$

Let $\mathbb{H}$ be the Hamilton quaternion algebra, which coincides with $B \otimes_{\mathbb{Q}} \mathbb{R}$. In what follows we consider the real Lie group $G = GL_2(\mathbb{H})$, which is nothing but $\mathcal{G}(\mathbb{R})$. The Lie group G admits an Iwasawa decomposition

$$G = Z^+NAK,$$

where

$$Z^+ := \left\{ \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} \;\middle|\; c \in \mathbb{R}_+^\times \right\}, \quad N := \left\{ n(x) = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \;\middle|\; x \in \mathbb{H} \right\}, \tag{1.1}$$

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$$A := \left\{ a_y := \begin{bmatrix} \sqrt{y} & 0 \\ 0 & \sqrt{y}^{-1} \end{bmatrix} \;\middle|\; y \in \mathbb{R}_+^\times \right\}, \quad K := \{k \in G \mid {}^t\bar{k}k = 1_2\}.$$

The subgroup Z^+ is contained in the center of G and K is a maximal compact subgroup of G, which is isomorphic to the definite symplectic group $\mathrm{Sp}^*(2)$. Let us consider the quotient G/Z^+K, which is realized as

$$\left\{ \begin{bmatrix} y & x \\ 0 & 1 \end{bmatrix} \;\middle|\; y \in \mathbb{R}_+^\times,\ x \in \mathbb{H} \right\}.$$

We remark that this gives a realization of the 5-dimensional real hyperbolic space. In fact, automorphic forms we are going to construct are regarded as Maass cusp forms on the real hyperbolic space.

We now introduce several spaces of auromorphic forms. For $r \in \mathbb{C}$ and the congruence subgroup $\Gamma_0(2) \subset \mathrm{SL}_2(\mathbb{R})$ we denote by $S(\Gamma_0(2), -(\frac{1}{4} + \frac{r^2}{4}))$ the space of Maass cusp forms of weight 0 on the complex upper half plane $\mathfrak{h}$ whose eigenvalue with respect to the hyperbolic Laplacian is $-(\frac{1}{4} + \frac{r^2}{4})$. For this space we should remark that r can be assumed to be a real number in view of the Selberg conjecture for $\Gamma_0(2)$ (cf. [9, Corollary 11.5]).

Let $\mathcal{O}$ be the Hurwitz order of B, which is a maximal order of B. For the discrete subgroup $\Gamma := GL_2(\mathcal{O}) \subset \mathrm{GL}_2(\mathbb{H})$ and $r \in \mathbb{C}$ we denote by $\mathcal{M}(\Gamma, r)$ the space of smooth functions F on $\mathrm{GL}_2(\mathbb{H})$ satisfying the following conditions:

1. $\Omega \cdot F = -(\frac{r^2}{4} + 1)F$, where Ω is the Casimir operator (cf. [13, (2.3)]),
2. for any $(z, \gamma, g, k) \in Z^+ \times \Gamma \times G \times K$, we have $F(z\gamma gk) = F(g)$,
3. F is of moderate growth.

We now explain the explicit construction of our lifting. Let $f \in S(\Gamma_0(2), -(\frac{1}{4} + \frac{r^2}{4}))$ be a Maass cusp form for $\Gamma_0(2)$ and let us assume that f is an eigenfunction of the Atkin-Lehner involution. By ϵ we denote the signature for the involution. Let S be the dual lattice of $\mathcal{O}$ in B with respect to $\frac{1}{2}\mathrm{tr}$. We have $S = \varpi_2\mathcal{O}$ with the uniformizer $\varpi_2 := 1 + i$ at 2. We introduce

$$S^{\mathrm{prim}} := \{\beta \in S \setminus \{0\} \mid \varpi_2 \mid \beta,\ \varpi_2^2 \nmid \beta,\ d \nmid \beta \text{ for all odd integer } d\},$$

where "$d \nmid \beta$" for $d \in \mathbb{Z}$ means that β is not a multiple of an element in S by d. We then know that any $\beta \in S \setminus \{0\}$ can be expressed as

$$\beta = \varpi_2{}^u d\beta_0,$$

where u is a non-negative integer, d an odd integer and $\beta_0 \in S^{\mathrm{prim}}$.

Let $\{c(n)\}_{n \in \mathbb{Z}\setminus\{0\}}$ be Fourier coefficients of f. From $c(n)$'s we define numbers $A(\beta)$'s by

$$A(\beta) := |\beta| \sum_{t=0}^{u} \sum_{n|d} (-\epsilon)^t c(-\frac{|\beta|^2}{2^{t+1}n^2})$$

for $\beta \in S \setminus \{0\}$, where we put $|\beta| := \sqrt{\nu(\beta)}$ for $\beta \in B$. We define a Maass cusp form F_f on $\mathrm{GL}_2(\mathbb{H})$ by the Fourier expansion

$$F_f(n(x)a_y) := \sum_{\beta \in S \setminus \{0\}} A(\beta) y^2 K_{\sqrt{-1}r}(2\pi|\beta|y) e^{2\pi\sqrt{-1}\,\mathrm{Re}(\beta x)},$$

where $K_{\sqrt{-1}r}$ denotes the K-Bessel function parametrized by $\sqrt{-1}r$. We are now able to state our first result:

Theorem 1.1 *Let f be a non-zero Mass cusp form which is an eigenfunction of the Atkin-Lehner involution. Then F_f is a non-zero cusp form on $\mathrm{GL}_2(\mathbb{H})$.*

1.2 Cuspidal representations generated by F_f's

Our study also deals with the cuspidal representations generated by F_f's. They turn out to be CAP representations of $\mathrm{GL}_2(B)$ and provide counterexamples of the Generalized Ramanujan conjecture. To be more precise, assume that f is a Hecke eigenform. We can regard F_f as a cusp form on the adele group $\mathcal{G}(\mathbb{A})$ with $\mathcal{G} = \mathrm{GL}_2(B)$. We can show that F_f is a Hecke eigenform, namely our lifting is Hecke equivariant. Then the strong multiplicity one theorem proved by Badulescu and Renard [1], [2] implies that F_f generates an irreducible cuspidal representation $\pi := \otimes'_{p \leq \infty} \pi_p$ of $\mathcal{G}(\mathbb{A})$. By our detailed study on Hecke eigenvalues of F_f we can determine local representations π_p for every $p < \infty$. We can also determine π_p explicitly at $p = \infty$ by the calculation of the infinitesimal action of the Casimir operator. For this we note that every π_p is unramified (at $p < \infty$) or spherical (at $p = \infty$). We can show that π_p (respectively π_∞) is non-tempered at every odd prime p (respectively tempered at $p = \infty$). If we further assume that f is a new form, we can also show the non-temperedness of π_p at $p = 2$. These lead to our another result as follows:

Theorem 1.2 *(1) Let f be a non-zero Hecke eigen cusp form and F_f be the lift. Let σ and π_F be irreducible cuspidal representations generated by f and $F = F_f$ respectively. Then π_F is nearly equivalent to an irreducible component of $\mathrm{Ind}_{P_2(\mathbb{A})}^{\mathrm{GL}_4(\mathbb{A})}(|\det|^{-1/2}\sigma \times |\det|^{1/2}\sigma)$. Here P_2 is the standard parabolic subgroup of GL_4 with Levi subgroup $\mathrm{GL}_2 \times \mathrm{GL}_2$. Namely π_F is a CAP representation.*
(2) The cuspidal representations π_F's are counterexamples of the Ramanujan conjecture.

2 Proof of the results

In this section we overview the proofs of Theorem 1.1 and 1.2.

2.1 Proof of Theorem 1.1

(1) Automorphy of F_f with respect to Γ

We follow the approach by [14] for the proof. The fundamental tool to show the automorphy of F_f is the converse theorem by Maass [12]. Let Γ_T be the discrete subgroup of Γ generated by

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \quad (\beta \in \mathcal{O}).$$

By $M(\Gamma_T, r)$ we denote the space of Γ_T-invariant automorphic forms on $GL_2(\mathbb{H})$ defined by replacing Γ with Γ_T.

Theorem 2.1 (Maass) *Let $\{A(\beta)\}_{\beta\in S\setminus\{0\}}$ be a sequence of complex numbers such that*

$$A(\beta) = O(|\beta|^\kappa) \quad (\exists\kappa > 0)$$

and put

$$F(n(x)a_y) := \sum_{\beta\in S\setminus\{0\}} A(\beta)y^2 K_{\sqrt{-1}r}(2\pi|\beta|y)e^{2\pi\sqrt{-1}\,\mathrm{Re}(\beta x)}.$$

For a harmonic polynomial P on $\mathbb{H}$ of degree l we introduce

$$\xi(s,P) := \pi^{-2s}\Gamma(s+\frac{\sqrt{-1}r}{2})\Gamma(s-\frac{\sqrt{-1}r}{2})\sum_{\beta\in S\setminus\{0\}} A(\beta)\frac{P(\beta)}{|\beta|^{2s}},$$

which converges for $\mathrm{Re}(s) > \frac{l+4+\kappa}{2}$. Let $\{P_{l,\nu}\}_\nu$ be a basis of harmonic polynomials on $\mathbb{H}$ of degree l.

Then $F \in \mathcal{M}(\Gamma_T; r)$ is equivalent to the condition that, for any l, ν, the $\xi(s, P_{l,\nu})$ satisfies the following three conditions.

1. *it has analytic continuation to the whole complex plane.*
2. *it is bounded on any vertical strip of the complex plane.*
3. *the functional equation*
$$\xi(2+l-s, P_{l,\nu}) = (-1)^l\xi(s,\hat{P}_{l,\nu})$$
holds, where $\hat{P}(x) := P(\bar{x})$ for $x \in \mathbb{H}$.

We apply this theorem to $F = F_f$. The Fourier expansion of F_f immediately implies the automorphy of F_f with respect to $\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}$ for $u \in \mathcal{O}^\times$ and $\beta \in \mathcal{O}$. Since Γ has

$$\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \quad (u \in \mathcal{O}^\times,\ \beta \in \mathcal{O})$$

as a set of generators, we complete the proof of the Γ-automorphy if we see that F_f satisfies the three conditions in the theorem. We remark that the set of Γ-cusps consists of only one element (cf. [13, Lemma 2.3]). The Fourier expansion therefore implies that F_f is a cusp form if its Γ-automorphy is verified.

We have thus known that it suffice to show the validity of the three conditions for F_f. We need to study analytic properties of the Dirichlet series $\xi(s, P_{l,\nu})$ attached to F_f and harmonic polynomials $P_{l,\nu}$. For that purpose we introduce the Raknin-Selberg L-series

$$I(s) := \int_{\Gamma_0(2)\backslash\mathfrak{h}} f(z)\Theta_{l,\nu}(z)\tilde{E}_\infty(z,s)y^{\frac{l+2}{2}}\frac{dxdy}{y^2},$$

in which the following theta series and the Eisenstein series are involved:

$$\Theta_{l,\nu}(z) := \sum_{\beta\in S} P_{l,\nu}(\beta) e^{2\pi\sqrt{-1}\frac{|\beta|^2}{2}z},$$

$$\tilde{E}_\infty(z,s) := (4\pi)^{\frac{l}{2}}\frac{\Gamma(s+\frac{1}{2}+l)}{\Gamma(s)}(\pi^{-s}\Gamma(s)\zeta(2s))\frac{1}{2}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(2)}(\frac{cz+d}{|cz+d|})^{l+2}(\frac{\mathrm{Im}(z)}{|cz+d|^2})^s.$$

We can now assume that the harmonic polynomials $\{P_{l,\nu}\}_\nu$ are eigenfunctions in the following sense:

$$P_{l,\nu}(\frac{1+i}{\sqrt{2}}x) = \epsilon_{l,\nu}P_{l,\nu}(x).$$

Here $\epsilon_{l,\nu}$ is an eighth root of unity. The following proposition enables us to complete the proof of the automorphy.

Proposition 2.2 *Recall that ϵ denotes the signature of f with respect to the Atkin-Lehner involution. We have*

$$\xi(s+\frac{l}{2}+\frac{1}{2},P_{l,\nu}) = \begin{cases} 2^{1-\frac{l}{2}}\pi^{-(l+1)}(2^s-\epsilon\epsilon_{l,\nu})I(s) & \text{if } \epsilon_{l,\nu}\in\{\pm1\}, \\ 0 & \text{if } \epsilon_{l,\nu}\notin\{\pm1\}) \end{cases}$$

The problem is then reduced to the analytic properties of $I(s)$. This is studied by virtue of good analytic properties of automorphic forms involved in $I(s)$, which leads to the proof of the automorphy of F_f with respect to Γ.

(2) Non-vanishing of F_f

To verify the non-vanishing assertion on F_f as in Theorem 1.1 we need the lemma as follows:

Lemma 2.3 *Let $f\in S(\Gamma_0(2);-(\frac{1}{4}+\frac{r^2}{4}))$ with Fourier coefficients $c(n)$ and with eigenvalue ϵ of the Atkin Lehner involution. Then, there exist $N>0, N\in\mathbb{Z}$, such that $c(-N)\neq 0$.*

Assume that $c(n)=0$ for all $n<0$. Set $f_1(z)=(f(z)+f(-\bar{z}))/2$ and $f_2(z)=(f(z)-f(-\bar{z}))/2$, which are even or odd Maass cusp form respectively. Then, f_1 and f_2 have the same eigenvalue ϵ of the Atkin Lehner involution as f. We see that the L-functions for f_1 and f_2 satisfy $L(s,f_1)=L(s,f_2)$ since f_1 and f_2 have the same Fourier coefficients for positive integers. On the other hand, $L(s,f_1)$ and $L(s,f_2)$ however satisfy functional equations with the gamma factors shifted by 1. We then know that, if $L(s,f_1)\neq 0$, we obtain an identity of gamma factors, which can be checked to be impossible. This implies the lemma.

We can then prove the non-vanishing property of F_f as in the theorem. Let N_0 be the smallest positive integer such that $c(-N_0)\neq 0$. Let $\beta_0\in\mathcal{O}$ be such that $|\beta_0|^2=N_0$. Choose, $\beta=\varpi_2\beta_0$. Then, by the choice of N_0 and definition of $A(\beta)$, we see that $A(\beta)=\sqrt{2N_0}c(-N_0)\neq 0$, as required.

2.2 Proof of Theorem 1.2

For each place $p \le \infty$ let $G_p := \mathrm{GL}_2(B_p)$ with $B_p = B \otimes_{\mathbb{Q}} \mathbb{Q}_p$. For a finite prime $p \neq 2$, we have $\mathrm{GL}_2(B_p) \simeq \mathrm{GL}_4(\mathbb{Q}_p)$. Let $\mathcal{O}_p$ be the p-adic completion of $\mathcal{O}$ for $p < \infty$. For a finite prime $p \neq 2$, $\mathcal{O}_p \simeq M_2(\mathbb{Z}_p)$ and $\mathrm{GL}_2(\mathcal{O}_p) \simeq \mathrm{GL}_4(\mathbb{Z}_p)$. Set $K_p = \mathrm{GL}_2(\mathcal{O}_p)$ for $p < \infty$.

We first remark that we can regard a cusp form $F = F_f$ as a cusp form Φ_F on the adele group $\mathcal{G}(\mathbb{A})$ since the class number $\mathcal{G}$ is one, i.e. $\mathcal{G}(\mathbb{A}) = \mathcal{G}(\mathbb{Q})\mathrm{GL}_2(\mathbb{H})U$ with $U := \prod_{p<\infty} K_p$. We study the cuspidal representation π_F of $\mathcal{G}(\mathbb{A})$ generated by Φ_F in detail.

(1) Hecke equivariance

We denote by $\mathcal{H}_p$ the Hecke algebra for $\mathrm{GL}_2(B_p)$ with respect to $\mathrm{GL}_2(\mathcal{O}_p)$ for $p < \infty$. According to [15, Section 8, Theorem 6], $\mathcal{H}_p$ has the following generators:

$$\begin{cases} \{\varphi_1^{\pm 1},\ \varphi_2\} & \text{if } p = 2, \\ \{\phi_1^{\pm 1},\ \phi_2,\ \phi_3,\ \phi_4\} & \text{if } p \neq 2. \end{cases}$$

Here $\varphi_1,\ \varphi_2$ denote the characteristic functions for

$$K_2 \begin{bmatrix} \varpi_2 & 0 \\ 0 & \varpi_2 \end{bmatrix} K_2,\ K_2 \begin{bmatrix} \varpi_2 & 0 \\ 0 & 1 \end{bmatrix} K_2$$

respectively, and $\phi_1,\ \phi_2,\ \phi_3,\ \phi_4$ denote the characteristic functions for

$$K_p \begin{bmatrix} p & & & \\ & p & & \\ & & p & \\ & & & p \end{bmatrix} K_p,\ K_p \begin{bmatrix} p & & & \\ & p & & \\ & & p & \\ & & & 1 \end{bmatrix} K_p,\ K_p \begin{bmatrix} p & & & \\ & p & & \\ & & 1 & \\ & & & 1 \end{bmatrix} K_p,\ K_p \begin{bmatrix} p & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} K_p$$

respectively when $p \neq 2$. Recall that ϖ_2 denotes a prime element of B_2.

We first state our result on the Hecke equivariance of $F = F_f$ at $p = 2$.

Proposition 2.4 *Let $f \in S(\Gamma_0(2); -(\frac{1}{4} + \frac{r^2}{4}))$ be a new form with Hecke eigenvalue λ_p for $p = 2$ and an eigenfunction of the Atkin Lehner involution with eigenvalue ϵ. Then*

$$(K_2 \begin{bmatrix} \varpi_2 & \\ & 1 \end{bmatrix} K_2)F = -3\sqrt{2}\epsilon F.$$

We next state the Hecke equivariance of $F = F_f$ at odd primes.

Proposition 2.5 *Let $f \in S(\Gamma_0(2); -(\frac{1}{4} + \frac{r^2}{4}))$ be a Hecke eigenform with Hecke eigenvalue λ_p for every odd prime p. For an odd prime p we then have*

$$(K_p \begin{bmatrix} p & & & \\ & p & & \\ & & p & \\ & & & 1 \end{bmatrix} K_p)F = (K_p \begin{bmatrix} p & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} K_p)F = p(p+1)\lambda_p F$$

$$(K_p \begin{bmatrix} p & & & \\ & p & & \\ & & 1 & \\ & & & 1 \end{bmatrix} K_p)F = (p^2\lambda_p^2 + p^3 + p)F.$$

To obtain these propositions we first calculate the actions of Hecke operators on the adelization Φ_F of $F = F_f$. We can interpret such actions in the non-adelic manner and see the influence of the actions on the Fourier expansion of F_f. We can relate such influence to the recurrence relation satisfied by Fourier coefficients of f. We then verify that F_f is a Hecke eigenform if so is f, and obtain the formula for Hecke eigenvalues in the above propositions.

(2) Cuspidal representation generated by F_f (Determination of local components)

The Hecke equivariance from the previous section enables us to determine the local components of the automorphic representation corresponding to the lifting. This will lead us to the conclusion that we have obtained a CAP representation and have found a couterexample of the Ramanujan conjecture.

Let π_F be the irreducible cuspidal automorphic representation of $\mathcal{G}(\mathbb{A})$ generated by the right translates of Φ_F. Note that the irreducibility follows from the strong multiplicity one result for $\mathcal{G}(\mathbb{A})$ (see [1], [2]). The representation π_F is cuspidal since F is a cusp form. Let $\pi_F = \otimes'_p \pi_p$, where π_p is an irreducible admissible representation of $\mathcal{G}(\mathbb{Q}_p)$ for $p < \infty$ and π_∞ is an irreducible admissible representation of $\mathcal{G}(\mathbb{R})$. Recall $U = \prod_{p<\infty} K_p$ where K_p is the maximal compact subgroup of $\mathcal{G}_p$. Hence, for $p < \infty$, the representation π_p is a spherical representation and can be realized as a subrepresentation of an unramified principal series representation, i.e. a representation induced from an unramified character of the Borel subgroup. The representation π_p is completely determined by the action of the Hecke algebra $H(\mathcal{G}_p, K_p)$ on the spherical vector in π_p, which in turn, is completely determined by the Hecke eigenvalues of F obtained in the previous section. See [3] for details. For $p = 2$ we need to assume that f is a new form for the determination of Hecke eigenvalue of F_f.

We first determine π_p for p odd. If p is an odd prime, then we have $\mathcal{G}_p = \mathrm{GL}_4(\mathbb{Q}_p)$ and $K_p = \mathrm{GL}_4(\mathbb{Z}_p)$. Given 4 unramified characters $\chi_1, \chi_2, \chi_3, \chi_4$ of $\mathbb{Q}_p^\times$, we obtain a character χ of the Borel subgroup P of upper triangular matrices in $\mathcal{G}$, by

$$\chi(\begin{bmatrix} a_1 & * & * & * \\ & a_2 & * & * \\ & & a_3 & * \\ & & & a_4 \end{bmatrix}) = \chi_1(a_1)\chi_2(a_2)\chi_3(a_3)\chi_4(a_4).$$

The modulus character δ_P is given by

$$\delta_P(\begin{bmatrix} a_1 & * & * & * \\ & a_2 & * & * \\ & & a_3 & * \\ & & & a_4 \end{bmatrix}) = |a_1^3 a_2 a_3^{-1} a_4^{-3}|,$$

where $|*|$ denotes the p-adic absolute value. The unramified principal representation corresponding to χ is given by $I(\chi)$ which consists of locally constant functions $f : \mathrm{GL}_4(\mathbb{Q}_p) \to \mathbb{C}$, satisfying

$$f(bg) = \delta_P(b)^{1/2}\chi(b)f(g), \text{ for all } b \in P, g \in \mathrm{GL}_4(\mathbb{Z}_p).$$

We can now give the statement of the determination of π_p for an odd p.

Proposition 2.6 *Let $f \in S(\Gamma_0(2); -(\frac{1}{4}+\frac{r^2}{4}))$ be a Hecke eigenform with Hecke eigenvalue λ_p for every odd prime p. Let $\pi_F = \otimes'_p \pi_p$ be the corresponding irreducible cuspidal automorphic representation of $\mathcal{G}(\mathbb{A})$. For an odd prime p, the representation π_p is the unique spherical constituent of the unramified principal series representation $I(\chi)$ where, up to the action of the Weyl group of GL_4, the character χ is given by*

$$\chi_1(p) = p^{1/2}\frac{\lambda_p + \sqrt{\lambda_p^2 - 4}}{2}, \quad \chi_2(p) = p^{1/2}\frac{\lambda_p - \sqrt{\lambda_p^2 - 4}}{2},$$
$$\chi_3(p) = p^{-1/2}\frac{\lambda_p + \sqrt{\lambda_p^2 - 4}}{2}, \quad \chi_4(p) = p^{-1/2}\frac{\lambda_p - \sqrt{\lambda_p^2 - 4}}{2}.$$

We next determine π_2. Recall that $B_2 = B \otimes_{\mathbb{Q}} \mathbb{Q}_2$, where B is a definite quaternion algebra over $\mathbb{Q}$ with discriminant 2 and $\mathcal{O}_2$ is the completion of the Hurwitz order $\mathcal{O}$ at 2. In this case $\mathcal{G}_2 = \mathrm{GL}_2(B_2)$ and $K_2 = \mathrm{GL}_2(\mathcal{O}_2)$. Given two unramified characters χ_1, χ_2 of $B_2^\times$, we obtain a character χ of the Borel subgroup of upper triangular matrices on $\mathcal{G}$ by

$$\chi(\begin{bmatrix} \alpha & * \\ 0 & \beta \end{bmatrix}) = \chi_1(\alpha)\chi_2(\beta).$$

The modulus character is given by

$$\delta(\begin{bmatrix} \alpha & * \\ 0 & \beta \end{bmatrix}) = |\alpha/\beta|^2.$$

Here, $|\,|$ is the 2-adic absolute value of the reduced norm of B_2. The unramified principal series representation corresponding to χ is given by $I(\chi)$ which consists of locally constant functions $f : \mathcal{G}_2 \to \mathbb{C}$, satisfying

$$f(bg) = \delta(b)^{1/2}\chi(b)f(g), \text{ for all } b \in \text{ Borel subgroup}, g \in \mathcal{G}_2.$$

The local component π_2 is determined as follows:

Proposition 2.7 *Let $f \in S(\Gamma_0(2); -(\frac{1}{4}+\frac{r^2}{4}))$ be a new form with Hecke eigenvalue λ_p for $p = 2$ and Atkin Lehner eigenvalue ϵ, for which $\lambda_2 = -\epsilon$ holds. Let $\pi_F = \otimes'_p \pi_p$ be the corresponding irreducible cuspidal automorphic representation of $\mathcal{G}(\mathbb{A})$. The representation π_2 is the unique spherical constituent of the unramified principal series representation $I(\chi)$ where, up to the action of the Weyl group, the character χ is given by*

$$\chi_1(\varpi_2) = -\sqrt{2}\epsilon, \qquad \chi_2(\varpi_2) = -1/\sqrt{2}\epsilon.$$

We finally determine π_∞. Let us note that $F = F_f \in \mathcal{M}(\mathrm{GL}_2(\mathcal{O});r)$ implies that the archimedean component π_∞ of π_F is spherical. Namely, π_∞ has a K_∞-invariant vector, where we put $K_\infty := K$ with K as in (1.1).

We now introduce $M_\infty := \left\{ \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} \;\middle|\; u_1, u_2 \in \mathbb{H}^1 \right\}$, where $\mathbb{H}^1 := \{x \in \mathbb{H} \mid \nu(x) = 1\}$. Let P_∞ be the standard proper parabolic subgroup $\mathcal{G}_\infty = GL_2(\mathbb{H})$ given by

$$\left\{ \begin{pmatrix} a & * \\ 0 & d \end{pmatrix} \in \mathcal{G}_\infty \right\}.$$

We have $P_\infty := Z^+NAM_\infty$, where Z^+, N and A are as in (1.1). The group Z^+AM_∞ is nothing but the Levi subgroup of P_∞. We now note that the Langlands classification of real reductive groups (cf. [11]) implies that π_∞ has to be embedded into some principal series representation I_{P_∞} of $\mathcal{G}_\infty$ induced from a quasi-character of P_∞. Since π_∞ is spherical I_{P_∞} is also spherical. As π_∞ has the trivial central character, so does I_{P_∞}. Combining these with the Frobenius reciprocity for compact Lie groups (cf. [10, Theorem 9.9]), one can verify that the quasi-character of P_∞ inducing I_{P_∞} has to be trivial on Z^+M_∞, and that I_{P_∞} and π_∞ have a unique K_∞-invariant vector, up to constant multiples. For $s \in \mathbb{C}$ we introduce the quasi-character χ_s of P_∞ defined by

$$\chi_s\left(\begin{pmatrix} a & * \\ 0 & d \end{pmatrix} \right) = \nu(ad^{-1})^s.$$

We note that the quasi-characters of P_∞ trivial on Z^+M_∞ should be of this form. We furthermore introduce the modulus character δ_∞ of P_∞. The principal series representation I_{P_∞} is then expressed as

$$I_{P_\infty} = \mathrm{Ind}_{P_\infty}^{\mathcal{G}_\infty}(\delta_\infty \chi_s).$$

Proposition 2.8 *We have an isomorphism*

$$\pi_\infty \simeq \mathrm{Ind}_{P_\infty}^{\mathcal{G}_\infty}(\delta_\infty \chi_{\pm\sqrt{-1}r})$$

as $(\mathfrak{g}, K_\infty)$-modules, where recall that $\mathfrak{g}$ denotes the Lie algebra of $\mathcal{G}_\infty$.

The proof of this proposition starts with showing $s = \pm\sqrt{-1}r$ (for the real number r, see Section 1.1). This is settled by considering the infinitesimal action of the Casimir operator on a spherical vector. By Harish-Chandra [8, Section 41, Theorem 1] the spherical principal series representation $\mathrm{Ind}_{P_\infty}^{\mathcal{G}_\infty}(\delta_\infty \chi_{\pm\sqrt{-1}r})$ is an irreducible unitary representation. We therefore have $\pi_\infty \simeq \mathrm{Ind}_{P_\infty}^{\mathcal{G}_\infty}(\delta_\infty \chi_{\pm\sqrt{-1}r})$.

(3) Cuspidal representation generated by F_f (CAP property)

Let us first give the definition of CAP representations.

Definition 2.9 *Let G_1 and G_2 be two reductive algebraic groups over a number field such that $G_{1,v} \simeq G_{2,v}$ for almost all places v. Let P_2 be a parabolic subgroup of G_2 with Levi decomposition $P_2 = M_2N_2$. An irreducible cuspidal automorphic representation $\pi = \otimes'_v \pi_v$ of $G_1(\mathbb{A})$ is called* *cuspidal associated to parabolic (CAP) P_2, if there exists an irreducible cuspidal automorphic representation σ of M_2 such that $\pi_v \simeq \pi'_v$ for almost all places v, where $\pi' = \otimes'_v \pi'_v$ is an irreducible component of* $\mathrm{Ind}_{P_2(\mathbb{A})}^{G_2(\mathbb{A})}(\sigma)$.

See [6] and [14] for details on CAP representations defined for two groups instead of just one. Take $G_1 = \mathcal{G} = \mathrm{GL}_2(B)$ and $G_2 = \mathrm{GL}_4$. Here B is a definite quaternion algebra with discriminant 2. Since these groups are inner forms of each other, we have $G_{1,p} \simeq G_{2,p}$ for all odd primes p. Let P_2 be the standard parabolic of GL_4 with Levi subgroup $M_2 = \mathrm{GL}_2 \times \mathrm{GL}_2$. Let $f \in S(\Gamma_0(2); -(\frac{1}{4} + \frac{r^2}{4}))$ be a Hecke eigenform with Hecke eigenvalue λ_p for every odd prime p and Atkin Lehner eigenvalue ϵ. Let $\sigma = \otimes'_p \sigma_p$ be the irreducible cuspidal automorphic representation of GL_2 corresponding to f. For an odd prime p, the representation σ_p is the unramified principal series representation $I(\eta)$, where η is given by

$$\eta(\begin{bmatrix} a & b \\ & d \end{bmatrix}) = \eta_0(a)\eta_0^{-1}(d).$$

Here, η_0 is an unramified character of $\mathbb{Q}_p^\times$ such that $\eta_0(p)+\eta_0^{-1}(p) = \lambda_p$. For $p = 2$ assume that f is a new form. Then the representation σ_2 is the twist of the Steinberg representation of $\mathrm{GL}_2(\mathbb{Q}_2)$ by an unramified character η', with $\eta'(2) = -\epsilon$. The representation σ gives a representation $|\det|^{-1/2}\sigma \times |\det|^{1/2}\sigma$ of M_2. Form Proposition 2.6 we deduce the following.

Proposition 2.10 *Let $f \in S(\Gamma_0(2); -(\frac{1}{4} + \frac{r^2}{4}))$ be a Hecke eigenform with Hecke eigenvalue λ_p for every odd prime p and Atkin Lehner eigenvalue ϵ. Let $\sigma = \otimes'_p \sigma_p$ be the irreducible cuspidal automorphic representation of* GL_2 *corresponding to f. Let $\pi_F = \otimes'_p \pi_p$ be the corresponding irreducible cuspidal automorphic representation of $\mathcal{G}(\mathbb{A})$. Then π_F is CAP to an irreducible component of* $\mathrm{Ind}_{P_2(\mathbb{A})}^{G_2(\mathbb{A})}(|\det|^{-1/2}\sigma \times |\det|^{1/2}\sigma)$.

We can furthermore show that our cuspidal representations π_F's provide counterexamples of the Ramanujan conjecture.

Proposition 2.11 *Let $\pi_F = \otimes'_p \pi_p$ be as in Proposition 2.10. For every odd prime p (respectively $p = \infty$), π_p is non-tempered (respectively tempered). If we further assume that f is a new form, π_p is non-tempered for every finite prime p and tempered for $p = \infty$.*

The temperedness of π_∞ is due to Proposition 2.8 and [5, Remark 2.1.13]. For an odd prime p, the unramified chacters χ_i with $1 \leq i \leq 4$ are not unitary. This means that π_p is non-tempered (cf. [16]).

Let $p = 2$ and suppose that f is a new form. To see the non-temperedness of π_2 we verify that the matrix coefficient of π_2 defined by a spherical vector is not in $L^{2+\epsilon}(\mathcal{G}_p/Z_p)$ for any $\epsilon > 0$, where Z_p denotes the center of $\mathcal{G}_p$.

As a result of the argument so far, we have proved Theorem 1.2.

3 Concluding remarks

(1) Existence of non-zero lifts

Weyl's law for congruence subgroups of $SL_2(\mathbb{Z})$ by Selberg (cf. [9, Section 11.1]) implies that there exist Maass cusp forms for $\Gamma_0(2)$. This and Theorem 1.1 imply the existence of non-zero lifts F_f.

From Wayl's law (cf. [9, (11.5)]) we can deduce that there exist non-zero newforms in $S(\Gamma_0(2); -(\frac{r^2}{4}+\frac{1}{4}))$ for some $r \in \mathbb{R}$. Let $N_\Gamma(T)$ be the counting function of an orthogonal basis of the discrete spectrum for a congruence subgroup Γ as in [9, Section 11]. Put $N^*_{\Gamma_0(2)}(T)$ to be such counting function for newforms of $\Gamma_0(2)$. With the help of Casselman's local theory of oldforms and newforms (cf. [4]) we deduce

$$\begin{aligned} N^*_{\Gamma_0(2)}(T) &= \frac{\mathrm{Vol}(\mathfrak{h}/\Gamma_0(2)) - 2\mathrm{Vol}(\mathfrak{h}/SL_2(\mathbb{Z}))}{4\pi}T^2 + O(T\log T) \\ &= \frac{\mathrm{Vol}(\mathfrak{h}/SL_2(\mathbb{Z}))}{4\pi}T^2 + O(T\log T) \end{aligned}$$

from Weyl's law just mentioned. In view of Theorem 1.1 and Proposition 2.11, this lead to the existence of a non-zero cuspidal representation π_F whose local component π_p is non-tempered at every $p < \infty$.

(2) A remark on the non-temperedness of π_2

According to Tadić [17] the parabolic induction $I(\chi)$ for $p = 2$ has two composition factor, one of which is a unique essentially square integrable subquotient. Our non-tempered representation π_2 is the remaining non-square integrable composition factor. Besides our approach there seem several ways to prove that the non-square integrable composition factor is non-tempered. In fact, Marko Tadić pointed out that the non-temperedness is proved by using the classification of the non-unitary dual of $GL(n)$ over a division algebra (cf. [17]) or by Casselman's criterion on the temperedness of an admissible representation.

(3) Further remarks on representation theoretic aspect of our lifting

Let π' be the unique irreducible quotient of $\mathrm{Ind}_{P_2(\mathbb{A})}^{\mathrm{GL}_4(\mathbb{A})}(|\det|^{-1/2}\sigma \times |\det|^{1/2}\sigma)$. This is denoted by $\mathrm{MW}(\sigma, 2)$ in Section 18 of [2], which is a non-cuspidal, discrete series representation of $\mathrm{GL}_4(\mathbb{A})$. Since σ is not the image of a cuspidal representation of $B_{\mathbb{A}}^\times$ under the Jacquet-Langlands correspondence, π' is B-compatible according to Proposition 18.2, part (a) of [2]. Hence there exists a discrete series representation π of $\mathrm{GL}_2(B_{\mathbb{A}})$ which maps to π' under the Jacquet-Langlands correspondence. Also, from Proposition 18.2, part (b) of [2], the representation π has to be cuspidal. By the strong multiplicity one theorem for $\mathrm{GL}_2(B)$, the representation π has to be exactly the same as π_F obtained from the classical construction.

We remark that Grobner [7] has also obtained examples of CAP representations for $\mathrm{GL}_2(B)$ using the results of [2]. The example by Grobner [7] has a non-tempered local component at the archimedean place, while our cuspidal representation π has a tempered local component at the place as Theorem 2.11 says.

There seem many works on CAP representations. There are however quite a few works on an explicit construction of cusp forms with CAP properties. We should note that a construction of automorphic forms often needs difficulties different from those for studies of automorphic representations. We can thus stress the novelty of our work in that we obtain explicit formula for the lift in terms of Fourier expansions which are valid for non-Hecke eigenforms as well. In addition, the classical method immediately shows that the lifting is a linear non-zero map.

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