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On the Allan and Extended spectra in locally convex algebras

Hugo Arizmendi-Peimbert and Angel Carrillo-Hoyo

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Abstract

Here we study and compare the spectrum $\sigma_A(x)$ given in [1] by Allan and the extended spectrum $\Sigma(x)$, given in [7] by Zelazko for x in a unital locally convex algebra, using the new concept of pseudo-Q algebra that we introduce in [2]. This one is a generalization of the notion of pseudo-complete algebra given in [1]. Also we show that there exists an algebra A that it is pseudo-Q but it is not pseudo-complete.

1 Introduction

G.R. Allan introduces in [1] the concept of *spectrum* for elements in a locally convex algebra. His idea derives from the spectral theory of a closed operator T on a Banach space E. In this theory the spectrum is the set of all the complex numbers for which $\lambda I - T$ has no *bounded inverse.* Allan establishes a suitable definition of bounded element in a locally convex algebra that, according to his own words, "is justified by the theory which stems form it".

Similar ideas are discussed in [5] by L. Waelbroeck for unital commutative quasi-complete locally convex algebras.

A complex algebra A with a topology τ is a locally convex algebra if it is a Hausdorff locally convex space with a *separately continuous* multiplication, i.e. for any $x_0 \in A$, the mappings $x \to x_0x$ and $x \to xx_0$ are continuous. We say that the multiplication is *jointly continuous* if the map $A \times A \to A$, $(x, y) \to xy$, is continuous.

Throughout this paper $A = (A, \tau)$ will be a locally convex complex algebra with a unit e. A' will denote its topological dual and $\{\|\cdot\|_\alpha, \alpha \in \Lambda\}$ a family of seminorms defining the topology τ.

An element $x \in A$ is called *bounded* if the set $\{(\lambda x)^n : n = 0, 1, ...\}$ is bounded for some non-zero complex number λ. The set of all bounded elements is denoted by A_0. For each $x \in A_0$ it is defined the *radius of boundedness* $\beta(x)$ of x by

$$\beta(x) = \inf\left\{\lambda > 0 : \left\{\left(\lambda^{-1}x\right)^n\right\}_{n\geq 1} \text{ is bounded}\right\}$$

with the usual convention that $\inf \emptyset = \infty$.

We say that A is Q-algebra if the set $G(A)$ of all invertible elements of A is open. In [4] it is proved that a normed algebra $(A, \|\cdot\|)$ is a Q-algebra if and only if $(e-x)^{-1} = \sum_{n=0}^{\infty} x^n$ for every $x \in A$ such that $\|x\| < 1$.

A net (a_λ) in A is called *advertible with respect to* $a \in A$ if $a_\lambda a \to e$ and $aa_\lambda \to e$. Observe that if (a_λ) is convergent, then $a_\lambda \to a^{-1}$.

The algebra A is called *advertibly complete* if every Cauchy advertible net in A converges in A.

Proposition 1 *Let A be a Q-algebra, then it is advertibly complete.*

Proof. Let (a_λ) be an advertible Cauchy net with respect to $a \in A$. Since $G(A)$ is an open neighborhood of e, there exists λ_0 such that $a_{\lambda_0}a$ and aa_{λ_0} are invertible, hence a is invertible and $a_\lambda = a_\lambda aa^{-1} \to a^{-1}$. ■

For the functional calculus that Allan constructs in [1] some kind of completeness condition is essential. Thus, he introduces the *pseudo-completeness* concept, which is defined by a weaker condition than completeness.

Here we show that it is possible to use the even weaker notions of *pseudo-Qnees* or *advertible completeness* in order to construct a similar functional calculus, using some basic properties of the *Q-normed* algebras.

By $\mathfrak{B}_1$ it is denoted in [1] the collection of all subsets B of A such that

(i) B is absolutely convex and $B^2 \subset B$ and

(ii) B is bounded and closed.

We shall assume, without loss of generality, that each $B \in \mathfrak{B}_1$ contains the unit e.

For each $B \in \mathfrak{B}_1$, let $A(B)$ the subalgebra generated by B. Then from (i) and (ii)

$$A(B) = \{\lambda x : \lambda \in \mathbb{C}, x \in B\}$$

and the equation

$$\|x\|_B = \inf\{\lambda > 0 : x \in \lambda B\}$$

defines an algebra norm in B. We shall assume that $A(B)$ carries the topology induced by this norm. Since B is bounded in (A, τ) the norm topology in B is stronger than its topology as a subspace of (A, τ).

The algebra A is called *pseudo-complete* if each normed algebra $A(B)$, for $B \in \mathfrak{B}_1$, is a Banach algebra. If A is sequentially complete then A is pseudo-complete.

We shall say that A is a *pseudo-Q algebra* if each of the normed algebras $A(B)$, for $B \in \mathfrak{B}_1$, is a Q-algebra.

Proposition 2 *Let A be a Q-algebra or a completely advertibly algebra. Then every $A(B)$ is a Q-normed algebra for every $B \in \mathfrak{B}$, i.e. A is a pseudo-Q algebra.*

Proof. Since any Q-algebra is advertibly complete, it is sufficient to treat the case that A is advertibly complete. Let $B \in \mathfrak{B}_1$ and take $x \in A(B)$ such that $\|x\|_B < 1$. Then $\left(\sum_{n=0}^{m} x^n\right)_{m=1}^{\infty}$ is a Cauchy sequence in $A(B)$ and $(e - x)\sum_{n=0}^{m} x^n \to e$ and $\left(\sum_{n=0}^{m} x^n\right)(e - x) \to e$ in $A(B)$, hence also in A. Since A is completely advertible, then $(e - x)$ is invertible and

$$(e - x)^{-1} = \sum_{n=0}^{\infty} x^n.$$

Then, $A(B)$ is a Q-normed algebra ∎

In [1], it is introduced by Allan the spectrum $\sigma_A(x)$ of $x \in A$ as that subset of the Riemann sphere $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ defined as follows

(a) For $\lambda \neq \infty$, $\lambda \in \sigma_A(x)$ if and only if $\lambda e - x$ has no inverse belonging to A_0

(b) $\infty \in \sigma_A(x)$ if and only if $x \notin A_0$

In that paper it is shown that $\sigma_A(x)$ is a non void set for every $x \in A$.

We shall call $\sigma_A(x)$ the *Allan spectrum. The Allan radius* $r_A(x)$ is defined by

$$r_A(x) = \sup\{|\lambda| : \lambda \in \sigma_A(x)\},$$

where $|\infty| = \infty$.

The *resolvent set* of x, $\rho(x)$, is the complement of $\sigma_A(x)$ in $\mathbb{C}_\infty$.

On the other hand, W. Żelazko defines in [7] the concept of *extended spectrum* $\Sigma(x)$ for $x \in A$ in the following way.

As usual

$$\sigma(x) = \{\lambda \in \mathbb{C} : (\lambda e - x) \notin G(A)\}.$$

The *resolvent function*

$$R(\lambda, x) = (\lambda e - x)^{-1}$$

defined in $\mathbb{C}\setminus\sigma(x)$ is not always a continuous map in $\mathbb{C}$. Put

$$\sigma_d(x) = \{\lambda_0 \in \mathbb{C} : R(\lambda, x) \text{ is discontinuous at } \lambda = \lambda_0\}$$

and

$$\sigma_\infty(x) = \begin{cases} \emptyset & \text{if } \lambda \to R(1, \lambda x) \text{ is continuous at } \lambda = 0 \\ \infty & \text{otherwise} \end{cases}$$

Then the extended spectrum $\Sigma(x)$ of x is the union

$$\sigma(x) \cup \sigma_d(x) \cup \sigma_\infty(x).$$

In [7, Theorem 15.2] it is proved that if A is complete, then $\Sigma(x)$ is a non void set for each $x \in A$. However, we now prove that this is true for any unital locally convex algebra.

Theorem 3 *(Żelazko) Let A be a complex unital locally convex algebra with and $x \in A$, then $\Sigma(x)$ is a non-void subset of $\mathbb{C}_\infty$.*

Proof. Let us assume that there exists $x \in A$ such that $\Sigma(x) = \varnothing$, then $x - \lambda e$ is invertible for every $\lambda \in \mathbb{C}$. In particular, there exists $f \in A'$ such that $f(x^{-1}) \neq 0$.

We define a complex function F as follows

$$F(\lambda) = f\left((x - \lambda e)^{-1}\right).$$

We have that F is a complex entire function, since

$$\lim_{\lambda\to\lambda_0}\frac{F(\lambda)-F(\lambda_0)}{\lambda-\lambda_0}=f\left(\left((x-\lambda_0 e)^{-1}\right)^2\right),$$

because $\sigma_d(x)=\varnothing$.

Since $\sigma_\infty(x)=\varnothing$, we also have that

$$\lim_{|\lambda|\to\infty}|F(\lambda)|=\lim_{|\lambda|\to\infty}\left|f\left((x-\lambda e)^{-1}\right)\right|=\lim_{|\lambda|\to\infty}\left|\frac{1}{\lambda}f\left(\left(\frac{x}{\lambda}-e\right)^{-1}\right)\right|=0.$$

It follows from Liouville's theorem that $F(\lambda)\equiv 0$, which is a contradiction. ■

We give now an alternative proof of this theorem using one that appears in [6]. This paper is concerning the topologization of the field $C(t)$ of all rational functions on the indeterminate t over $\mathbb{C}$ in such way that the addition and multiplication are continuous operations. Note that every division algebra over $\mathbb{C}$, other than $\mathbb{C}$ itself, contains a subfield isomorphic to $C(t)$.

Let us recall the theorem just mentioned.

Theorem 4 *[6, Theorem 1](Williamson) Let A be a division algebra over $\mathbb{C}$, with a topology such that*

(i') there is one nonzero continuous linear functional;

(ii) addition and scalar multiplication are continuous;

(iii') multiplication (left or right) by an element of A is a continuous operation;

(iv') for each complex number λ_0 there is a non-negative integer $n(\lambda_0)$ such that $(\lambda-\lambda_0)^n\left\{(t-\lambda e)^{-1}-(t-\lambda_0 e)^{-1}\right\}$ is bounded for all λ near λ_0; and there is a non-negative integer n' such that $\lambda^{-n'}(t-\lambda e)^{-1}$ is bounded for all sufficiently large $|\lambda|$.

Then $A=\mathbb{C}$.

Proof. (of Theorem 3) Suppose $\Sigma(x)=\emptyset$, for some $x\in A$. Since $\sigma(x)=\emptyset$, then x is not a scalar multiple of e and we have that all rational functions $\frac{p(x)}{q(x)}$ are in A. Denote the algebra of these functions by $C(x)$. It is a division algebra that satisfies all the conditions of Williamson theorem because it is a locally convex, with continuous multiplication, and $\sigma_d(x)=\sigma_\infty(x)=\emptyset$. Therefore $C(x)=\mathbb{C}$, which is a contradiction. (Observe that we can take $n(\lambda_0)=n'=0$). ■

The *extended spectral radius* $R(x)$ is defined by

$$R(x)=\sup\{|\lambda|:\lambda\in\Sigma(x)\}.$$

where $|\infty|=\infty$.

2 Comparison between $\Sigma(x)$ and $\sigma_A(x)$

Then next two results are proved in [2].

Theorem 5 *If A is a pseudo-Q algebra, then $\Sigma(x)\subset\sigma_A(x)$ for any $x\in A$. Therefore, $R(x)\le r_A(x)$.*

Corollary 6 *If A is a pseudo-Q algebra, then $\sigma_A(x)$ is a closed set in $\mathbb{C}_\infty$ and it is compact if $\infty\notin\sigma_A(x)$.*

Lemma 7 *[3, Lemma 2.2] Let $x\in A$ be such that the extended spectral radius $R(x)$ is finite. Then for each $f\in A'$ the function $F(\lambda)=f(R(1,\lambda x))$ is holomorphic in the complex open disc $D(0,\delta)$ centered in 0 with radius $\delta=\frac{1}{R(x)}$ if $R(x)>0$ and $D(0,\delta)=\mathbb{C}$, otherwise. Furthermore,*

$$F^{(n)}(\lambda)=n!\ f(R(1,\lambda x))^{n+1}x^n$$

for every $\lambda\in D(0,\delta)$ and $n=0,1,2,...$

In particular,

$$F^{(n)}(0)=n!\ f(x^n)$$

for all $n=0,1,...$

Theorem 8 *If A is pseudo-Q algebra and $x\in A$, then $\Sigma(x)=\sigma_A(x)$ if $\Sigma(x)$ is closed in $\mathbb{C}_\infty$.*

Proof. Assume that $\Sigma(x)$ is closed in $\mathbb{C}_\infty$. By Theorem 5 we only have to prove that $\lambda_0 \notin \Sigma(x)$ implies $\lambda_0 \notin \sigma_A(x)$.

Let $\lambda_0 \notin \Sigma(x)$, with $\lambda_0 \neq \infty$, then $\lambda_0 e - x \in G(A)$. We shall show that $(\lambda_0 e - x)^{-1}$ is a bounded element. Since $\Sigma(x)$ is closed, there exists an open disc $D(\lambda_0)$ around λ_0 such that $\lambda e - x \in G(A)$ if $\lambda \in D(\lambda_0)$. We also know that $R(\lambda, x)$ is continuous at $\lambda = \lambda_0$. Using the identity

$$(\lambda e - x)^{-1} - (\lambda_0 e - x)^{-1} = (\lambda_0 - \lambda)(\lambda e - x)^{-1}(\lambda_0 e - x)^{-1},$$

we obtain

$$\lim_{\lambda \to \lambda_0} \frac{R(\lambda, x) - R(\lambda, x)}{\lambda - \lambda_0} = -R(\lambda_0, x)^2.$$

Then for any $f \in A'$ we get

$$\lim_{\lambda \to \lambda_0} \frac{f(R(\lambda, x)) - f(R(\lambda_0, x))}{\lambda - \lambda_0} = -f\left(R(\lambda_0, x)^2\right),$$

which implies that $R(\lambda, x)$ is weakly holomorphic in $\lambda = \lambda_0$. By [1, (3.8) Theorem] we conclude that $(\lambda_0 e - x)^{-1}$ is bounded in A. Therefore $\lambda_0 \notin \sigma_A(x)$.

If $\infty \notin \Sigma(x)$, then some neighborhood of ∞ does not intersect $\Sigma(x)$ and we have that $R(x) < \infty$. Let $f \in A'$. By Lemma 7, the Taylor expansion of $F(\lambda) = f((R(1, \lambda x)))$ around 0 is

$$F(\lambda) = f(e) + \lambda f(x) + \frac{2\lambda^2}{2!} f(x^2) + ...$$

for $|\lambda| < \frac{1}{R(x)}$. In particular, $\lim f(\lambda_0^n x^n) = 0$ for some $\lambda_0 > 0$ and then$\{f(\lambda_0^n x^n) : n \geq 1\}$ is bounded; therefore $\{(\lambda_0 x)^n : n \geq 1\}$ is bounded. Thus $x \in A_0$ and $\infty \notin \sigma_A(x)$. ■

Example 9 *In [7, 10.9 Example] it is given a complex complete metrizable locally convex algebra W (Williamson's algebra) with a jointly continuous multiplication that contains a subalgebra isomorphic with the field $A = C(t)$ of all rational functions of the indeterminate t over the complex field $\mathbb{C}$. Obviously A is a Q-algebra, therefore it is a pseudo-Q algebra. We claim that it is not pseudo-complete. Assume the contrary. It can be proved that $r_A(t) = 0$. Therefore, $\{t^n : n = 0, 1, ...\}$ is a bounded and idempotent set. Then, there exists an absolutely convex closed idempotent subset B of A that contains $\{t^n : n = 0, 1, ...\}$ and then, $\|t^n\|_B \leq 1$. According our assumption $A(B)$ is a Banach algebra, hence $\sum_{n=0}^{\infty} a_n t^n \in A(B) \subset C(t)$ for every complex sequence (a_n) such that the series $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence greater than 1, but this is impossible because there exists such series for which the holomorphic function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is not a rational function and then the series $\sum_{n=0}^{\infty} a_n t^n$ can not belongs to A. This proves our claim.*

Example 10 *Let X be a completely regular Hausdorff space and let $B_0(X)$ be the family of all bounded functions on X vanishing at infinity. We denote by $(C_b(X), \beta)$ the locally convex algebra of all complex bounded continuous functions on X with the usual operations and endowed with the strict topology β, which is given by the family of seminorms*

$$\|f\|_\varphi = \sup_{x \in X} |f(x)\varphi(x)|$$

for each $f \in C_b(X)$ and $\varphi \in B_0(X)$. Every linear multiplicative continuous functional on $C_b(X)$ is a point evaluation T_x, i.e., $T_x(f) = f(x)$ for all $x \in X$, and every linear multiplicative functional on $C_b(X)$ is a point evaluation $T_{\widetilde{x}}$ with $\widetilde{x} \in \beta(X)$, where $\beta(X)$ is the Stone-Čech compactification of X. That is $\mathfrak{M}(C_b(X)) = X$ and $\mathfrak{M}^{\#}(C_b(X)) = \beta(X)$.

The algebra $(C_b(X), \beta)$ is complete if X is a k-space, i.e. $F \subset X$ is closed if and only if $F \cap K$ is closed for every compact $K \subset X$.

We have that $(C_b(X))_0 = C_b(X)$ and $\beta(f) = \|f\|_\infty = \sup_{x \in X} |f(x)|$ for each $f \in C_b(X)$, because $|\lambda| > \|f\|_\infty$ implies $\left\|\left(\frac{f}{\lambda}\right)^n\right\|_\varphi \to 0$ for every $\varphi \in B_0(X)$. On the other hand, we have that $\sigma_A(f) = \Sigma(f) = cl(f(X))$ for all $f \in C_b(X)$, where cl denotes the closure operator in $\mathbb{C}_\infty$, since $(\lambda 1 - f)(x) = 0$ for $\lambda \in f(X)$ and $(\lambda 1 - f)^{-1}$ is not bounded for every $\lambda \in cl(f(X)) \diagdown f(X)$.

Theorem 11 *Let A be a pseudo-Q algebra. Then*

$$R(x) = \beta(x) = r_A(x)$$

for every $x \in A$.

Proof. First we prove that $\beta(x) \geq r_A(x)$. If $\beta(x) = \infty$ we are done. Let $\beta(x) < r < \infty$ and $\lambda \in \mathbb{C}$ with $|\lambda| > r$. Since $\beta(x) = \inf\{\|x\|_B : B \in \mathfrak{B}_1\}$ we have that there exists $B \in \mathfrak{B}_1$ such that $\|x\|_B < |\lambda|$. Being $A(B)$ a Q-algebra, we have that $\left(e - \frac{x}{\lambda}\right)^{-1} \in A(B)$ and there exists $M > 0$ such that $\left\|\frac{(\lambda e - x)^{-1}}{M}\right\|_B < 1$, hence $\frac{(\lambda e - x)^{-1}}{M} \in B$ and consequently $\left\{\left(\frac{(\lambda e - x)^{-1}}{M}\right)^n : n \geq 1\right\}$ is a bounded set since B is bounded and idempotent: Therefore $\lambda \in r_A(x)$ and we have that $r \geq r_A(x)$. This implies that $\beta(x) \geq r_A(x)$.

We also have that $R(x) \leq r_A(x)$, since $\Sigma(x) \subset \sigma_A(x)$ and from [7, 15.6 Theorem] and [1, (2.18) Proposition] we get

$$R(x) = \beta'(x) = \beta''(x) = \beta(x),$$

where

$$\beta'(x) = \sup_{f \in A'} \left(\limsup |f(x^n)|^{1/n}\right)$$

and

$$\beta''(x) = \sup_{\|\cdot\|_\alpha} \left(\limsup \|x^n\|_\alpha^{1/n}\right).$$

Therefore, we obtain the result. ∎

3 Algebras with continuous inversion

Let A be a locally convex algebra in which the map $x \to x^{-1}$ is continuous relative to the set of invertible elements. In this case we have that $\sigma_d(x) = \sigma_\infty(x) = \varnothing$ for every $x \in A$.

The first part of the next result is proved in [1, (4.1) Theorem] and the last one in [2].

Theorem 12 *Let A be a locally convex unital algebra with continuous inversion, and let $x \in A$. Then*

$$\sigma(x) \subset \sigma_A(x) \subset cl(\sigma(x))$$

and if A is pseudo-Q algebra, then

$$\sigma_A(x) = \sigma(x).$$

Example 13 *Let $A = H(D(0,1))$ be the algebra of all holomorphic functions in the complex unit open disc centered at 0, endowed with the open-compact topology τ_k, which can be given by the sequence of seminorms $\{\|\cdot\|_n = 1, 2, ...\}$, where $\|f\|_n = \max_{|z| \leq r_n} |f(z)|$ and $0 < r_1 < r_2 < ... < 1$ is an increasing of positive numbers tending to 1. Here we have that the spectrum $\sigma_A(z)$ of z is the closed disc $\overline{D(0,1)}$ since each element $(\lambda - z)^{-1}$ with $|\lambda| = 1$ is not bounded in A. Nevertheless, $\Sigma(z) = \sigma(z) = D(0,1)$, since in A the inversion $x \to x^{-1}$ is continuous on $G(A)$.*

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