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ONE-STEP EXTENSIONS OF SUBNORMAL 2-VARIABLE WEIGHTED SHIFTS

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(BASED ON JOINT WORK WITH R. CURTO AND J. YOON.)

1. Introduction

Consider the following reconstruction-of-the-measure problem:

Problem 1.1 (A). *Given two probability measures μ_1 and μ_2 on $\mathbb{R}_+^2$, find necessary and sufficeint conditions for the existence of a probability measure μ on $\mathbb{R}_+^2$ such that*

$$\frac{s\, d\mu(s,t)}{\int s\, d\mu(s,t)} = d\mu_1(s,t) \text{ and } \frac{t\, d\mu(s,t)}{\int t\, d\mu(s,t)} = d\mu_2(s,t). \tag{1.1}$$

Note that (1.1) implies that $td\mu_1(s,t) = \lambda s d\mu_2(s,t)$ for some $\lambda > 0$.

In this talk, we solve this interpolation problem using techniques from multivariable operator theory, namely the theory of 2-variable weighted shifts.

Definition 1.2. $T \in \mathcal{B}(\mathcal{H})$: *normal if* $T^*T = TT^*$,

subnormal if $T = N|_{\mathcal{H}}$, *where* N *normal and* $N(\mathcal{H}) \subseteq \mathcal{H}$,

hyponormal if $[T^*, T] := T^*T - TT^* \geq 0$.

Definition 1.3. $\mathbf{T} \equiv (T_1, \cdots, T_n)$: *hyponormal if*

$$\begin{aligned}[\mathbf{T}^*, \mathbf{T}] : &= ([T_j^*, T_i])_{i,j=1}^k \\ &= \begin{pmatrix} [T_1^*, T_1] & [T_2^*, T_1] & \cdots & [T_n^*, T_1] \\ [T_1^*, T_2] & [T_2^*, T_2] & \cdots & [T_n^*, T_2] \\ \vdots & \vdots & \ddots & \vdots \\ [T_1^*, T_n] & [T_2^*, T_n] & \cdots & [T_n^*, T_n] \end{pmatrix} \geq 0.\end{aligned}$$

Definition 1.4. *The n-tuple* $\mathbf{T} \equiv (T_1, T_2, \cdots, T_n)$ *is said to be normal if* $\mathbf{T}$ *is commuting and each* T_i *is normal, and* $\mathbf{T}$ *is subnormal if* $\mathbf{T}$ *is the restriction of a normal n-tuple to a common invariant subspace.*

- Clearly, normal $\Longrightarrow$ subnormal $\Longrightarrow$ hyponormal.
- Normality(sub-, hypo-) of $\mathbf{T}$ is not affected by permuting of the operators T_i.
- If $(T_1, \cdots, T_n)$ is normal(sub-, hypo-) then so is $(k_1T_1, \cdots, k_nT_n)$ for any $k_1, \cdots, k_n \in \mathbb{C}$.
- If $(T_1, \cdots, T_n)$ is normal(sub-, hypo-) then any operator in $LS\{T_1, \cdots, T_n\}$ is normal(sub-, hypo-).

Problem 1.5 (Lifting Problem for Commuting Subnormals). *Find necessary and sufficient conditions for a pair of subnormal operators on a Hilbert space to admit commuting normal extensions i.e., to be subnormal.*

Necessary Conditions: Commuting

Sufficient Conditions: Doubly commuting, either T_1 or T_2 is normal, either T_1 or T_2 is isometry,$\cdots$

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Besides their relevance for the construction of examples and counterexamples in Hilbert space operator theory, weighted shifts can also be used to detect properties such as subnormality, via the Lambert-Lubin Criterion([15, 17]):

Theorem 1.6 ([15]). *If $T \in \mathcal{B}(\mathcal{H})$ is one-one, then T is subnormalif and only if T_x is subnormal for all $x(\neq 0) \in \mathcal{H}$ where T_x is the weighted shift with weights $\{\frac{\|T^{n+1}x\|}{\|T^n x\|}\}_{n=0}^{\infty}$.*

Theorem 1.7 ([17]). *If $T_1, T_2 \in \mathcal{B}(\mathcal{H})$ are commuting and one-one, then $\mathbf{T} \equiv (T_1, T_2)$ is subnormal if and only if $\mathbf{T}_x$ is subnormal for all $x(\neq 0) \in \mathcal{H}$ where $\mathbf{T}_x$ is the 2-variable weighted shift with weights $\alpha_{m,n} := \frac{\|T_1^{m+1}T_2^n x\|}{\|T_1^m T_2^n x\|}$ and $\beta_{m,n} := \frac{\|T_1^m T_2^{n+1} x\|}{\|T_1^m T_2^n x\|}$.*

Thus, to study the subnormality of commuting pairs, we focus on weighted shifts in the sequel.

Example 1.8 (1-variable weighted shift). For a bounded sequence $a \equiv \{a_n\}_{n=0}^{\infty}$ of positive real numbers (called *weights*), let $W_a : \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$ be the associated unilateral weighted shift, defined by $W_a e_n := a_n e_{n+1}$ (all $n \geq 0$), where $\{e_n\}_{n=0}^{\infty}$ is the canonical orthonormal basis in $\ell^2(\mathbb{Z}_+)$.

For a weighted shift W_a, *the moments of a* are given as

$$\gamma_k \equiv \gamma_k(a) := \begin{cases} 1, & \text{if } k = 0 \\ a_0^2 \cdots a_{k-1}^2, & \text{if } k \geq 1. \end{cases}$$

It is easy to see that W_a is never normal, and that it is hyponormal if and only if $a_0 \leq a_1 \leq \cdots$.

We shall often write shift$(a_0, a_1, \cdots)$ to denote the weighted shift W_a.

Example 1.9 (2-variable weighted shift). For $\alpha \equiv \{\alpha_{\mathbf{k}}\}, \beta \equiv \{\beta_{\mathbf{k}}\} \in \ell^\infty(\mathbb{Z}_+^2)$, we define the 2-variable weighted shift $W_{(\alpha,\beta)} \equiv (W_\alpha, W_\beta)$ on $\ell^2(\mathbb{Z}_+^2)$ by

$$W_\alpha e_{\mathbf{k}} := \alpha_{\mathbf{k}} e_{\mathbf{k}+\varepsilon_1} \quad \text{and} \quad W_\beta e_{\mathbf{k}} := \beta_{\mathbf{k}} e_{\mathbf{k}+\varepsilon_2},$$

where $\varepsilon_1 := (1,0), \varepsilon_2 := (0,1)$ and $\{e_{\mathbf{k}} : \mathbf{k} \in \mathbb{Z}_+^2\}$ is the canonical orthonormal basis of $\ell^2(\mathbb{Z}_+^2)$.

In an entirely similar way one can define multivariable weighted shifts.

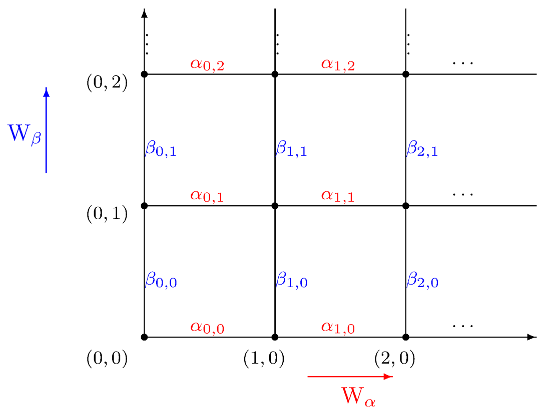
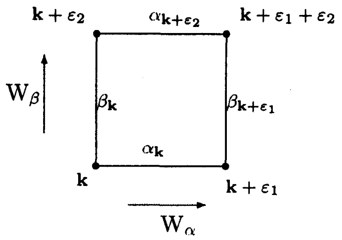


FIGURE 1. Weight diagram for 2-variable weighted shift $W_{(\alpha,\beta)}$

Clearly,

$$W_\alpha W_\beta = W_\beta W_\alpha \iff \alpha_{\mathbf{k}}\beta_{\mathbf{k}+\varepsilon_1} = \beta_{\mathbf{k}}\alpha_{\mathbf{k}+\varepsilon_2} \quad (\forall \mathbf{k} \in \mathbb{Z}_+^2). \tag{1.2}$$



In the sequel, we assume that all 2-variable weighted shifts $W_{(\alpha,\beta)}$ are commuting, i.e., it satisfies condition (1.2).

Given $\mathbf{k} \in \mathbb{Z}_+^2$, the *moments* $\gamma_{\mathbf{k}} \equiv \gamma_{\mathbf{k}}(\alpha,\beta)$ of (α,β) of order $\mathbf{k}$ is defined by

$$\begin{cases} 1 & \text{if } \mathbf{k} \equiv (k_1,k_2) = (0,0) \\ \alpha_{(0,0)}^2 \cdots \alpha_{(k_1-1,0)}^2 & \text{if } k_1 \geq 1 \text{ and } k_2 = 0 \\ \beta_{(0,0)}^2 \cdots \beta_{(0,k_2-1)}^2 & \text{if } k_1 = 0 \text{ and } k_2 \geq 1 \\ \alpha_{(0,0)}^2 \cdots \alpha_{(k_1-1,0)}^2 \cdot \beta_{(k_1,0)}^2 \cdots \beta_{(k_1,k_2-1)}^2 & \text{if } k_1 \geq 1 \text{ and } k_2 \geq 1. \end{cases}$$

We remark that, due to the commutativity condition (1.2), $\gamma_{\mathbf{k}}$ can be computed using any nondecreasing path from $(0,0)$ to $\mathbf{k}$.

Question 1.10. *Which weighted shifts are subnormal?*

Theorem 1.11 (Berger's Theorem(1-variable)). *W_a is subnormal if and only if there exists a probability measure ξ (called the Berger measure of W_a) supported in $[0, \|W_a\|^2]$ such that $\gamma_k(a) = \int s^k \, d\xi(s)$ $(k \geq 0)$.*

Theorem 1.12 (Berger's Theorem(2-variable)([14])). *$W_{(\alpha,\beta)}$ is subnormal if and only if there is a probability measure μ (called the Berger measure of $W_{(\alpha,\beta)}$) supported in the 2-dimensional rectangle $R = [0, \|W_\alpha\|^2] \times [0, \|W_\beta\|^2]$ such that*

$$\gamma_{\mathbf{k}}(\alpha,\beta) = \int_R s^{k_1} t^{k_2} d\mu(s,t), \forall \mathbf{k} \equiv (k_1,k_2) \in \mathbb{Z}_+^2.$$

2. Auxiliary Lemmas

For a 2-variable weighted shift $W_{(\alpha,\beta)}$, we let $\mathcal{M}$ (resp. $\mathcal{N}$) be the invariant subspace of $\ell^2(\mathbb{Z}_+^2)$ spanned by the canonical orthonormal basis vectors associated to indices $\mathbf{k} = (k_1,k_2)$ with $k_1 \geq 0$ and $k_2 \geq 1$ (resp. $k_1 \geq 1$ and $k_2 \geq 0$).

We consider the following problem:

Problem 2.1 (B). *Assume that $W_{(\alpha,\beta)}|_{\mathcal{M}}$ and $W_{(\alpha,\beta)}|_{\mathcal{N}}$ are subnormal with the Berger measures $\mu_{\mathcal{M}}$ and $\mu_{\mathcal{N}}$, respectively. Find necessary and sufficient conditions on $\mu_{\mathcal{M}}$, $\mu_{\mathcal{N}}$ and β_{00} for the subnormality of $W_{(\alpha,\beta)}$.*

Note that Problem (B) is equivalent to Problem (A).

If W_a is subnormal with Berger measure ξ, and if we let for fixed $i \geq 1$,

$$\mathcal{L}_i := \bigvee \{e_n : n \geq i\}$$

then the Berger measure ξ_i of $W_a|_{\mathcal{L}_i}$ is $\frac{s^i}{\gamma_i} d\xi(s)$.

Lemma 2.2 (1-variable subnormal backward extension ([5])). *If $W_a|_{\mathcal{L}_1}$ is subnormal with Berger measure ξ_1 then W_a is subnormal if and only if*

$$\frac{1}{s} \in L^1(\xi_1) \text{ and } a_0^2 \left\| \frac{1}{s} \right\|_{L^1(\xi_1)} \leq 1. \tag{2.1}$$

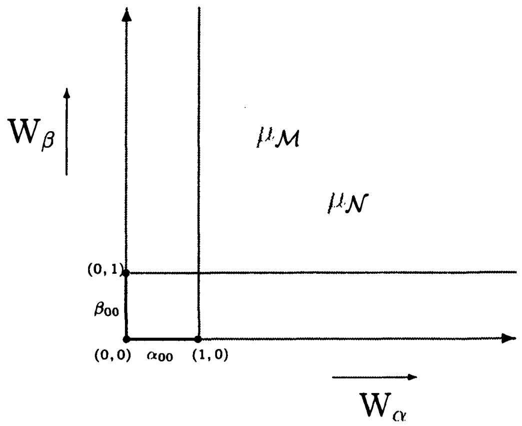


FIGURE 2. 2-variable weighted shift $W_{(\alpha,\beta)}$ in Problem (B)

In this case, the Berger measure ξ of W_a is $d\xi(s) = \frac{a_0^2}{s} d\xi_1(s) + (1 - a_0^2 \left\|\frac{1}{s}\right\|_{L^1(\xi_1)}) d\delta_0(s)$.

- Let μ and ν be two positive measures on $\mathbb{R}_+$. We say that $\mu \leq \nu$ if $\mu(E) \leq \nu(E)$ for each Borel subset $E \subseteq \mathbb{R}_+$.
- Let μ be a probability measure on $\mathbb{R}_+ \times \mathbb{R}_+$ and assume that $\frac{1}{t} \in L^1(\mu)$. The *extremal measure* μ_{ext} (which is also a probability measure) on $\mathbb{R}_+ \times \mathbb{R}_+$ is given by
$$d\mu_{ext}(s,t) := (1 - \delta_0(t)) \frac{1}{t \left\|\frac{1}{t}\right\|_{L^1(\mu)}} d\mu(s,t).$$
Here δ_0 denotes Dirac measure at 0.
- Given a measure μ on $X \times Y$, the *marginal measure* μ^X is a measure on X given by
$$\mu^X := \mu \circ \pi_X^{-1},$$
where $\pi_X : X \times Y \to X$ is the canonical projection onto X.

Lemma 2.3 (2-variable subnormal backward extension ([10])). *Assume that $W_{(\alpha,\beta)}|_{\mathcal{M}}$ is subnormal with the Berger measure $\mu_{\mathcal{M}}$ and that* shift $(\alpha_{00}, \alpha_{10}, \cdots)$ *is subnormal with Berger measure ξ. Then $W_{(\alpha,\beta)}$ is subnormal if and only if the following conditions hold:*

(i) $\frac{1}{t} \in L^1(\mu_{\mathcal{M}})$;
(ii) $\beta_{00}^2 \leq (\left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})})^{-1}$;
(iii) $\beta_{00}^2 \left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})} (\mu_{\mathcal{M}})_{ext}^X \leq \xi$.

Moreover, if $\beta_{00}^2 \left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}_1})} = 1$, then $(\mu_{\mathcal{M}})_{ext}^X = \xi$.

Lemma 2.4 ([11]). *Let μ be the Berger measure of a subnormal 2-variable weighted shift $W_{(\alpha,\beta)}$, and let ξ be the Berger measure of the associated 0-th horizontal 1-variable $shift(\alpha_{00}, \alpha_{10}, \cdots)$. Then $\xi = \mu^X$.*

3. MAIN RESULT AND APPLICATION

We provides a concrete solution of Problem (B) in terms of $\mu_{\mathcal{M}}$, $\mu_{\mathcal{N}}$ and β_{00}.

Theorem 3.1 (Main Theorem). *Assume that $W_{(\alpha,\beta)}|_{\mathcal{M}}$ and $W_{(\alpha,\beta)}|_{\mathcal{N}}$ are subnormal with associated Berger measures $\mu_{\mathcal{M}}$ and $\mu_{\mathcal{N}}$, respectively, and let $c := \frac{\int s d\mu_{\mathcal{M}}}{\int t d\mu_{\mathcal{N}}} \equiv \frac{\alpha_{01}^2}{\beta_{10}^2}$. Then $W_{(\alpha,\beta)}$ is subnormal if and only if the following*

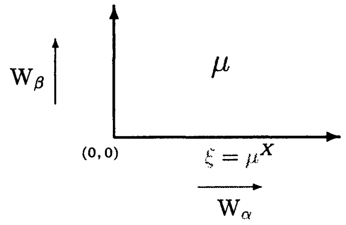


FIGURE 3. 2-variable weighted shift $W_{(\alpha,\beta)}$ in Lemma 2.4

conditions hold:
(i) $\frac{1}{t} \in L^1(\mu_{\mathcal{M}})$ *and* $\frac{1}{s} \in L^1(\mu_{\mathcal{N}})$;
(ii) $\beta_{00}^2 \left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})} \leq 1$;
(iii) $\beta_{00}^2 \left\{\left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})} (\mu_{\mathcal{M}})_{ext}^X + c\left\|\frac{1}{s}\right\|_{L^1(\mu_{\mathcal{N}})} \delta_0 - \frac{c}{s}(\mu_{\mathcal{N}})^X\right\} \leq \delta_0$.

For a measure μ with $\frac{1}{s} \in L^1(\mu)$, we write $d\widetilde{\mu}(s) := \frac{1}{s\left\|\frac{1}{s}\right\|_{L^1(\mu)}} d\mu(s)$.

Lemma 3.2 ([8]). *Let $W_{(\alpha,\beta)}$ be the 2-variable weighted shift given in Figure 4. Then $W_{(\alpha,\beta)}|_{\mathcal{M}}$ is subnormal if and only if $\psi := \eta_1 - \alpha_{01}^2 \left\|\frac{1}{s}\right\|_{L^1(\sigma)} \tau$ is a positive measure. In this case, the Berger measure of $W_{(\alpha,\beta)}|_{\mathcal{M}}$ is*

$$\mu_{\mathcal{M}} = \alpha_{01}^2 \left\|\frac{1}{s}\right\|_{L^1(\sigma)} \widetilde{\sigma} \times \tau + \delta_0 \times \psi.$$

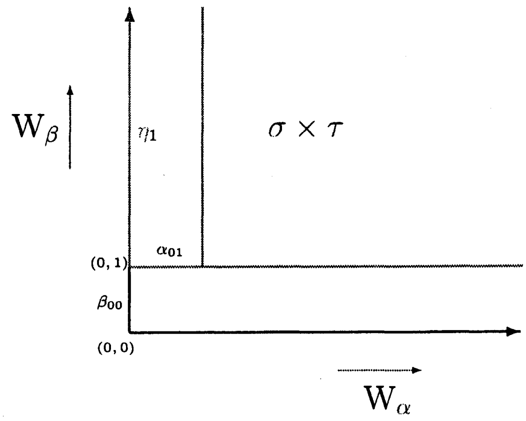


FIGURE 4. 2-variable weighted shift $W_{(\alpha,\beta)}$ in Lemma 3.2

As a special case of Main Theorem, we have:

Theorem 3.3 (The case when $W_{(\alpha,\beta)}$ has a core of tensor form). *Assume that $W_{(\alpha,\beta)}|_{\mathcal{M}}$ and $W_{(\alpha,\beta)}|_{\mathcal{N}}$ are subnormal with associated Berger measures $\mu_{\mathcal{M}}$ and $\mu_{\mathcal{N}}$, respectively, and let $\rho := \mu_{\mathcal{M}}^X$. Also assume that $\mu_{\mathcal{M}\cap\mathcal{N}} = \sigma \times \tau$ for some 1-variable probability measures σ and τ. Then $\rho = \mu_{\mathcal{M}}^X = (\mu_{\mathcal{M}})_{ext}^X$, and hence $W_{(\alpha,\beta)}$ is subnormal if and only if the following conditions hold:*
(i) $\frac{1}{t} \in L^1(\mu_{\mathcal{M}})$ *and* $\frac{1}{s} \in L^1(\mu_{\mathcal{N}})$;

(ii) $\beta_{00}^2 \left\| \frac{1}{t} \right\|_{L^1(\mu_{\mathcal{M}})} \leq 1$;

(iii) $\left(\beta_{00}^2 \left\| \frac{1}{t} \right\|_{L^1(\mu_{\mathcal{M}})}\right) \rho \leq \xi$, *where* ξ *is the Berger measure of* shift $(\alpha_{00}, \alpha_{10}, \cdots)$.

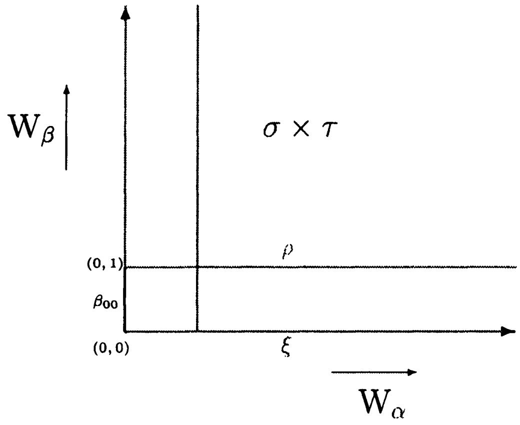


FIGURE 5. 2-variable weighted shift $W_{(\alpha,\beta)}$ in Theorem 3.3

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