A regular homotopy version of the Goldman-Turaev Lie bialgebra, the Enomoto-Satoh traces and the divergence cocycle in the Kashiwara-Vergne problem

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Introduction

This is an announcement on my research in progress, which introduces a refinement of the Goldman-Turaev Lie bialgebra. Its goal is to interpret the divergence cocycle in the Kashiwara-Vergne problem [1] and the Enomoto-Satoh obstructions for the surjectivity of the Johnson homomorphisms (= the Enomoto-Satoh traces) [3] as some part of a regular homotopy version of the Turaev cobracket.

In my previous work joint with Yusuke Kuno [8] we proved that the Morita traces are included in the Turaev cobracket. The Enomoto-Satoh traces [3] are refinements of the Morita traces, and closely related to the divergence cocycle in the Kashiwara-Vergne problem. Enomoto [2] proved that the graded quotient of the Turaev cobracket does *not* include the Enomoto-Satoh traces. This fact seems to come from the fact that the Turaev cobracket is defined to be invariant under the birth-death move of a monogon in free loops. This is the reason why we consider the *regular* homotopy set of *immersed* free loops on a surface. On the other hand, the first term of the Enomoto-Satoh traces is just the Earle class k on the mapping class group. Furuta gave an explicit cocycle for the Earle class in terms of a framing of the tangent bundle of the surface. For details, see [10] §4. Our construction is inspired by Furuta's construction.

A regular homotopy version of the Goldman-Turaev Lie bialgebra

Let S be a compact connected oriented C^∞ surface with $\partial S \neq \emptyset$. We denote by $\hat{\pi}^+ = \hat{\pi}^+(S)$ the *regular* homotopy set of free *immersed* loops on S. The infinite cyclic group $\langle r \rangle$ acts on the set $\hat{\pi}^+$ by inserting a (positive) monogon into a loop. The action is free, and the orbit space $\hat{\pi} = \hat{\pi}(S) := \hat{\pi}^+(S)/\langle r \rangle$ equals the free homotopy set of free loops on S. We denote by $\Phi : \hat{\pi}^+ \to \hat{\pi}$ the quotient map, which can be regarded as the map forgetting smooth structures on immersed loops. The rational group ring $\mathbb{Q}\langle r \rangle$ is naturally identified with

the Laurent polynomial ring $\mathbb{Q}[r, r^{-1}]$. The $\mathbb{Q}$-free vector space over the set $\hat{\pi}^+$, $\mathbb{Q}\hat{\pi}^+$, is a free $\mathbb{Q}\langle r \rangle$-module. We denote by $\mathbb{Q}\langle r \rangle 1$ the linear span of the regular homotopy classes of null-homotopic immersed free loops on S.

Since S is connected and its boundary is non-empty, the tangent bundle TS is trivial. We call the homotopy class f of a global trivialization $TS \cong S \times \mathbb{R}^2 \overset{\mathrm{pr}_2}{\to} \mathbb{R}^2$ a *framing* of S. If we fix a framing f, then we can define the (global) rotation number $\mathrm{rot}_f : \hat{\pi}^+ \to \mathbb{Z}$. The map $\tilde{\Phi}_f := (\Phi, \mathrm{rot}_f) : \hat{\pi}^+ \to \hat{\pi} \times \mathbb{Z}$ is a bijection. We define $s_f : \hat{\pi} \to \hat{\pi}^+$ by $s_f(\alpha) := \tilde{\Phi}_f^{-1}(\alpha, 0)$ for $\alpha \in \hat{\pi}$, and $\varepsilon_f : \mathbb{Q}\hat{\pi}^+ \to \mathbb{Q}\langle r \rangle$ by $\varepsilon_f(\beta) := r^{\mathrm{rot}_f(\beta)}$ for $\beta \in \hat{\pi}^+$.

The *regular* Goldman bracket $[\,,\,]^+ : \mathbb{Q}\hat{\pi}^+ \otimes_{\mathbb{Q}\langle r \rangle} \mathbb{Q}\hat{\pi}^+ \to \mathbb{Q}\hat{\pi}^+$ is defined in the same way as the original one [4]. The *regular* Turaev cobracket $\delta^+ : \mathbb{Q}\hat{\pi}^+ \to \mathbb{Q}\hat{\pi}^+ \otimes_{\mathbb{Q}\langle r \rangle} \mathbb{Q}\hat{\pi}^+$ is also defined in a similar way to the original one [12]. The triple $(\mathbb{Q}\hat{\pi}^+, [\,,\,]^+, \delta^+)$ is a Lie bialgebra. For any embedded loop $\alpha \in \hat{\pi}^+$ and $n \in \mathbb{Z}$ we have $\delta^+(\alpha^n) = 0$. In particular, the cobracket δ^+ vanishes on $\mathbb{Q}\langle r \rangle 1$. Hence we obtain the induced operation

$$\delta^+ : \mathbb{Q}\hat{\pi}^+/\mathbb{Q}\langle r \rangle 1 \to \mathbb{Q}\hat{\pi}^+ \otimes_{\mathbb{Q}\langle r \rangle} \mathbb{Q}\hat{\pi}^+.$$

In the original case [12] the target of δ is $(\mathbb{Q}\hat{\pi}/\mathbb{Q}1)^{\otimes 2}$, since the cobracket has to be invariant under the birth-death move of a monogon, which we can ignore in the context of regular homotopy.
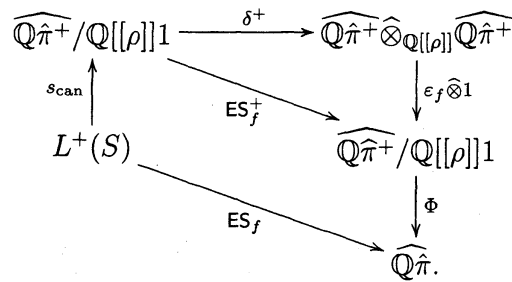
We number the connected components of the boundary $\partial S = \coprod_{a=0}^{n} \partial_a S$, where $n = \sharp\pi_0(\partial S) - 1$. For each a we choose a point $*_a \in \partial_a S$ and an inward vector $v_a \in T_{*_a}S$. We define a $\mathbb{Q}$-linear small category $\mathbb{Q}\Pi^+ S|_E$ whose object set is $E := \{*_a\}_{a=0}^n$, and whose morphism vector space from $*_a$ to $*_b$ is the $\mathbb{Q}$-free vector space over the set $\Pi^+ S(v_a, -v_b) := \{\ell : [0,1] \to S$; an immersed path in S from $*_a$ to $*_b$ with $\ell(0) = v_a$ and $\ell(1) = -v_b\}$ modulo regular homotopy. The infinite cyclic group $\langle r \rangle$ acts on the set $\Pi^+ S(v_a, -v_b)$ by inserting monogons into paths. If we fix a framing f of S, we have a group isomorphism $\Pi^+ S(v_a, -v_a) \cong \pi_1(S, *_a) \times \mathbb{Z}$. We denote by $\mathrm{Der}_\partial(\mathbb{Q}\Pi^+ S|_E)$ the Lie algebra of $\mathbb{Q}\langle r \rangle$-linear derivations of the category $\mathbb{Q}\Pi^+ S|_E$ annihilating all loops parallel to some boundary component. In the same way as in [6] we can define a $\mathbb{Q}\langle r \rangle$-Lie algebra homomorphism $\sigma^+ : \mathbb{Q}\hat{\pi}^+/\mathbb{Q}\langle r \rangle 1 \to \mathrm{Der}_\partial(\mathbb{Q}\Pi^+ S|_E)$.

Now we take completions of $\mathbb{Q}\hat{\pi}^+$ and $\mathbb{Q}\Pi^+ S|_E$ with respect to the augmentation ideal of the group ring $\mathbb{Q}\Pi^+ S(v_a, -v_a)$, and denote them by $\widehat{\mathbb{Q}\hat{\pi}^+}$ and $\widehat{\mathbb{Q}\Pi^+ S}|_E$, respectively. Recall the completed group ring $\widehat{\mathbb{Q}\langle r \rangle}$ is naturally identified with the ring of formal power series in $\rho := \log r$. In other words, we have $\widehat{\mathbb{Q}\langle r \rangle} = \mathbb{Q}[[\log r]] = \mathbb{Q}[[\rho]]$. The bracket $[\,,\,]^+$ and the cobracket δ^+ induce a natural Lie bialgebra structure on the completion $\widehat{\mathbb{Q}\hat{\pi}^+}$.

The Enomoto-Satoh traces

Recall that the completed Goldman-Turaev Lie bialgebra $\widehat{\mathbb{Q}\hat{\pi}}$ introduced in [7] has a decreasing filtration $\{\widehat{\mathbb{Q}\hat{\pi}}(m)\}_{m=1}^\infty$, and that the $\mathbb{Q}$-linear category $\widehat{\mathbb{Q}\Pi S}|_E$ admits a coproduct Δ [7]. Then we introduce a Lie subalgebra $L^+(S) := \{u \in \widehat{\mathbb{Q}\hat{\pi}}(3); (\sigma(u)\hat{\otimes}1 + 1\hat{\otimes}\sigma(u))\Delta = \Delta\sigma(u)\} \subset \widehat{\mathbb{Q}\hat{\pi}}$. We can prove that the restriction of the map $s_f : \widehat{\mathbb{Q}\hat{\pi}} \to \widehat{\mathbb{Q}\hat{\pi}^+}/\mathbb{Q}[[\rho]]1$ to the subalgebra $L^+(S)$ does not depend on the choice of a framing f. So we denote it by $s_{\mathrm{can}} : L^+(S) \to \widehat{\mathbb{Q}\hat{\pi}^+}/\mathbb{Q}[[\rho]]1$, and call it the *canonical section*. Then we define the maps $\mathsf{ES}_f^+ : \widehat{\mathbb{Q}\hat{\pi}^+}/\mathbb{Q}[[\rho]]1 \to \widehat{\mathbb{Q}\hat{\pi}^+}/\mathbb{Q}[[\rho]]1$ and $\mathsf{ES}_f : L^+(S) \to \widehat{\mathbb{Q}\hat{\pi}}$ by the following commutative

diagram

$$\begin{array}{ccc}
\widehat{\mathbb{Q}\hat{\pi}^+}/\mathbb{Q}[[\rho]]1 & \xrightarrow{\delta^+} & \widehat{\mathbb{Q}\hat{\pi}^+}\widehat{\otimes}_{\mathbb{Q}[[\rho]]}\widehat{\mathbb{Q}\hat{\pi}^+} \\
\scriptstyle{s_{\mathrm{can}}}\Big\uparrow & \xrightarrow{\mathsf{ES}_f^+} & \Big\downarrow{\scriptstyle \varepsilon_f\otimes 1} \\
L^+(S) & & \widehat{\mathbb{Q}\hat{\pi}^+}/\mathbb{Q}[[\rho]]1 \\
& \xrightarrow{\mathsf{ES}_f} & \Big\downarrow{\scriptstyle \Phi} \\
& & \widehat{\mathbb{Q}\hat{\pi}}.
\end{array}$$

In the case the boundary ∂S is connected, the graded quotient of the map ES_f

$$\mathrm{gr}(\mathsf{ES}_f) : \mathrm{gr}(L^+(S)) \to \mathrm{gr}(\widehat{\mathbb{Q}\hat{\pi}})$$

is exactly the Enomoto-Satoh traces. On the other hand, if S is of genus 0, the Lie algebra $L^+(S)$ is isomorphic to an extension of the positive part of the special derivation algebra $\mathfrak{sder}_n$, and $\widehat{\mathbb{Q}\hat{\pi}}$ to the space $\mathfrak{tr}_n$ in [1]. Then the graded quotient $\mathrm{gr}(\mathsf{ES}_f)$ equals the restriction of the divergence cocycle div in the Kashiwara-Vergne problem [1]. The proof of these facts is based on a tensorial description of the homotopy intersection form by Massuyeau and Turaev [9]. Hence the Enomoto-Satoh traces and the divergence cocycle are interpreted as some part of the regular Turaev cobracket.

The mapping class group

The homomorphism $\sigma^+ : \mathbb{Q}\hat{\pi}^+/\mathbb{Q}\langle r\rangle 1 \to \mathrm{Der}_\partial(\mathbb{Q}\Pi^+S|_E)$ induces a $\mathbb{Q}[[\rho]]$-Lie algebra homomorphism $\sigma^+ : \widehat{\mathbb{Q}\hat{\pi}^+}/\mathbb{Q}[[\rho]]1 \to \mathrm{Der}_\partial(\widehat{\mathbb{Q}\Pi^+S}|_E)$. Then it is a Lie algebra isomomorphism

$$\sigma^+ : \widehat{\mathbb{Q}\hat{\pi}^+}/\mathbb{Q}[[\rho]]1 \xrightarrow{\cong} \mathrm{Der}_\partial(\widehat{\mathbb{Q}\Pi^+S}|_E). \tag{1}$$

Moreover, for any framing f of S, the map $\tilde{\Phi}_f$ induces an isomorphism

$$\tilde{\Phi}_f : \widehat{\mathbb{Q}\hat{\pi}^+}/\mathbb{Q}[[\rho]]1 \xrightarrow{\cong} \widehat{\mathbb{Q}\hat{\pi}}\widehat{\otimes}\mathbb{Q}[[\rho]]. \tag{2}$$

Let $\mathcal{I}^L(S)$ be the largest Torelli group in the sense of Putman [11]. By the isomorphism (1) we can define the *geometric* Johnson homomorphism

$$\tau^+ : \mathcal{I}^L(S) \to \widehat{\mathbb{Q}\hat{\pi}^+}/\mathbb{Q}[[\rho]]1$$

in the same way as in [7]. Applying a regular homotopy version of the logarithm formula for Dehn twists [6][7][9] to Putman's generators of $\mathcal{I}^L(S)$ [11], we can prove

$$\delta^+ \circ \tau^+ = 0 : \mathcal{I}^L(S) \to \widehat{\mathbb{Q}\hat{\pi}^+}\widehat{\otimes}_{\mathbb{Q}[[\rho]]}\widehat{\mathbb{Q}\hat{\pi}^+}. \tag{3}$$

In fact, the cobracket δ^+ vanishes at any power of any embedded loop. Let C be a simple closed curve in the interior of S with $\pm[C] = 0 \in H_1(S;\mathbb{Z})$. In other words, C is a bounding simple closed curve. Then the Dehn twist t_C along C satisfies $\tau(t_C) \in L^+(S)$. Choose a framing f of S. Let h be the genus of the subsurface bounded by C. Then, under the isomorphism (2), the regular homotopy version of the logarithm formula says

$$\tilde{\Phi}_f(\tau^+(t_C)) = \frac{1}{2}|(\log(C) + (1-2h)\rho)^2| = \frac{1}{2}|(1-2h)\rho\log(C) + (\log(C))^2| = \frac{1}{2}|(\log(C))^2|,$$

since C is null-homologous. We define the subgroup $\mathcal{K}(S) \subset \mathcal{I}^L(S)$ by those generated by such Dehn twists. In view of a theorem of Johnson [5], $\mathcal{K}(S)$ is the Johnson kernel if the boundary of S is connected.

Consequently, for any S, we obtain the commutative diagram

$$
\begin{array}{ccccc}
\mathcal{I}^L(S) & \xrightarrow{\tau^+} & \widehat{\mathbb{Q}\hat{\pi}^+}/\mathbb{Q}[[\rho]]1 & \xrightarrow{\delta^+} & \widehat{\mathbb{Q}\hat{\pi}^+} \otimes_{\mathbb{Q}[[\rho]]} \widehat{\mathbb{Q}\hat{\pi}^+} \\
\text{incl.} \uparrow & & s_{\text{can}} \uparrow & & \downarrow \varepsilon_f \otimes 1 \\
\mathcal{K}(S) & \xrightarrow{\tau} & L^+(S) & & \widehat{\mathbb{Q}\hat{\pi}^+}/\mathbb{Q}[[\rho]]1 \\
& & & & \downarrow \Phi \\
& & & & \widehat{\mathbb{Q}\hat{\pi}}.
\end{array}
$$

with ES_f^+ and ES_f labeling the diagonal arrows.

In particular, we have $\mathsf{ES}_f \circ \tau|_{\mathcal{K}(S)} = 0 : \mathcal{K}(S) \to \widehat{\mathbb{Q}\hat{\pi}}$. This gives a geometric proof for the fact that the Enomoto-Satoh traces are obstructions for the surjectivity of the Johnson homomorphims.

The genus 0 case

Let $n \geq 2$ be an integer. Here we study the framed pure braid group FP_n on n strands on the 2-disk. This is nothing but the largest Torelli group of $S := \Sigma_{0,n+1}$. By capping each of the boundary components except one by the surface $\Sigma_{1,1}$ we obtain an embedding of the surface $\iota : S \hookrightarrow \hat{S} := \Sigma_{n,1}$, which induces a Lie algebra homomorphism $\iota : \widehat{\mathbb{Q}\hat{\pi}}(S) \to \widehat{\mathbb{Q}\hat{\pi}}(\hat{S})$. The pull-back of the Enomoto-Satoh trace $\mathrm{gr}(\mathsf{ES}_f)$ by the map ι equals the divergence cocycle div in the Kashiwara-Vergne problem [1] up to some low degree map $H_1(S)^{\otimes 2} \to H_1(S)$. Here f is any framing of $\hat{S}$. In this case we have $\mathcal{K}(S) = \{1\}$. Instead we consider the commutator subgroup $[FP_n, FP_n]$. Choose a framing f of S. For any simple closed curve C_i, $i = 1, 2$, in S, we have

$$
\tilde{\Phi}_f(\tau^+(t_{C_i})) = \frac{1}{2}\mathrm{rot}_f(C_i)\rho|\log(C_i)| + \frac{1}{2}|(\log(C_i))^2|.
$$

Since the genus of S is zero, the homology group $H_1(S; \mathbb{Q})$ is spanned by the boundary loops. In particular, $|\log(C_i)|$ is in the center of $\widehat{\mathbb{Q}\hat{\pi}}$. Hence we have

$$
\left[\tilde{\Phi}_f(\tau^+(t_{C_1})), \tilde{\Phi}_f(\tau^+(t_{C_2}))\right] = \left[\frac{1}{2}|(\log(C_1))^2|, \frac{1}{2}|(\log(C_2))^2|\right],
$$

which is independent of the choice of the framing f. Thus we have $\tau^+|_{[FP_n, FP_n]} = s_{\text{can}} \circ \tau :$ $[FP_n, FP_n] \to \widehat{\mathbb{Q}\hat{\pi}^+}/\mathbb{Q}[[\rho]]1$, and $(\mathsf{ES}_f \circ \tau)(\varphi) = 0$ for any $\varphi \in [FP_n, FP_n]$.

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