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# Markoff graphs

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## 1 Introduction

It is known that the discrete part of the Markoff spectrum for $\mathbb{Q}$ is characterized by simple closed geodesics on the once punctured torus having the highest symmetry (Theorem 3.4). This is suggested by an analogy between the equation $p^2+q^2+r^2=3pqr$ whose positive integer solutions represent the discrete part of the Markoff spectrum for $\mathbb{Q}$ and the equation $X^2+Y^2+Z^2=XYZ$ which describes the moduli space of the once punctured torus. In this note we represent this analogy, in other words, the analogy between the structure of the solutions of these equations by means of a graph. We call it the *Markoff graph*, which is used, for example, in [8] and [4].

The tiling $\{3, \infty\}$ of the upper half-plane $\mathbb{H}^2$ by regular ideal triangles has natural bipartite structure and the once punctured torus having the highest symmetry consists of two adjacent triangles of this tiling (§2). The dual graph of $\{3, \infty\}$ defines the Markoff graph. It naturally provides the set of the positive integer solutions of $p^2+q^2+r^2=3pqr$ and the set of matrices whose axes project to simple closed geodesics on the once punctured torus having the highest symmetry (§4). The obtained matrices are in $\mathrm{SL}(2, \mathbb{Z})$ and have explicit forms (Theorem 3.2), which is one of the advantages by making use of the Markoff graph.

We can apply this approach to a geometric interpretation (Theorem 3.7) of the Markoff spectrum for $\mathbb{Q}(i)$ developed in [1]. The positive integer solutions as $x, y_1, y_2$ of the equation $2x^2+y_1^2+y_2^2=4y_1y_2x$ give two sequences of the Markoff spectrum for $\mathbb{Q}$ which are not in the discrete part, and those as $x$ give the discrete part of the Markoff spectrum for $\mathbb{Q}(i)$ (except for one value). We can also find an analogy between this equation and $X^2+Y^2+Z^2=XYZ$ (§3).

To represent the analogy between the structure of the solutions of these equations, we need the tiling $\{4, \infty\}$ of $\mathbb{H}^2$ by ideal hyperbolic squares and its natural bipartite structure (§2). The dual graph of this tiling defines the two-color Markoff graph, which is new as far as we know. It naturally provides the set of the positive integer solutions of $2x^2+y_1^2+y_2^2=4y_1y_2x$ and the set of matrices whose axes project to simple closed geodesics on the once punctured torus corresponding to the ideal hyperbolic square with the second highest symmetry (§5). The obtained matrices are in the Hecke group generated by $\begin{pmatrix} 1 & \sqrt{2} \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and have explicit forms (Theorem 3.5).

The bipartite structure of the tiling $\{4, \infty\}$ leads us to obtain a twice punctured torus which is a double cover of the once punctured torus corresponding to the ideal hyperbolic square. Immersing this surface into the Borromean rings complement, we can finally obtain the geometric interpretation of the Markoff spectrum for $\mathbb{Q}(i)$. For a construction of a model of the Borromean rings complement and an immersion of the twice punctured torus into the model, we refer the reader to [1].

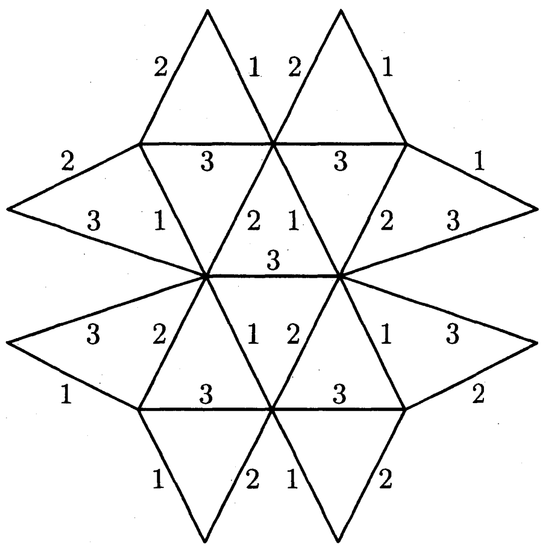


Figure 1: Tiling $\{3, \infty\}$ and labeling in the same cyclic order

The construction of the Markoff graph shows a basic idea of this approach. We explain in parallel how to build the Markoff graph and the two-color Markoff graph. This is an introductory note to [1], [2], and [3], which are joint works with I. R. Aitchison and B. Rittaud.

## 2 Tiling of hyperbolic spaces

Let $\mathbb{H}^2 = \{z = x + iy \in \mathbb{C} \mid y > 0\}$ be the upper half-plane endowed with the hyperbolic metric $ds^2 = (dx^2 + dy^2)/y^2$. A geodesic in $\mathbb{H}^2$ is a semicircle or a ray perpendicular to the real axis. The upper half-plane $\mathbb{H}^2$ is tiled by the ideal hyperbolic triangle with vertices 0, 1, and $\infty$, and its images by the action of the modular group $\mathrm{PSL}(2, \mathbb{Z})$. Since this tiling is done by regular ideal triangles, $\infty$ at each vertex, it is denoted by $\{3, \infty\}$. We depict it as in Figure 1. Label the edges of a triangle in $\{3, \infty\}$ by $1, 2, 3$, and transfer the labels to edge-adjacent triangles with the same cyclic order $1, 2, 3$ (see Figure 1). As a consequence, the cyclic order is preserved, but in passing through an edge to an adjacent triangle, the labels are rotated by $\pi$. We thus have a natural 2-coloring of triangles of $\{3, \infty\}$. The symmetry group of $\mathbb{H}^2$ respecting both this labeling and bipartite structure has fundamental domain consisting of two adjacent triangles forming a hyperbolic ideal rhombus, with opposite edges receiving the same edge-label. The quotient by the symmetry group is a once punctured torus. Considering a free group $\Gamma_3 = \langle A_3, B_3 \rangle$ generated by

$$A_3 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \text{ and } B_3 = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix},$$

the once punctured torus consisting of two adjacent ideal hyperbolic triangles is realized as the quotient space $\mathbb{H}^2/\Gamma_3$.

Generally, if two elements $g$ and $h$ in $\mathrm{PSL}(2, \mathbb{R})$ are hyperbolic, their axes intersect, and the commutator $[h, g]$ is parabolic, then the quotient space of $\mathbb{H}^2$ by a two generator Fuchsian group $\langle g, h \rangle$ is regarded as a once punctured torus. If we set $X = \mathrm{tr}(g)$, $Y = \mathrm{tr}(h)$, and $Z = \mathrm{tr}(gh)$, the conditions on $g$ and $h$ above are equivalent to the one: $X, Y$, and $Z$ satisfy

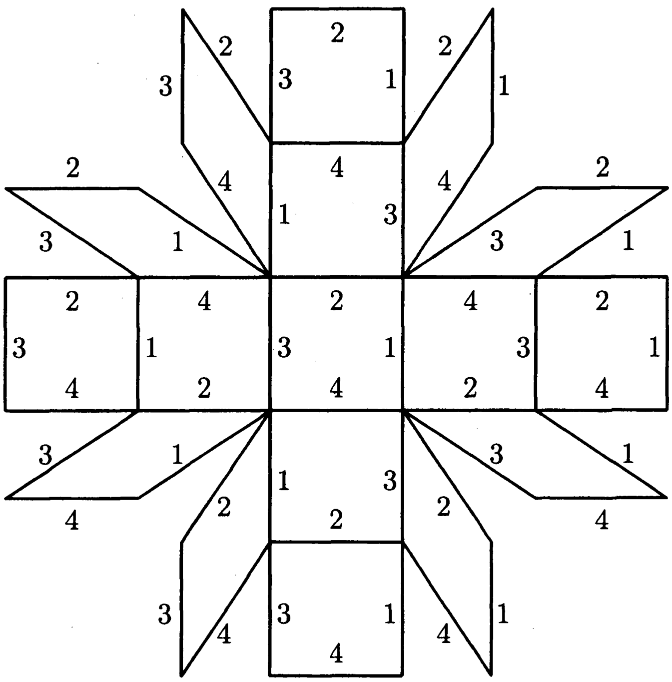

Figure 2: Tiling $\{4,\infty\}$ and labeling in the same cyclic order

Fricke's moduli equation $X^2+Y^2+Z^2=XYZ$ and all of them are greater than 2. We call such a Fuchsian group, the quotient space by which is a once punctured torus, a *Fricke group* and denoted by $(X,Y,Z)$. The group $\Gamma_3$ is an example of the Fricke group, and has $(3,3,3)$.

Let us consider another example of the Fricke group: $\Gamma_4=\langle A_4,B_4\rangle$ generated by

$$A_4=\begin{pmatrix} 2\sqrt{2} & 1 \\ -1 & 0 \end{pmatrix} \text{ and } B_4=\begin{pmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{pmatrix}$$

having the coordinates $(2\sqrt{2},2\sqrt{2},4)$. The quotient space $\mathbb{H}^2/\Gamma_4$ is regarded as a once punctured torus by the ideal hyperbolic square; $\mathbb{H}^2$ is tiled by this once punctured torus and its images by the action of $\Gamma_4$. Since this tiling is done by regular ideal squares, $\infty$ at each vertex, it is denoted by $\{4,\infty\}$. We depict it as in Figure 2. As with the tiling of $\mathbb{H}^2$ by regular triangles, we label every square with labels from $\{1,2,3,4\}$ in the same cyclic order. We also have a natural 2-coloring of squares of $\{4,\infty\}$ (see Figure 2). The quotient by the symmetry group of $\mathbb{H}^2$ respecting both this labeling and bipartite structure is a twice punctured torus consisting of two adjacent ideal hyperbolic squares. Considering a group $\Gamma_4^2=\langle P_4,Q_4,R_4\rangle$ generated by

$$P_4=B_4A_4^{-1}=\begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 3 \end{pmatrix},\quad Q_4=A_4^{-1}B_4^{-1}=\begin{pmatrix} 1 & -\sqrt{2} \\ -\sqrt{2} & 3 \end{pmatrix},$$

$$R_4=B_4^2=\begin{pmatrix} 3 & 2\sqrt{2} \\ 2\sqrt{2} & 3 \end{pmatrix},$$

the twice punctured torus consisting of two adjacent ideal hyperbolic squares is realized as the quotient space $\mathbb{H}^2/\Gamma_4^2$.

Now consider the upper half-space $\mathbb{H}^3 = \{z + jt \mid z = x + iy \in \mathbb{C}, t > 0\}$ with the hyperbolic metric $ds^2 = (dx^2 + dy^2 + dt^2)/t^2$. A geodesic in $\mathbb{H}^3$ is a semicircle or a ray perpendicular to the boundary $\mathbb{C}$ of $\mathbb{H}^3$, which is realized as the intersection of hemispheres (or half-planes). The upper half-space $\mathbb{H}^3$ is tiled by the ideal hyperbolic octahedron with vertices 0, 1, $i$, $1+i$, $(1+i)/2$ and $\infty$, and its images by the action of the Picard group $\mathrm{PSL}(2, \mathbb{Z}[i])$.

There are various subgroups of the Picard group leading to quotient spaces of $\mathbb{H}^3$ related to links. Here we focus on the Borromean rings complement obtained by gluing together two ideal hyperbolic octahedra. The following elements of $\mathrm{SL}(2, \mathbb{Z}[i])$

$$P_\infty = \begin{pmatrix} 1 & 1-i \\ 0 & 1 \end{pmatrix}, \quad P_0 = \begin{pmatrix} 1 & 0 \\ -1+i & 1 \end{pmatrix}, \quad \text{and } P_1 = \begin{pmatrix} i & 1-i \\ -1+i & 2-i \end{pmatrix}$$

generate the group $\Gamma_{\mathrm{BR}}$ with relations:

$$\Gamma_{\mathrm{BR}} = \left\langle P_\infty, P_0, P_1 \;\middle|\; \begin{array}{l} P_\infty(P_1^{-1}P_0P_1P_0^{-1}) = (P_1^{-1}P_0P_1P_0^{-1})P_\infty \\ P_0(P_\infty^{-1}P_1P_\infty P_1^{-1}) = (P_\infty^{-1}P_1P_\infty P_1^{-1})P_0 \\ P_1(P_0^{-1}P_\infty P_0P_\infty^{-1}) = (P_0^{-1}P_\infty P_0P_\infty^{-1})P_1 \end{array} \right\rangle.$$

The quotient space $\mathbb{H}^3/\Gamma_{\mathrm{BR}}$ is a geometric model of the Borromean rings complement consisting of two adjacent ideal hyperbolic octahedra in the above tiling (see §3 of [1]).

We can immerse the twice punctured torus $\mathbb{H}^2/\Gamma_4^2$ into the Borromean rings complement taking the conjugate $U\Gamma_4^2U^{-1}$ of $\Gamma_4^2$ by $U = \begin{pmatrix} (1-i)/\sqrt{2} & i \\ 0 & 1+i \end{pmatrix}$ (see §4 of [1]). This plays an important role in giving a geometric interpretation of the Markoff spectrum for $\mathbb{Q}(i)$ discussed in the next section.

We sometimes find naturally a twice punctured torus in a hyperbolic 3-manifold realized by a subgroup of the Picard group. See, for example, [12].

# 3 Markoff spectra

Let $f(\xi, \eta) = a\xi^2 + b\xi\eta + c\eta^2$ be a binary indefinite quadratic form with real coefficients and with discriminant $D(f) = b^2 - 4ac$. We define $m(f) = \inf_{(\xi,\eta)\in\mathbb{Z}^2-\{(0,0)\}} |f(\xi, \eta)|$. The set

$$\mathcal{M} = \left\{ \sqrt{D(f)}/m(f) \;\middle|\; (a, b, c) \in \mathbb{R}^3, D(f) > 0 \right\}$$

is called the *Markoff spectrum* for the rational number field $\mathbb{Q}$. The discrete part of this spectrum is described in the form:

$$\mathcal{M} \cap [0, 3) = \left\{ \sqrt{9 - \frac{4}{k^2}} \;\middle|\; k \in \mathcal{K} \right\},$$

where $\mathcal{K} = \{1, 2, 5, 13, 29, \ldots\}$ is the set of positive integer solutions (called *Markoff numbers*) of Markoff's equation

$$p^2 + q^2 + r^2 = 3pqr \tag{1}$$

(see [9]).

Markoff's equation is obtained from Fricke's moduli equation and the coordinates $(3, 3, 3)$ of $\Gamma_3$: to get Markoff's, set $X = 3x$, $Y = 3y$, and $Z = 3z$ in Fricke's. Using this link, we can define a bijection between the set of the Markoff numbers and a set of primitive elements of the group $\Gamma_3$. This correspondence is first studied by H. Cohn (see [5], [7]). An element $g$ is *primitive* in a two generator free group $\Gamma$ if there is another element $h$ which generates $\Gamma$ together with $g$. The existence of the correspondence between the Markoff numbers and the primitive elements of $\Gamma_3$ allows us to interpret the Markoff spectrum for $\mathbb{Q}$ by means of geodesics on the once punctured torus $\mathbb{H}^2/\Gamma_3$.

A geodesic on a quotient space of $\mathbb{H}^2$ is defined as the projection of a geodesic in $\mathbb{H}^2$. A geodesic is called *simple* if it has no self-intersections.

**Theorem 3.1** ([10]). *A geodesic $\gamma$ in $\mathbb{H}^2$ is the axis of a primitive element of a Fricke group $\Gamma$ if and only if $\gamma$ projects to a simple closed geodesic on the once punctured torus $\mathbb{H}^2/\Gamma$.*

For a Markoff number $k$, a corresponding primitive element $N_k$ of $\Gamma_3$ has an explicit form:

**Theorem 3.2.** *For each $k \in \mathcal{K}$,*

$$N_k = \begin{pmatrix} a & b \\ k & d \end{pmatrix}, \quad (a, b, d) \in \mathbb{Z}^3, \quad \mathrm{tr}(N_k) = 3k, \quad \det(N_k) = 1.$$

We define a form by the fixed point equation of the action of $N_k$: $f_{N_k}(\xi, \eta) = k\xi^2 - (a - d)\xi\eta - b\eta^2$.

**Theorem 3.3.** $m(f_{N_k}) = f_{N_k}(1, 0) = k$. *Hence, $f_{N_k}$ attains* $\dfrac{\sqrt{D(f_{N_k})}}{m(f_{N_k})} = \sqrt{9 - \dfrac{4}{k^2}}$.

The *Euclidean height* of a geodesic $\gamma$ in $\mathbb{H}^2$ with endpoints $\alpha$ and $\beta$ is defined as $|\alpha - \beta|/2$ if both $\alpha$ and $\beta$ are finite or $\infty$ otherwise. Denote this by $h_E(\gamma)$. For the axis $\gamma_{N_k}$ of $N_k$, we get $h_E(\gamma_{N_k}) = \sqrt{9 - (4/k^2)}/2$ and it attains the maximum of the set $\{h_E(g(\gamma_{N_k})) \mid g \in \Gamma_3\}$.

The following is deduced from these theorems and facts:

**Theorem 3.4** ([5], [6], [7], see also [10]). *The discrete part of the Markoff spectrum for $\mathbb{Q}$ is given by the twice maximal Euclidean height of the lifts of the simple closed geodesics on the once punctured torus $\mathbb{H}^2/\Gamma_3$.*

In this note, we show that a bijection between the Markoff numbers and primitive elements of $\Gamma_3$ is naturally defined by means of the Markoff graph which is the dual graph of the tiling $\{3, \infty\}$ of $\mathbb{H}^2$, and prove Theorem 3.2 (see §4). Theorem 3.3 is obtained from the construction of the Markoff graph and Theorem 3.2. Moreover, we show that to the following Markoff type spectra we can apply the method using the dual graph of the tiling $\{4, \infty\}$ and obtain similar results to the Markoff spectrum for $\mathbb{Q}$.

Let $\mathbb{Q}(i)$ denote the imaginary quadratic number field whose ring of integers is the set of Gaussian integers $\mathbb{Z}[i]$. The Markoff spectrum for $\mathbb{Q}(i)$ is defined in the same way as $\mathcal{M}$: for $f(\xi, \eta) = a\xi^2 + b\xi\eta + c\eta^2$

$$\mathcal{M}_1 = \left\{ \sqrt{|D(f)|}/m_1(f) \,\middle|\, (a, b, c) \in \mathbb{C}^3, D(f) \neq 0 \right\},$$

where $m_1(f) = \inf_{(\xi,\eta) \in \mathbb{Z}[i]^2 - \{(0,0)\}} |f(\xi, \eta)|$. The discrete part of this spectrum is described in the form

$$\mathcal{M}_1 \cap [0, 4) = \left\{ \sqrt{4 - \frac{1}{\lambda^2}} \,\middle|\, \lambda \in \mathcal{N}(\Lambda) \right\} \cup \left\{ \sqrt{\frac{3}{5}\sqrt{41}} \right\},$$

where $\mathcal{N}(\Lambda) = \{1, 5, 29, 65, 169, \cdots\}$ is the set of positive integer solutions $x$ of Vulakh's equation $2x^2 + y_1^2 + y_2^2 = 4y_1y_2x$, which is equivalently written by

$$\begin{cases} x_1 + x_2 &= 2y_1y_2 \\ 2x_1x_2 &= y_1^2 + y_2^2 \end{cases} \tag{2}$$

(see [13] and [15]).

It is also known that the positive integer solutions of (2) give two sequences of the Markoff spectrum for $\mathbb{Q}$ which are not in the discrete part (see [11] and [14]). That is, let $\mathcal{N}(M) = \{1, 3, 11, 17, 41, \cdots\}$ be the set of positive integer solutions $y_1$, $y_2$ of (2), then we have

$$\left\{ \sqrt{16 - \frac{4}{\lambda^2}} \;\middle|\; \lambda \in \mathcal{N}(\Lambda) \right\} \cup \left\{ \sqrt{16 - \frac{8}{m^2}} \;\middle|\; m \in \mathcal{N}(M) \right\} \subset \mathcal{M} \cap [0, 4). \tag{3}$$

Vulakh's equation is obtained from Fricke's moduli equation and the coordinates $(2\sqrt{2}, 2\sqrt{2}, 4)$ of $\Gamma_4$: to get Vulakh's, set $X = 4x$, $Y = 2\sqrt{2}y_1$, and $Z = 2\sqrt{2}y_2$ in Fricke's. We define a bijection between a set of primitive elements of $\Gamma_4$ and the sets $\mathcal{N}(\Lambda)$, $\mathcal{N}(M)$ of positive integer solutions of (2) by means of the two-color Markoff graph which is the dual graph of the tiling $\{4, \infty\}$ of $\mathbb{H}^2$ (see §5).

Primitive elements of $\Gamma_4$ corresponding to elements of $\mathcal{N}(\Lambda)$ and $\mathcal{N}(M)$ have explicit forms:

**Theorem 3.5.** *(i) For each $\lambda \in \mathcal{N}(\Lambda)$,*

$$\Lambda_\lambda = \begin{pmatrix} a & b\sqrt{2} \\ \lambda\sqrt{2} & c \end{pmatrix}, \ (a, b, c) \in \mathbb{Z}^3, \ \mathrm{tr}(\Lambda_\lambda) = 4\lambda, \ \det(\Lambda_\lambda) = 1.$$

*(ii) For each $m \in \mathcal{N}(M)$,*

$$M_m = \begin{pmatrix} \alpha\sqrt{2} & \beta \\ m & \gamma\sqrt{2} \end{pmatrix}, \ (\alpha, \beta, \gamma) \in \mathbb{Z}^3, \ \mathrm{tr}(M_m) = 2\sqrt{2}m, \ \det(M_m) = 1.$$

Let $\gamma_{\Lambda_\lambda}$ denote the axis of $\Lambda_\lambda$ and let $\gamma_{M_m}$ denote the axis of $M_m$. Then we have $h_E(\gamma_{\Lambda_\lambda}) = \sqrt{16 - (4/\lambda^2)}/(2\sqrt{2})$ and $h_E(\gamma_{M_m}) = \sqrt{16 - (8/m^2)}/(2\sqrt{2})$. It can be proved that they attain the maximum of the sets $\{h_E(g(\gamma_{\Lambda_\lambda})) \mid g \in \Gamma_4\}$ and $\{h_E(g(\gamma_{M_m})) \mid g \in \Gamma_4\}$, respectively. Moreover, Theorem 3.1 ensures that the projection of $\gamma_{\Lambda_\lambda}$ and $\gamma_{M_m}$ on the once punctured torus $\mathbb{H}^2/\Gamma_4$ are simple closed.

Take the conjugate of the matrices $\Lambda_\lambda$, $M_m$ of Theorem 3.5 by $V = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{pmatrix}$ and denote them by $\hat{\Lambda}_\lambda$ and $\hat{M}_m$. They are described as:

$$\hat{\Lambda}_\lambda = V\Lambda_\lambda V^{-1} = \begin{pmatrix} a & 2b \\ \lambda & c \end{pmatrix}, \quad \hat{M}_m = VM_mV^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 2\alpha & 2\beta \\ m & 2\gamma \end{pmatrix}.$$

We define quadratic forms from these matrices considering the fixed point equations:

$$f_{\hat{\Lambda}_\lambda}(\xi, \eta) = \lambda\xi^2 - (a - c)\xi\eta - 2b\eta^2, \quad f_{\hat{M}_m}(\xi, \eta) = m\xi^2 - 2(\alpha - \gamma)\xi\eta - 2\beta\eta^2.$$

**Theorem 3.6.** *$m(f_{\hat{\Lambda}_\lambda}) = f_{\hat{\Lambda}_\lambda}(1, 0) = \lambda$ and $m(f_{\hat{M}_m}) = f_{\hat{M}_m}(1, 0) = m$.*

Hence, $f_{\hat{\Lambda}_\lambda}$ attains $\dfrac{\sqrt{D(f_{\hat{\Lambda}_\lambda})}}{m(f_{\hat{\Lambda}_\lambda})} = \sqrt{16 - \dfrac{4}{\lambda^2}}$ and $f_{\hat{M}_m}$ attains $\dfrac{\sqrt{D(f_{\hat{M}_m})}}{m(f_{\hat{M}_m})} = \sqrt{16 - \dfrac{8}{m^2}}$.
We thus obtain an analogy of Theorem 3.4 for the sequences in (3) (see [3]).

Since the twice punctured torus $\mathbb{H}^2/\Gamma_4^2$, which is a double cover of the once punctured torus $\mathbb{H}^2/\Gamma_4$, is immersed in the Borromean rings complement, we can apply the same argument as above to the Markoff spectrum for $\mathbb{Q}(i)$ and obtain similar results (see [1]).

The conjugate $\tilde{\Lambda}_\lambda$ of the matrix $\Lambda_\lambda$ of Theorem 3.5 by $U$ is an element of $\mathrm{SL}(2, \mathbb{Z}[i])$ and has the form:

$$\tilde{\Lambda}_\lambda = U\Lambda_\lambda U^{-1} = \begin{pmatrix} \tilde{a} + \lambda i & \tilde{b} + \tilde{c}i \\ 2\lambda i & \tilde{d} - \lambda i \end{pmatrix}, \ (\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}) \in \mathbb{Z}^4, \ \mathrm{tr}(\tilde{\Lambda}_\lambda) = 4\lambda, \ \det(\tilde{\Lambda}_\lambda) = 1.$$

Let $\gamma_{\tilde{\Lambda}_\lambda}$ denote the axis of $\tilde{\Lambda}_\lambda$. For a geodesic $\gamma$ in $\mathbb{H}^3$, we define $h_E(\gamma)$ in the same way as for a geodesic in $\mathbb{H}^2$. Then we have $h_E(\gamma_{\tilde{\Lambda}_\lambda}) = \sqrt{4 - (1/\lambda^2)}/2$. As an analogy of Theorem 3.4, we obtain:

**Theorem 3.7.** *The discrete part of the Markoff spectrum for $\mathbb{Q}(i)$, except for one value, is given by the twice maximal Euclidean height of the axis of $\tilde{\Lambda}_\lambda$, which projects to a simple closed geodesic on the twice punctured torus immersed in the Borromean rings complement.*

# 4 Markoff graph

A simplicial tree all of whose vertices have degree 3 is called a binary tree. Let $\Sigma_2$ be a binary tree properly embedded in the plane. A connected component of the complement of $\Sigma_2$ is called a *face*. Let $V(\Sigma_2)$ denote the set of vertices of $\Sigma_2$.

For a vertex $v \in V(\Sigma_2)$ we label the three faces around $v$ by solutions $p, q, r$ of (1) as in the left of Figure 3. (Here we do not suppose that the solutions are integers.) Then it is said that $v$ satisfies the *vertex relation* (1) and $v$ is denoted by $(p, q, r)$, where we do not consider the order of $p, q, r$.

A vertex $v$ has three edges. The endpoints of these edges, other than $v$, are called the *neighbor* vertices of $v$. Let us consider the neighbor vertex $v'$ of $v$ connecting by the edge between the faces $p$ and $r$. Denoting a new face by $q'$, the faces $p$, $r$, and $q'$ meet at the vertex $v'$, which is now denoted by $(p, r, q')$. If both $v$ and $v'$ satisfy the vertex relation (1) and $q' \neq q$, then we get the following lemma:

**Lemma 4.1.** *Two vertices $(p, q, r)$ and $(p, q', r)$ satisfy $q + q' = 3pr$.*

*Proof.* By the vertex relation of $(p, q, r)$, we have $p^2 + q^2 + r^2 = 3pqr$, by that of $(p, q', r)$, $p^2 + (q')^2 + r^2 = 3pq'r$. The aimed relation is obtained from the subtraction of these equations. □

We call $q + q' = 3pr$ the *relation of the edge* between $p$ and $r$ (or simply the *edge relation*). We now have, if $v$ and $v'$ satisfy (1) and $q' \neq q$, then $v'$ is written by $(p, r, 3pr - q)$.

When $v$ and the other neighbor vertices of $v$ satisfy the vertex relation (1), we obtain from the same argumentation the following result (see the left of Figure 3):

- For $v = (p, q, r)$ and $v'' = (p'', q, r)$, the relation of the edge between $q$ and $r$ is $p + p'' = 3qr$ and $v''$ is written by $(q, r, 3qr - p)$.
- For $v$ and $v''' = (p, q, r''')$, the relation of the edge between $p$ and $q$ is $r + r''' = 3pq$ and $v'''$ is written by $(p, q, 3pq - r)$.

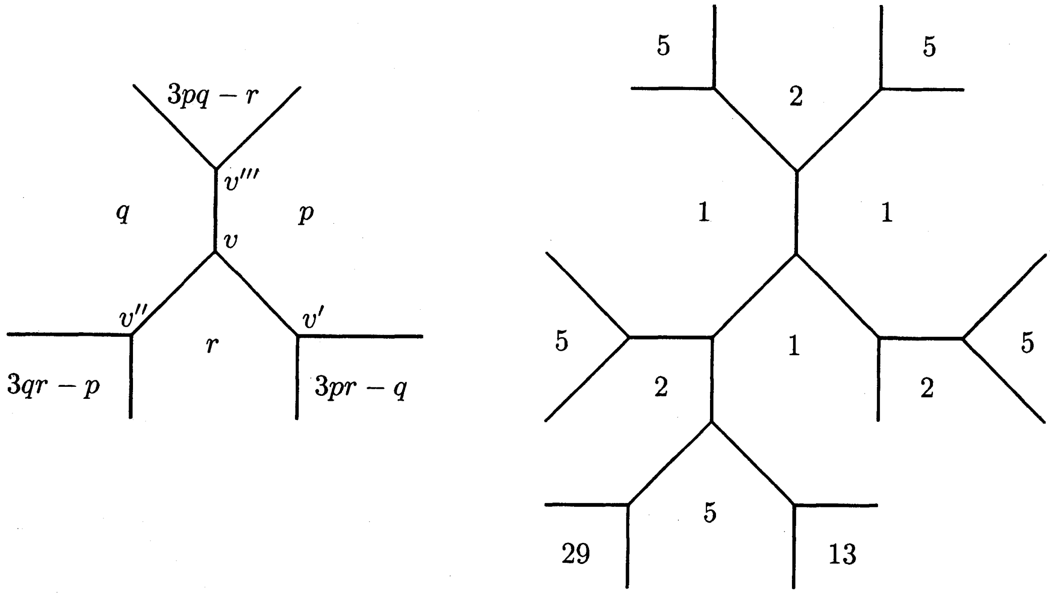


Figure 3: Labeling and edge relations, an example of the Markoff graph of numbers

Using these edge relations inductively, we can build from a vertex satisfying the vertex relation (1) a labeled binary tree $\Sigma_2$ all of whose vertices satisfy the relation (1). We call this a *Markoff graph* of numbers. If we take $(p,q,r) = (1,1,1)$ as an initial triple, we obtain the Markoff graph of numbers all of whose faces are labeled by positive integers (see the right of Figure 3).

For a binary tree $\Sigma_2$ embedded in the plane, we now think of directed edges so that for each vertex one edge is entering and two edges are going out. We call a vertex reached by ascending against the entering edge the *up* vertex of $v$, a vertex reached by following the right (or left) edge the *right* (or *left*) vertex of $v$. (They are in Figure 4 denoted by $v_u$, $v_r$, and $v_l$, respectively.)

If matrices correspond to faces around a vertex $v$ as in the left of Figure 4, that is, $B$ and $A$ correspond to the right and left faces of the entering edge of $v$, and the product $BA$ of them corresponds to the other face, then it is said that a triple of matrices $(B, A, C = BA)$ *corresponds* to the vertex $v$. Here the order of $B, A, C$ in the triple is important.

Given a vertex $v$ with a triple of matrices, we can obtain triples of matrices corresponding to the three neighbor vertices of $v$. Those corresponding to the right and left vertices are uniquely determined: if $(B, A, C = BA)$ corresponds to $v$, the triple $(B, C, BC)$ corresponds to the right vertex $v_r$ and $(C, A, CA)$ to the left vertex $v_l$. A triple of matrices corresponding to the up vertex $v_u$ has two possibilities: $(B, B^{-1}A, A)$ and $(B, BA^{-1}, A)$ (see the center and the right of Figure 4).

Using this operation inductively, we can build from a vertex $v$ with a triple of matrices a binary tree $\Sigma_2$ all of whose vertices have corresponding triples of matrices. We call this a *Markoff graph of matrices* (see, for example, Figure 5).

There exits an isomorphism between Markoff graphs of matrices and numbers.

**Lemma 4.2.** Suppose that a triple of matrices $(B, A, C)$ corresponds to a vertex $v \in V(\Sigma_2)$ and a triple of matrices $(B', A', C')$ corresponds to a neighbor vertex $v'$ of $v$. If $(p,q,r)$ defined by $\mathrm{tr}(A) = 3p$ , $\mathrm{tr}(B) = 3q$, and $\mathrm{tr}(C) = 3r$ satisfies the vertex relation (1), then $(p', q', r')$ defined by $\mathrm{tr}(A') = 3p'$, $\mathrm{tr}(B') = 3q'$, and $\mathrm{tr}(C') = 3r'$ also satisfies (1).

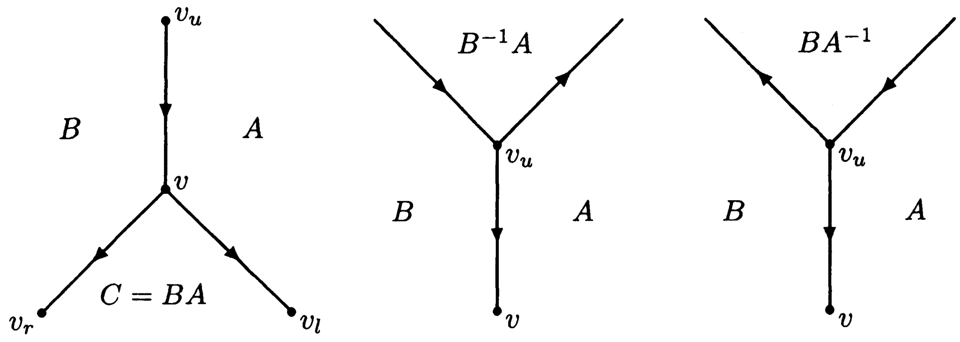


Figure 4: Labeling by matrices

*Proof.* We show the case that $v'$ is the left vertex of $v = (B, A, C = BA)$. Then we can write $(B', A', C') = (C, A, CA)$. We immediately get $p' = p$ and $q' = r$. We obtain $r' = 3pr - q$ from

$$\begin{aligned} 3r' &= \mathrm{tr}(C') = \mathrm{tr}(CA) = \mathrm{tr}(C)\mathrm{tr}(A) - \mathrm{tr}(CA^{-1}) = \mathrm{tr}(C)\mathrm{tr}(A) - \mathrm{tr}(B) \\ &= 3r \cdot 3p - 3q = 3(3rp - q), \end{aligned}$$

which is the relation of the edge between $p$ and $r$.

In the same way, we can prove the case that $v'$ is the right vertex of $v$.

Suppose that $v'$ is the up vertex of $v$. If $(B', A', C') = (B, B^{-1}A, A)$, then $q' = q$ and $r' = p$. We obtain $p' = 3pq - r$ from

$$\begin{aligned} 3p' &= \mathrm{tr}(A') = \mathrm{tr}(B^{-1}A) = \mathrm{tr}(B^{-1})\mathrm{tr}(A) - \mathrm{tr}(B^{-1}A^{-1}) = \mathrm{tr}(B)\mathrm{tr}(A) - \mathrm{tr}(C) \\ &= 3q \cdot 3p - 3r = 3(3pq - r), \end{aligned}$$

which is the relation of the edge between $p$ and $q$.

We can prove in the same way the case $(B', A', C') = (BA^{-1}, A, B)$. □

In the proof we used the well-known trace relations of matrices in $\mathrm{SL}(2, \mathbb{C})$:

$$\mathrm{tr}(N) = \mathrm{tr}(N^{-1}), \quad \mathrm{tr}(NM) = \mathrm{tr}(MN), \quad \mathrm{tr}(MN) = \mathrm{tr}(M)\mathrm{tr}(N) - \mathrm{tr}(MN^{-1})$$

for $M, N \in \mathrm{SL}(2, \mathbb{C})$.

Let us consider the Markoff graph of numbers building from a vertex $(1, 1, 1)$. The integers occurring in the faces of this graph are Markoff numbers. We define the initial matrices $N_1$, $N_2$ using elements of $\Gamma_3$:

$$N_1 = B_3^{-1} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad N_2 = B_3^{-1}A_3^{-1}B_3^{-1} = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}. \tag{4}$$

Let us consider the Markoff graph of matrices building from a vertex $(N_2, N_1, N_2N_1)$. The traces of the matrices around the vertex $v_0$ in Figure 5 are 3, 6, and 15 which are three times of 1, 2, and 5. Hence, in the sense of Lemma 4.2, the Markoff graph of numbers building from $(1, 1, 1)$ (the right of Figure 3) is isomorphic to the Markoff graph of matrices building from $(N_2, N_1, N_2N_1)$ (Figure 5).

By the definition of the initial matrices and the construction of the graph, we can prove that the matrices occurring in the Markoff graph are primitive in $\Gamma_3$. Moreover, a matrix

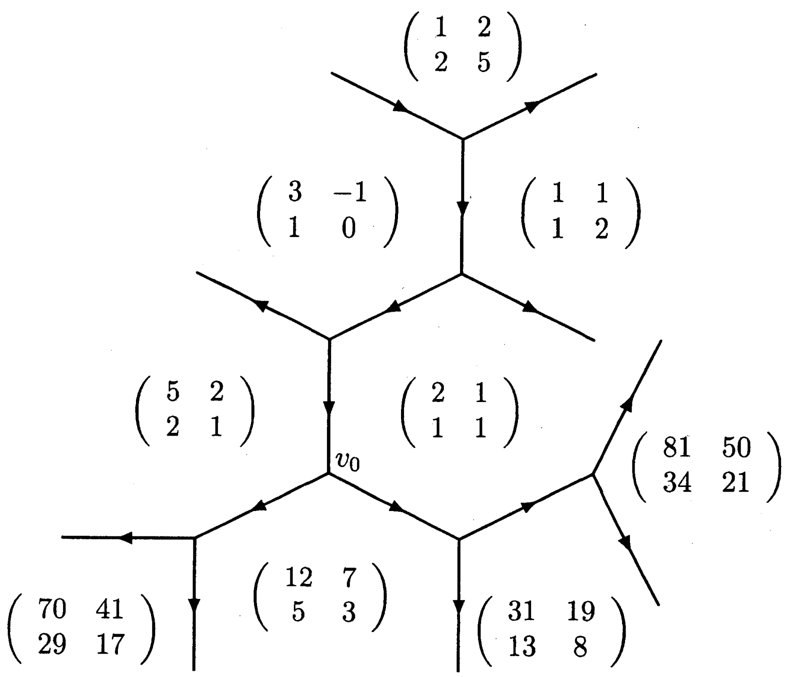


Figure 5: An example of the Markoff graph of matrices

$N_k$ corresponding to a Markoff number $k$ is described explicitly as in Theorem 3.2. Note that a matrix $N_k$ is not unique for a $k$, as we see in Figure 5.

*Proof of Theorem 3.2.* We can directly check that the initial matrices (4) satisfy the form of the theorem. It is almost evident from the construction of the Markoff graph of matrices that the condition $\det(N_k) = 1$ is satisfied. Using Lemma 4.2 inductively, we can prove $\operatorname{tr}(N_k) = 3k$. Hence, to complete a proof of Theorem 3.2, we have only to show that the $(2,1)$-entry of $N_k$ is equal to $k$.

Suppose that a triple of matrices $(N_{k_1}, N_{k_0}, N_{k_2})$ corresponding to a vertex $v$ in the Markoff graph of matrices building from $(N_2, N_1, N_2N_1)$ is described as in Figure 6. Of course, $k_0^2 + k_1^2 + k_2^2 = 3k_0k_1k_2$. From $N_{k_2} = N_{k_1}N_{k_0}$ we get the relations:

$$a_2 = a_0a_1 + k_0b_1, \quad b_2 = b_0a_1 + d_0b_1, \quad k_2 = a_0k_1 + k_0d_1, \quad d_2 = b_0k_1 + d_0d_1. \tag{5}$$

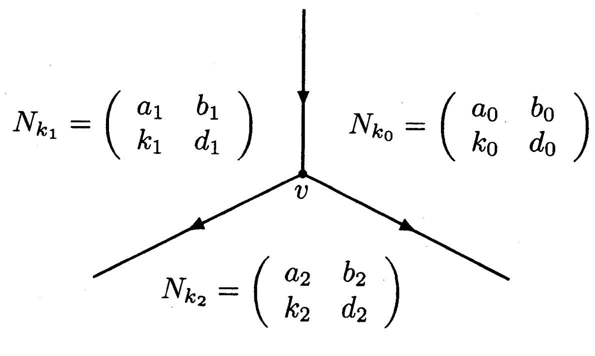


Figure 6: Triple of matrices corresponding to $v$

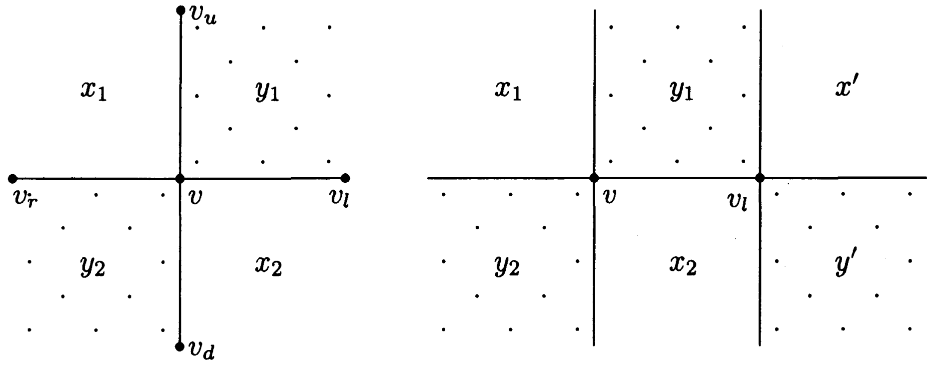


Figure 7: Labeling of the faces around a vertex. Black faces are represented by dots.

We can directly check that the matrices around $v_0$ in Figure 5 satisfy these relations.

We prove that the $(2,1)$-entry of $N_{k_1}N_{k_2}$, which is a new matrix in the right vertex of $v$, is equal to $3k_1k_2 - k_0$, that is, $k_1a_2 + d_1k_2 = 3k_1k_2 - k_0$:

$$\begin{aligned} k_1a_2 + d_1k_2 &= k_1(a_0a_1 + k_0b_1) + d_1(a_0k_1 + k_0d_1) \\ &= a_0k_1(a_1 + d_1) + k_0(b_1k_1 + d_1^2) = 3a_0k_1^2 + k_0(3d_1k_1 - 1) \\ &= 3k_1k_2 - k_0. \end{aligned}$$

In the same way, we can prove that the $(2,1)$-entry of $N_{k_2}N_{k_0}$, which is a new matrix in the left vertex of $v$, is equal to $3k_0k_2 - k_1$.

A new matrix occurring in the up vertex of $v$ is either $N_{k_1}^{-1}N_{k_0}$ or $N_{k_1}N_{k_0}^{-1}$. We prove that the $(2,1)$-entry $a_1k_0 - a_0k_1$ of $N_{k_1}^{-1}N_{k_0}$ is equal to $3k_0k_1 - k_2$:

$$3k_0k_1 - k_2 = 3k_0k_1 - (a_0k_1 + k_0d_1) = (3k_0 - a_0)k_1 - k_0(3k_1 - a_1) = -a_0k_1 + k_0a_1.$$

In the same way, we can prove that the $(2,1)$-entry of $N_{k_1}N_{k_0}^{-1}$ is equal to $3k_0k_1 - k_2$.

The proof of Theorem 3.2 is finished. □

# 5 Two-color Markoff graph

A simplicial tree all of whose vertices have degree 4 is called a ternary tree. Let $\Sigma_3$ be a ternary tree properly embedded in the plane. A connected component of the complement of $\Sigma_3$ is called a *face*. Let $V(\Sigma_3)$ denote the set of vertices of $\Sigma_3$.

We color the faces of $\Sigma_3$ by black and white such that they make a noncommutative infinite checkerboard, that is, any two adjacent faces of $\Sigma_3$ have different colors. Two black faces and two white faces meet at each vertex of $\Sigma_3$, every edge of $\Sigma_3$ is between a black face and a white one. For a vertex $v \in V(\Sigma_3)$ we label the two white and two black faces around $v$ by solutions $x_1$, $x_2$ and $y_1$, $y_2$ of (2), respectively, as in the left of Figure 7. (The black faces are dotted ones in the figure. Here we do not need that the solutions are integers.) Then it is said that $v$ satisfies the *vertex relation* (2) and $v$ is denoted by $(x_1, x_2; y_1, y_2)$, where we do not consider the order of $x_1, x_2$ and $y_1, y_2$.

A vertex $v$ has four edges. In Figure 7, we call a vertex reached by following the edge between the faces $x_1$ and $y_1$ (or $x_2$ and $y_2$) the *up* (or *down*) vertex of $v$; a vertex reached by following the edge between $x_1$ and $y_2$ (or $x_2$ and $y_1$) the *right* (or *left*) vertex of $v$. (They are in Figure 7 denoted by $v_u$, $v_d$, $v_r$, and $v_l$, respectively. Going down from $v_u$ to $v_d$, if turning to the right, we arrive at $v_r$, if turning to the left, we arrive at $v_l$. That is the reason of naming.) These vertices are called the *neighbor* vertices of $v$.

Let us consider the left vertex $v_l$ of $v$. Labeling new faces of $v_l$ (i.e., faces neither $x_2$ nor $y_1$) by $x'$ and $y'$, the vertex $v_l$ is represented as $(x_2, x'; y_1, y')$ (see the right of Figure 7). Note that a white face is labeled by $x'$ and a black one is by $y'$. If both $v$ and $v_l$ satisfy the vertex relation (2), $x' \neq x_1$, and $y' \neq y_2$, then we get:

**Lemma 5.1.** *Two vertices* $(x_1, x_2; y_1, y_2)$ *and* $(x_2, x'; y_1, y')$ *satisfy* $y_2 + y' = 4x_2y_1$.

*Proof.* By the vertex relation of $(x_1, x_2; y_1, y_2)$, we have $x_1+x_2 = 2y_1y_2$ and $2x_1x_2 = y_1^2+y_2^2$, by that of $(x_2, x'; y_1, y')$, $x_2 + x' = 2y_1y'$ and $2x_2x' = y_1^2 + (y')^2$. The aimed relation immediately follows from these equations. □

We call $y_2 + y' = 4x_2y_1$ the *relation of the edge* between $y_1$ and $x_2$ (or simply the *edge relation*). The representation $x' = 2y_1(4x_2y_1 - y_2) - x_2$ follows from this and the first equation of (2). We thus conclude that, if two neighbor vertices $v$ and $v_l$ satisfy the vertex relation (2), $v_l$ is written by $(x_2, 2y_1(4x_2y_1 - y_2) - x_2; y_1, 4x_2y_1 - y_2)$.

When $v$ and the right (up or down) vertex of $v$ satisfies the vertex relation (2), we obtain from the same argumentation the following result:

- For $v = (x_1, x_2; y_1, y_2)$ and the right vertex $v_r = (x_1, x'; y_2, y')$, the relation of the edge between $x_1$ and $y_2$ is $y_1 + y' = 4x_1y_2$ and $v_r$ is written by $(x_1, 2y_2(4x_1y_2 - y_1) - x_1; y_2, 4x_1y_2 - y_1)$.
- For $v$ and the up vertex $v_u = (x_1, x'; y_1, y')$, the relation of the edge between $x_1$ and $y_1$ is $y_2 + y' = 4x_1y_1$ and $v_u$ is written by $(x_1, 2y_1(4x_1y_1 - y_2) - x_1; y_1, 4x_1y_1 - y_2)$.
- For $v$ and the down vertex $v_d = (x_2, x'; y_2, y')$, the relation of the edge between $x_2$ and $y_2$ is $y_1 + y' = 4x_2y_2$ and $v_d$ is written by $(x_2, 2y_2(4x_2y_2 - y_1) - x_2; y_2, 4x_2y_2 - y_1)$.

Using these edge relations inductively, we can build from a vertex satisfying the vertex relation (2) a labeled ternary tree $\Sigma_3$ all of whose vertices satisfy the relation (2). We call this a *two-color Markoff graph* of numbers. If we take $(x_1, x_2; y_1, y_2) = (1, 1; 1, 1)$ as an initial quadruple, we obtain the two-color Markoff graph of numbers all of whose faces are labeled by positive integers (see Figure 8).

For a ternary tree $\Sigma_3$ embedded in the plane, we now think of directed edges so that for each vertex one edge is entering and three edges are going out. For a vertex $v \in V(\Sigma_3)$ we label four faces around $v$ by $a, b, c, d$ as in the left of Figure 9. In the same way as above, a vertex reached by following the edge between the faces $a$ and $b$ is called the up vertex; a vertex reached by following the edge between $b$ and $c$ ($c$ and $d$, or $d$ and $a$ ) is called the right (down or left) vertex. (They are in Figure 9 denoted by $v_u$, $v_r$, $v_d$, and $v_l$, respectively.) Suppose that faces of $\Sigma_3$ are colored by black and white so that any two adjacent faces have different colors. If faces $a, b, c, d$ are black, white, black, white, then $v$ is called *type I*. If faces $a, b, c, d$ are white, black, white, black, then $v$ is called *type II* (see Figure 9). If matrices correspond to faces around a vertex $v$ of type $I$ as in the center of Figure 9, that is, $\Lambda$ and $M$ correspond to the faces $b$ and $a$, the product $\Lambda M$ of them corresponds to the face $c$, and the product $\Lambda M^2$ of the matrices of the faces $c$ and $a$ corresponds to the face $d$, then it is said that a quadruple of matrices $(\Lambda, \Lambda M^2; M, \Lambda M)$ *corresponds* to the vertex $v$ of type $I$ (or simply $v$ of type $I$ *has* $(\Lambda, \Lambda M^2; M, \Lambda M)$). Note that a quadruple consists of ordered pairs of two matrices of white faces and those of black faces following the going down direction of edges. In the same way, if matrices correspond to faces around a vertex $v$ of type $II$ as in the right of Figure 9, then it is said that a quadruple of matrices

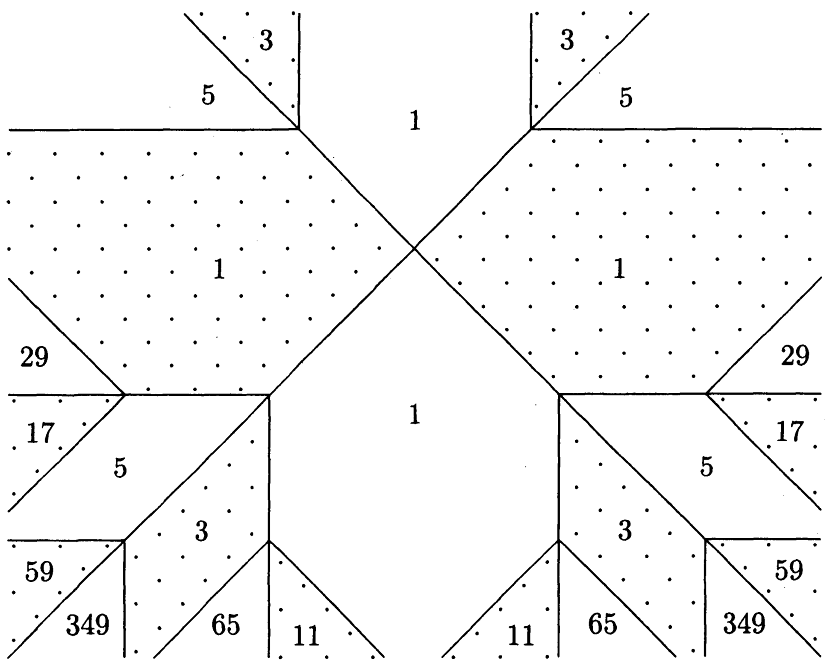


Figure 8: The two-color Markoff graph of numbers built from a vertex $(1,1;1,1)$. We can find horizontal and vertical axes of symmetry.

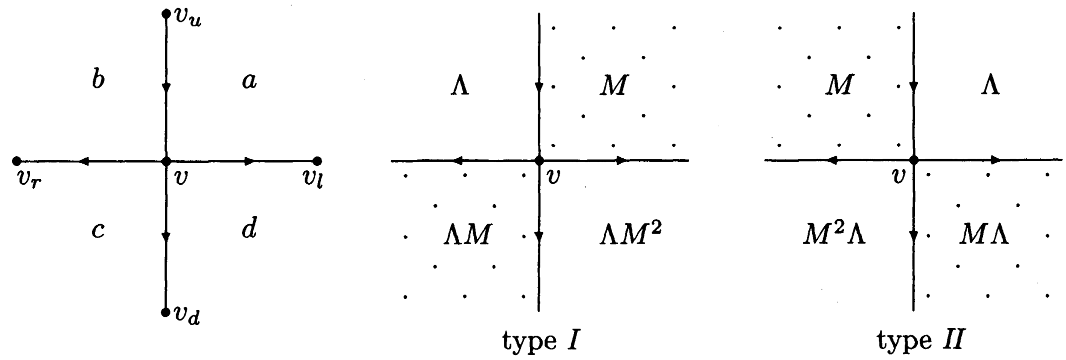


Figure 9: Labeling of the faces around a vertex and types of vertices

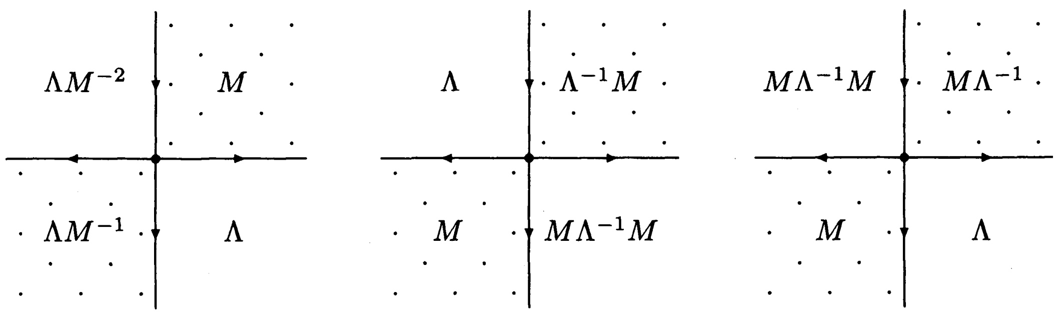


Figure 10: Quadruple of matrices corresponding to a vertex of type $I$

$(\Lambda, M^2\Lambda; M, M\Lambda)$ *corresponds* to the vertex $v$ of type $II$. In both cases of vertices of type $I$ and of type $II$, if we know two matrices corresponding to two adjacent faces, we can obtain from definition the matrices corresponding to the other faces. (Suppose that we know $\Lambda$ and $M$ in a vertex of type $I$. Figure 10 shows the other three cases than the center of Figure 9.)

Given a vertex $v$ of type $I$ or of type $II$ with a quadruple of matrices, we get quadruples of matrices and types corresponding to the four neighbor vertices of $v$. Except for the up vertex, quadruples of matrices corresponding to neighbor vertices are uniquely determined. For example, if the vertex $v$ is of type $I$ and has $(\Lambda_1, \Lambda_2; M_1, M_2)$, then the right vertex is of type $I$ and has $(\Lambda_1, \Lambda_1 M_2^2; M_2, \Lambda_1 M_2)$, the down vertex is of type $II$ and has $(\Lambda_2, M_2^2\Lambda_2; M_2, M_2\Lambda_2)$, and the left vertex is of type $I$ and has $(\Lambda_2, \Lambda_2 M_1^2; M_1, \Lambda_2 M_1)$. (Note that the notation $\Lambda_1$, $\Lambda_2$, $M_1$, $M_2$ from here to the end of a proof of Lemma 5.2 is different from that of Theorem 3.5.) If the vertex $v$ is of type $II$ and has $(\Lambda_1, \Lambda_2; M_1, M_2)$, we can write down similarly. In this case, the right vertex is of type $II$, the down vertex is of type $I$, and the left vertex is of type $II$.

There are three possibilities of quadruple of matrices corresponding to the up vertex of $v$. If $v$ is of type $I$ and has $(\Lambda_1, \Lambda_2; M_1, M_2)$, one of the followings takes place:

(i) If the up vertex is of type $I$ and $v$ is its right vertex, $(\Lambda_1, M_1\Lambda_1^{-1}M_1; \Lambda_1^{-1}M_1, M_1)$ corresponds to the up vertex.

(ii) If the up vertex is of type $I$ and $v$ is its left vertex, $(\Lambda_1 M_1^{-2}, \Lambda_1; M_1, \Lambda_1 M_1^{-1})$ corresponds to the up vertex.

(iii) If the up vertex is of type $II$ and $v$ is its down vertex, $(M_1\Lambda_1^{-1}M_1, \Lambda_1; \Lambda_1 M_1^{-1}, M_1)$ corresponds to the up vertex.

If $v$ is of type $II$ and has $(\Lambda_1, \Lambda_2; M_1, M_2)$, we can write down similarly.

Using this operation inductively, we can build from a vertex $v$ with type and a quadruple of matrices a ternary tree $\Sigma_3$ all of whose vertices have corresponding quadruples of matrices. We call this a *two-color Markoff graph of matrices.*

There exits an isomorphism between two-color Markoff graphs of matrices and numbers.

**Lemma 5.2.** *Suppose that a quadruple of matrices $(\Lambda_1, \Lambda_2; M_1, M_2)$ corresponds to a vertex $v \in V(\Sigma_3)$ and a quadruple of matrices $(\Lambda_1', \Lambda_2'; M_1', M_2')$ corresponds to a neighbor vertex $v'$ of $v$. If $(x_1, x_2; y_1, y_2)$ defined by $\mathrm{tr}(\Lambda_1) = 4x_1$, $\mathrm{tr}(\Lambda_2) = 4x_2$, $\mathrm{tr}(M_1) = 2\sqrt{2}y_1$, and $\mathrm{tr}(M_2) = 2\sqrt{2}y_2$ satisfies the vertex relation (2), then $(x_1', x_2'; y_1', y_2')$ defined by $\mathrm{tr}(\Lambda_1') = 4x_1'$, $\mathrm{tr}(\Lambda_2') = 4x_2'$, $\mathrm{tr}(M_1') = 2\sqrt{2}y_1'$, and $\mathrm{tr}(M_2') = 2\sqrt{2}y_2'$ also satisfies (2).*

*Proof.* We show the case that $v$ is of type $I$. In the same way, we can prove the case that $v$ is of type $II$.

If $v$ is of type $I$, we can write $M_2 = \Lambda_1 M_1$ and $\Lambda_2 = \Lambda_1 M_1^2$. If $v'$ is the right vertex of $v$, then $v'$ is of type $I$ and $(\Lambda_1', \Lambda_2'; M_1', M_2') = (\Lambda_1, \Lambda_1 M_2^2; M_2, \Lambda_1 M_2)$. We immediately get $x_1' = x_1$ and $y_1' = y_2$. The relation $y_2' = 4x_1y_2 - y_1$ of the edge between $x_1$ and $y_2$ is obtained from

$$\begin{aligned} 2\sqrt{2}y_2' &= \mathrm{tr}(\Lambda_1 M_2) = \mathrm{tr}(\Lambda_1)\mathrm{tr}(M_2) - \mathrm{tr}(\Lambda_1 M_2^{-1}) = \mathrm{tr}(\Lambda_1)\mathrm{tr}(M_2) - \mathrm{tr}(M_1^{-1}) \\ &= 4x_1 \cdot 2\sqrt{2}y_2 - 2\sqrt{2}y_1 = 2\sqrt{2}(4x_1y_2 - y_1). \end{aligned}$$

We get $x_2' = 2y_2(4x_1y_2 - y_1) - x_1$ by

$$\begin{aligned} 4x_2' &= \mathrm{tr}(\Lambda_1 M_2^2) = \mathrm{tr}(\Lambda_1 M_2)\mathrm{tr}(M_2) - \mathrm{tr}(\Lambda_1) \\ &= 2\sqrt{2}(4x_1y_2 - y_1) \cdot 2\sqrt{2}y_2 - 4x_1 = 4(2y_2(4x_1y_2 - y_1) - x_1). \end{aligned}$$

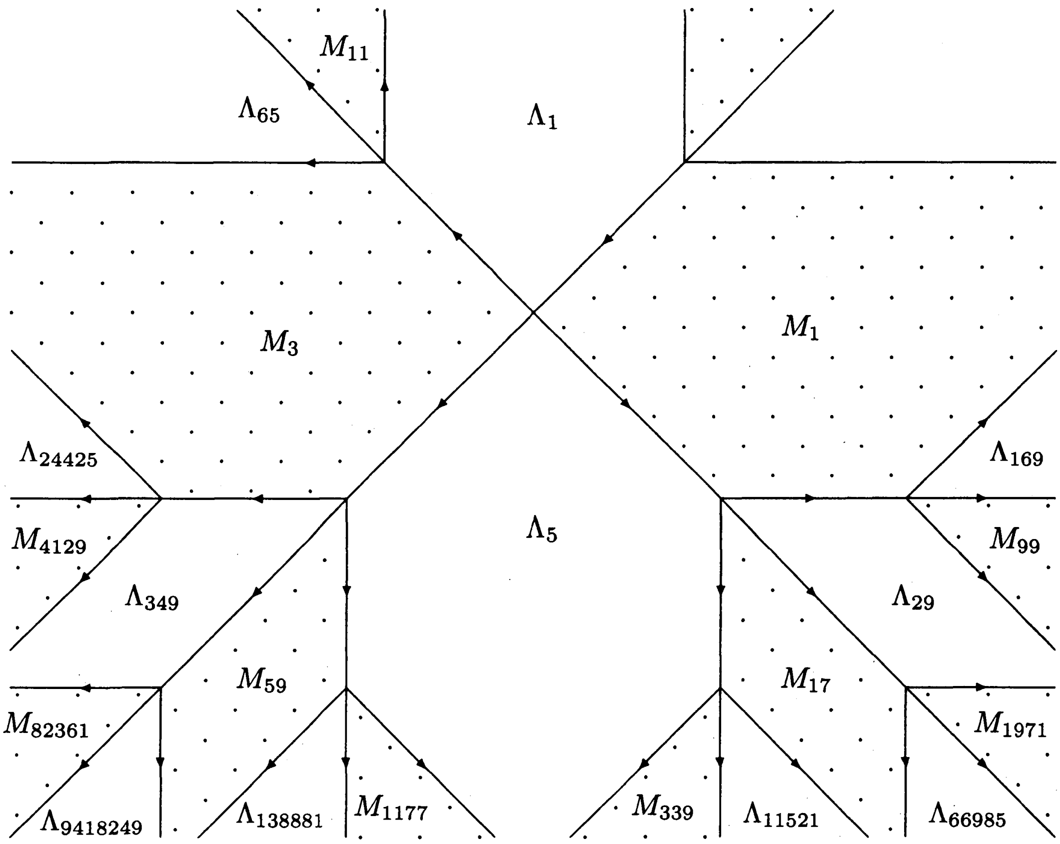


Figure 11: The two-color Markoff graph of matrices built from a vertex $(\Lambda_1, \Lambda_5; M_1, M_3)$ of type $I$. Edges without a pile are edges whose direction are not uniquely determined.

The quadruple $(x_1', x_2'; y_1', y_2')$ indeed satisfies the vertex relation (2).

We can prove in the same way the case that $v'$ is the down and left vertex of $v$.

Suppose that $v'$ is the up vertex of $v$. Let us consider case (i) mentioned above. Since $(\Lambda_1', \Lambda_2'; M_1', M_2') = (\Lambda_1, M_1\Lambda_1^{-1}M_1; \Lambda_1^{-1}M_1, M_1)$, we immediately get $x_1' = x_1$ and $y_2' = y_1$. The relation $y_1' = 4x_1y_1 - y_2$ of the edge between $x_1$ and $y_1$ is obtained from

$$\begin{aligned} 2\sqrt{2}y_1' &= \mathrm{tr}(\Lambda_1^{-1}M_1) = \mathrm{tr}(\Lambda_1^{-1})\mathrm{tr}(M_1) - \mathrm{tr}(\Lambda_1^{-1}M_1^{-1}) = \mathrm{tr}(\Lambda_1)\mathrm{tr}(M_1) - \mathrm{tr}(M_2) \\ &= 4x_1 \cdot 2\sqrt{2}y_1 - 2\sqrt{2}y_2 = 2\sqrt{2}(4x_1y_1 - y_2). \end{aligned}$$

We get $x_2' = 2y_1(4x_1y_1 - y_2) - x_1$ by

$$\begin{aligned} 4x_2' &= \mathrm{tr}(M_1\Lambda_1^{-1}M_1) = \mathrm{tr}(M_1\Lambda_1^{-1})\mathrm{tr}(M_1) - \mathrm{tr}(M_1\Lambda_1^{-1}M_1^{-1}) \\ &= \mathrm{tr}(M_1\Lambda_1^{-1})\mathrm{tr}(M_1) - \mathrm{tr}(\Lambda_1) = 4(2y_1(4x_1y_1 - y_2) - x_1). \end{aligned}$$

The quadruple $(x_1', x_2'; y_1', y_2')$ also satisfies the vertex relation (2).

We can prove cases (ii) and (iii) in the same way. $\square$

Let us consider the two-color Markoff graph of numbers building from a vertex $(1, 5; 1, 3)$. The set of integers occurring in the white and black faces is equal to $\mathcal{N}(\Lambda)$ and $\mathcal{N}(M)$, respectively. We define the initial matrices $M_1$ and $\Lambda_1$ using elements of $\Gamma_4$:

$$M_1 = B_4 = \begin{pmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{pmatrix} \text{ and } \Lambda_1 = B_4A_4 = \begin{pmatrix} 3 & \sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix}. \tag{6}$$

Let us consider the two-color Markoff graph of matrices building from a vertex $(\Lambda_1, \Lambda_5 = M_3M_1; M_1, M_3 = \Lambda_1 M_1)$ of type $I$ (see Figure 11). Since

$$M_3 = \begin{pmatrix} 4\sqrt{2} & 5 \\ 3 & 2\sqrt{2} \end{pmatrix} \text{ and } \Lambda_5 = \begin{pmatrix} 13 & 9\sqrt{2} \\ 5\sqrt{2} & 7 \end{pmatrix},$$

we get $(\mathrm{tr}(\Lambda_1), \mathrm{tr}(\Lambda_5); \mathrm{tr}(M_1), \mathrm{tr}(M_3)) = (4, 20; 2\sqrt{2}, 6\sqrt{2})$. Hence, in the sense of Lemma 5.2, the two-color Markoff graph of numbers building from $(1, 5; 1, 3)$ (Figure 8) is isomorphic to the two-color Markoff graph of matrices building from $(\Lambda_1, \Lambda_5; M_1, M_3)$ of type $I$ (Figure 11).

By the definition of the initial matrices and the construction of the graph, we can prove that the matrices occurring in the two-color Markoff graph are primitive in $\Gamma_4$. Moreover, matrices $\Lambda_\lambda$ corresponding to $\lambda \in \mathcal{N}(\Lambda)$ and $M_m$ to $m \in \mathcal{N}(M)$ are described explicitly as in Theorem 3.5, which is proved in the same way as Theorem 3.2.

A proof of a similar theorem for the conjugate of $\Lambda_\lambda$ and $M_m$ by $V' = \begin{pmatrix} \sqrt{2} & -1 \\ 0 & 1 \end{pmatrix}$ was done in [1], where we used the Vulakh-Schmidt tree, instead of the two-color Markoff graph, to define quadruple of matrices. The two-color Markoff graph of matrices describes naturally and clearly the definition of matrices $\Lambda_\lambda$, $M_m$ and makes the proof of Theorem 3.5 simpler.

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