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# Non-hyperbolic automatic groups and groups acting on CAT(0) cube complexes

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## 1. Introduction

This note is a brief summary of my talk that I gave at RIMS workshop "Complex Analysis and Topology of Discrete Groups and Hyperbolic Spaces." See [4] for detail.

If a group $G$ has a finite $K(G,1)$ and does not contain any Baumslag–Solitar groups, is $G$ hyperbolic? (See [1].) This is one of the most famous questions on hyperbolic groups. Probably, many people expect that the answer is negative, and it would be better to restrict our attention to some good class of groups. In this talk, we consider automatic groups. If an automatic group $G$ does not contain any $\mathbb{Z}+\mathbb{Z}$ subgroups, is $G$ hyperbolic? Our problem is listed in [5] and attributed to Gersten. Note that, if the group is the fundamental group of a closed 3-manifold, our question corresponds to the so-called "weak hyperbolization" of 3-manifolds.

In this talk, we define the notion of "$n$-tracks of length $n$", which suggests a clue of the existence of $\mathbb{Z}+\mathbb{Z}$ subgroup, and show its existence in every non-hyperbolic automatic groups with mild conditions. As an application, we show that if a group acts freely, cellularly, properly discontinuously and cocompactly on a CAT(0) cube complex and its quotient is "weakly special", then the above question is answered affirmatively. See [4] for detail.

## 2. Automatic group

Let $G$ be a finitely generated group with a set of generators $A$. Let $w$ be a word over $A$. We denote by $w(t)$ the prefix of $w$ with length $t$. The image of $w$ in $G$ by the natural projection is denoted by $\overline{w}$. We denote by $\overline{w(t_1,t_2)}$ the subpath of the image of $w$ in the Cayley graph $\Gamma(G,A)$ from the vertex $\overline{w(t_1)}$ to the vertex $\overline{w(t_2)}$.

Now, we recall the concept of automatic structure. See [2] for detail. We denote by $\varepsilon$ the identity element of $G$. A special letter $\$ \notin A$ is used to define the automatic structure of the group. A finite state automaton $M$ over an alphabet $A$ is a machine that determines "accept" or "reject" for a given word over $A$. The language given by all the accepted words of a finite state automaton $M$ is denoted by $L(M)$.

**Definition 2.1.** *An automatic structure on $G$ consists of a finite state automaton $W$ over $A$ and finite state automata $M_x$ over $(A\cup\{\$\})\times(A\cup\{\$\})$, for $x\in A\cup\{\varepsilon\}$, satisfying the following conditions:*

(1) *The natural projection from $L(W)$ to $G$ is surjective.*

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(2) *For $x \in A \cup \{\varepsilon\}$, we have $(w, w') \in L(M_x)$ if and only if $\overline{w\,x} = \overline{w'}$ and both $w$ and $w'$ are elements of $L(W)$.*

$W$ is called a *word acceptor*, and each $M_x$ is called a *compare automaton* for the automatic structure. An automatic group is one that admits an automatic structure.

## 3. Existence of $n$-tracks in non-hyperbolic automatic groups

Let $G$ be an automatic group with automatic structure $(A, W, \{M_x\}_{x \in A \cup \{\varepsilon\}})$ where $A$ is the set of generators with $A^{-1} = A$, $W$ the word acceptor and $M_x$ the compare automaton for $x \in A \cup \{\varepsilon\}$. The following is the key concept in this talk.

**Definition 3.1.** *Let $T = \{t_1, t_2, \ldots, t_n\}$ be a set of mutually disjoint $n$ paths of length $n$ in $\Gamma$. We call $T$ $n$-tracks of length $n$ if there exist $2n$ words $w_1, w'_1, w_2, w'_2, \ldots, w_n, w'_n$ of $L(W)$ and a positive integer $r$ such that $(w'_i, w_{i+1})$ is accepted by some compare automaton for $i = 1, 2, \ldots, n-1$, and that $t_i = \overline{w_i(r, r+n)} = \overline{w'_i(r, r+n)}$ for $i = 1, 2, \ldots, n$. See Fig. 1.*

We show the existence of tracks in every non-hyperbolic automatic groups with mild conditions.

**Theorem 3.2.** *Let $G$ be a weakly geodesically automatic group whose automatic structure is prefix closed and has the uniqueness property. If $G$ is not hyperbolic, then it contains $n$-tracks of length $n$ for any $n > 0$.*

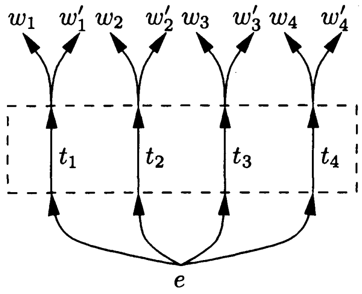


FIGURE 1. 4-track $T = \{t_1, t_2, t_3, t_4\}$ and its related paths

## 4. CAT(0) cube complexes

Does the existence of $n$-track of length $n$ for any $n$ imply the existence of $\mathbb{Z}+\mathbb{Z}$ subgroup? We do not have the complete answer. But, as an application of the theorem in the previous section, we give a partial answer to this question for the groups acting on CAT(0) cube complexes.

4.1. **Definitions.** In this subsection, we briefly review the notion of CAT(0) cube complex.

An $n$-cube is a copy of $[-1, 1]^n$. A cube complex is obtained from a collection of cubes of various dimensions by identifying certain subcubes. A flag complex is a simplicial complex with the property that every finite set of pairwise adjacent vertices spans a simplex. Let $X$ be a cube complex. The link of a vertex $v$ in $X$ is a complex built from simplices corresponding to the corners of cubes adjacent to $v$.

**Definition 4.1.** *A cube complex $X$ is nonpositively curved if, for each vertex $v$ in $X$, link$(v)$ is a flag complex.*

Gromov showed that a cube complex is CAT(0) if and only if it is simply connected and nonpositively curved. Many groups studied in combinatorial group theory act properly and cocompactly on CAT(0) cube complexes.

Let us recall the definition of hyperplane for cube complex. A midplane in a cube $[-1,1]^n$ is the subspace obtained by restricting exactly one coordinate to 0. Given an edge in a cube, there is a unique midplane which cuts the edge transversely. A hyperplane $H$ of a cube complex $X$ is obtained by developing the midplanes in $X$, i.e., identifying common subcubes of midplanes which cuts the same edge. These edges are said to be dual to $H$.

Let $X$ be a CAT(0) cube complex, and $V(X)$ its vertex set. Let $G$ be a group acting freely, cellularly, properly discontinuously and cocompactly on $X$. Let $G\backslash X$ denote the quotient of the complex $X$ by the action of $G$. The fundamental groupoid $\pi(G\backslash X)$ is the groupoid whose objects are the points of $G\backslash X$ and morphisms between points $v, v'$ are homotopy classes of paths in $G\backslash X$ beginning at $v$ and ending at $v'$. The multiplication in $\pi(G\backslash X)$ is induced by composition of paths.

A directed cube is a cube with two ordered diagonally opposite vertices specified. Let $A$ be the set of homotopy classes of the diagonal of all directed cubes in $G\backslash X$. The correspondence between $A$ and directed cubes in $G\backslash X$ is one to one. The directed cubes in $X$ can be labelled equivariantly by (the lifts of) $A$, so each cube-path in X defines a word in $A^*$. Let $\mathcal{L}$ be the subset of $A^*$ which corresponds to normal cube-paths.

**Lemma 4.2.** *Let $A$ and $\mathcal{L}$ be as above. Then we have:*

(1) *There exists an isometry between $\pi(G\backslash X)$ with the word metric given by $A$ and $V(X)$ with the metric given by normal cube paths. (Lemma 4.1 in* [6]*)*
(2) *$\mathcal{L}$ is regular over $A$. (Proposition 5.1 in* [6]*)*
(3) *$\mathcal{L}$ satisfies 1-fellow travel property. (Proposition 5.2 in* [6]*)*

*In particular, $(A, L)$ induces an automatic structure for $\pi(G\backslash X)$. (See Theorem 5.3 in* [6]*) This structure is prefix closed, weakly geodesically automatic with uniqueness property.*

The set of states of (non-deterministic) finite-state automaton for $\mathcal{L}$ is $A$. (Proposition 5.1 in [6]) Thus, There is a natural map from the set of states of the word acceptor of $\pi(G\backslash X)$ to $G\backslash X$ by taking the tail of directed cubes.

Let $v$ be a vertex in $G\backslash X$. The group $G$ is realized as a subgroupoid $\pi(G\backslash X, \{v\})$ whose object is $v$ only, and whose morphisms are all the morphisms of $\pi(G\backslash X)$ between $v$. It is easy to construct an automatic structure for the group $G = \pi(G\backslash X, \{v\})$ from the automatic structure for the groupoid $\pi(G\backslash X)$.

4.2. **Groups acting on CAT(0) cube complexes.** Let $G$ be a group acting freely, cellularly, properly discontinuously and cocompactly on a CAT(0) cube complex $X$.

Let $\mathcal{M}$ be the standard automaton for the automatic structure of the groupoid $\pi(G\backslash X)$ given in 4.1.

We use the same symbols as in the previous subsection. Let $(s, t, g)$ be a state in $\mathcal{M}$. Since $\mathcal{L}$ (the set of words corresponding to normal cube-paths) satisfies 1-fellow travel property, $g$ is in $A$ (the set of generators). Recall that $A$ consists of directed cubes in $G\backslash X$. We define the dimension the the state $(s, t, g)$, denoted by $\dim(s, t, g)$, as the dimension of $g$ as a (directed) cube. We also define dim(failure state $F$) $= +\infty$.

Let us introduce some notation. (See [3] for more details.) Let $\vec{a}, \vec{b}$ be oriented edges having a common initial (or terminal) vertex $v$. Oriented edges $\vec{a}$ and $\vec{b}$ are said to directly osculate at $v$ if they are not adjacent in $\mathrm{link}(v)$. Let $a, b$ be (unoriented) edges having a common end point $v$. Edges $a$ and $b$ are said to osculate at $v$ if they are not adjacent in $\mathrm{link}(v)$.

We consider hyperplanes in $G\backslash X$. From now on, we assume that each hyperplane in $G\backslash X$ is embedding.

A hyperplane $H$ is said to be 2-sided if its open cubical neighborhood is isomorphic to the product $H \times (-1, 1)$. If a hyperplane is not 2-sided, then it is said to be 1-sided. If $H$ is 2-sided, one can orient dual edges in a consistent way. A 2-sided hyperplane is said to directly self-osculate if it is dual to distinct oriented edges that directly-osculate. We say that 1-sided hyperplane self-osculates if it is dual to distinct (unoriented) edges that osculate.

We introduce the following notion:

**Definition 4.3.** *We say that a 2-sided hyperplane $H$ self-contacts if there are two vertices $u, v$ such that $d(u, v) = 1$ and $H$ directly self-osculates at $u$ and $v$. We say that a 1-sided hyperplane $H$ self-contacts if there are two vertices $u, v$ such that $d(u, v) = 1$ and $H$ self-osculates at $u$ and $v$.*

**Remark 4.4.** *By definition, if a cube complex is special in the sense of* [3], *then each hyperplane embeds, and it has no hyperplane of self-contact,*

This is our main theorem in this section.

**Theorem 4.5.** *Let $G$ be a group acting freely, cellularly, properly discontinuously and cocompactly on a CAT(0) cube complex $X$. If each hyperplane in $G\backslash X$ is embedding and does not self-contact and $G$ is not word hyperbolic, then, $G$ contains $\mathbb{Z} + \mathbb{Z}$ subgroup.*

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