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VISUALISATION OF STABILITY REGIONS FOR LOGISTIC DIFFERENCE EQUATIONS WITH MULTIPLE DELAYS

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ABSTRACT. We consider linearised stability of nonlinear difference equations by investigating location of roots of the associated characteristic equations. The characteristic equation is given as a polynomial equation with the order determined by the delay in the difference equation. For some polynomial equations we visualise all the sets of coefficients such that all the roots locate inside the unit circle in the complex plane. The information can be translated to understand stability of the nonlinear difference equation in terms of its original parameters. We present some examples that delay in the difference equation can stabilise the equilibrium of the equation.

1. INTRODUCTION: DELAY IN DIFFERENCE EQUATIONS

Let f be a mapping from $\mathbb{R}$ to $\mathbb{R}$. In the paper [12] the authors consider a difference equation with one single delay in a form

$$x_{n+1} = x_n f(x_{n-k}),\ n \in \mathbb{N}_+, \tag{1.1}$$

where $k \geq 1$ is a positive integer. Initial conditions are given as a sequence

$$(x_0, x_{-1}, \dots, x_{-k}) = (p_0, p_1 \dots, p_k) \in \mathbb{R}^{k+1}.$$

One can then compute $x_1 = p_0 f(p_k)$ and recursively construct the solution (as long as the solution exists in the domain of f). The equilibrium of (1.1) is given as a root of the equation

$$1 = f(x).$$

We denote the equilibrium by x^* assuming that it exists. If f is a differentiable function (at least around the equilibrium), one can linearise equation (1.1) around the equilibrium x^* to get a linear difference equation:

$$y_{n+1} = f(x^*) y_n + x^* f'(x^*) y_{n-k} = y_n + x^* f'(x^*) y_{n-k}. \tag{1.2}$$

The linearised equation (1.2) leads to the following characteristic equation:

$$\lambda^{k+1} = \lambda^k + x^* f'(x^*), \tag{1.3}$$

which is a polynomial equation of the $(k+1)$-th order. It is known that the equilibrium is asymptotically stable if all the roots of the equation (1.3) locate inside the unit circle in the complex plane (i.e., all the zeros of (1.3) have their magnitude less than one). We refer [6, 10] as general references for the stability theory of difference equations.

To be concrete, let us set

$$f(x) = \exp\{r(1-x)\},\ x \in \mathbb{R}_+,$$

where $r > 0$. Then equation (1.1) becomes the logistic equation

$$x_{n+1} = x_n \exp\{r(1 - x_{n-k})\}. \tag{1.4}$$

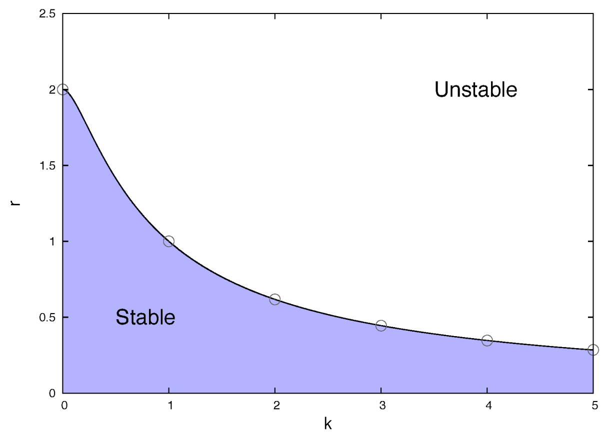


FIGURE 1.1. Stability region for the equilibrium of (1.4). A stable equilibrium can become unstable as k increases.

We refer [10, 12, 2] for the biological motivation for the difference equation (1.4). One can easily see that the equation (1.4) has the equilibrium

$$x^* = 1$$

and that linearisation leads to the following characteristic equation:

$$\lambda^{k+1} = \lambda^k - r.$$

In [12] it is shown that the equilibrium is asymptotically stable if

$$r < 2\cos\frac{k\pi}{(2k+1)}$$

and it is unstable if the converse inequality holds. Since the stable equilibrium becomes unstable as the parameter of delay k increases, see Figure 1.1, the result derives cliché that delayed negative feedback induces instability.

Does the delay *always* destabilise difference equations? Let us consider the following difference equation

$$x_{n+1} = x_n \exp\left[r\{1 - \alpha x_n - (1-\alpha)x_{n-1}\}\right], \tag{1.5}$$

where $\alpha \in [0,1]$. Stability of the equilibrium of (1.5) was studied in [14, 2]. Here, using (1.5) as an example, we would like to illustrate our approach for stability analysis of non-linear difference equation. Step of the analysis is analogue to the one proposed in [4, 3], where the authors study transcendental equations which are derived from continuous delay equations describing population dynamics.

For (1.5) the characteristic equation, associated to the equilibrium $x^* = 1$, becomes a quadratic polynomial equation:

$$\lambda^2 + a\lambda + b = 0 \tag{1.6}$$

where

$$a = r\alpha - 1, \tag{1.7a}$$

$$b = r(1-\alpha). \tag{1.7b}$$

Let us focus on the equation (1.6) in the "abstract" form and, for a moment, we forget the specification of coefficients (a,b) in terms of the original parameters (r,α) given by (1.7). Our aim is to obtain all the sets of (a,b) (in $\mathbb{R}^2$) such that all the roots locate inside the unit circle in the complex plane, which is exactly the *stability region* in the (a,b)-parameter plane.

Notice that there exists a root with $|\lambda| = 1$ at the boundary of the stability region, due to the continuity of the roots with respect to the coefficients (a,b). One can immediately see that if

$$1+a+b=0 \tag{1.8}$$

holds, then equation (1.6) has a root $\lambda = 1$ while if

$$1-a+b=0 \tag{1.9}$$

holds, then (1.6) has a root $\lambda = -1$. Those conditions can be visualised as two lines in the (a,b)-parameter plane that form a part of stability boundaries in the (a,b)-parameter plane, see Figure 1.2 (a).

Equation (1.6) of course could have a conjugate pair of complex roots with $|\lambda| = 1$. Suppose that $\lambda = e^{i\omega} = \cos\omega + i\sin\omega$, $\omega \in (0,\pi)$ solves equation (1.6). We get the following two equations

$$0 = \cos 2\omega + a\cos\omega + b, \tag{1.10a}$$

$$0 = \sin 2\omega + a\sin\omega. \tag{1.10b}$$

These equations (1.10) can be easily solved with respect to (a,b) as

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -2\cos\omega \\ 1 \end{pmatrix}, \quad \omega \in (0,\pi). \tag{1.11}$$

The equality (1.11) defines a parametric line by ω in the (a,b)-plane, which one can easily plot. Along the parametric line given by (1.11), the characteristic equation (1.6) has a root $\lambda = e^{i\omega}$ (and also $\lambda = e^{-i\omega}$), see again Figure 1.2 (a).

Now the (a,b)-parameter plane is decomposed into some regions by those three lines, where the characteristic equation (1.6) has a root with $|\lambda| = 1$. To determine the stability region, in each point in the (a,b)-parameter plane, one should count the number of roots that locate inside/outside the unit circle in $\mathbb{C}$. In general this can be done by applying Rouché's theorem as in [5]. For equation (1.6), however, one can easily verify that the number locating outside the unit circle in $\mathbb{C}$ as in Figure 1.2 (a) by elementary calculations. See also Theorem 1.3.4 in Chapter 1 in [10] for an explicit stability condition derived in a different way applying the Schur-Cohn criterion.

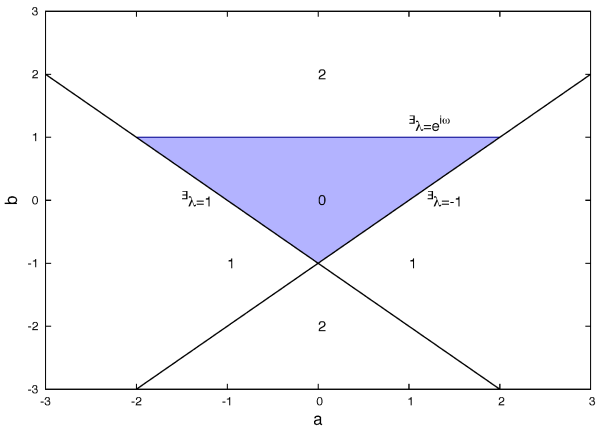
Finally let us interpret the stability region depicted in Figure 1.2 (a) in terms of the original parameters (r,α) in (1.5). According to (1.7) one can see that (1.8) amounts to that $r = 0$ holds, where the characteristic equation has a root $\lambda = 1$, and that (1.9) amounts to

$$r = \frac{2}{2\alpha - 1},$$

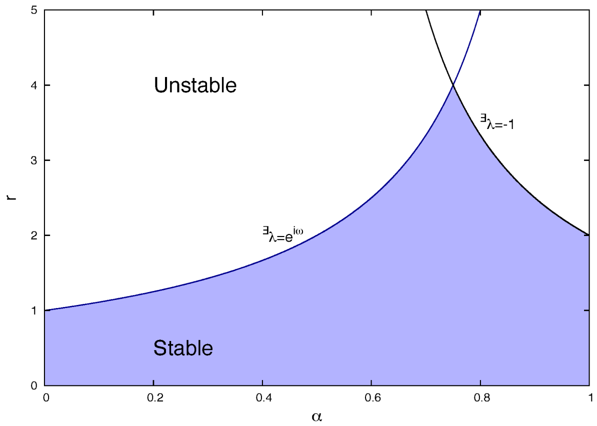
where the characteristic equation has a root $\lambda = -1$. The inverse mapping of (1.7) is

$$r = a+b+1,$$

$$\alpha = \frac{a+1}{a+b+1}.$$



(a) in the (a,b)-parameter plane



(b) in the (α,r)-parameter plane

FIGURE 1.2. Stability region for the characteristic equation (1.4). In (a) the number of roots that locate outside the unit circle in $\mathbb{C}$ is depicted.

Then one obtains the following curve corresponding to (1.11)

$$\begin{pmatrix} r \\ \alpha \end{pmatrix} = \begin{pmatrix} 2-2\cos\omega \\ \frac{1-2\cos\omega}{2-2\cos\omega} \end{pmatrix}, \quad \omega \in (0,\pi),$$

which can be explicitly expressed as

$$r = \frac{1}{1-\alpha}.$$

Therefore as in Figure 1.2 (b) we can deduce the stability region for the positive equilibrium of (1.5) in the (r,α)-parameter plane. Figure 1.2 (b) shows that stability threshold for r

changes non-monotonically with respect to α. Unstable equilibrium can become stable by reducing α from 1 up to $\frac{3}{4}$, showing that the term x_{n-1} contributes the stability! We note that the same figure actually appeared in [2].

2. General formulation

In this section we would like to generalise the idea described in the previous section. Our motivation comes from stability analysis of the logistic equation with multiple delays:

$$x_{n+1} = x_n \exp\left[r\left\{1 - \alpha x_n - \beta x_{n-j} - (1-\alpha-\beta)x_{n-k}\right\}\right], \tag{2.1}$$

where

$$r > 0,\ \alpha \geq 0,\ \beta \geq 0,\ \alpha+\beta \leq 1$$

and k and j are integers such that $k > j$.

One can easily see that equation (2.1) has the positive equilibrium $x^* = 1$. The characteristic equation for the positive equilibrium $x^* = 1$ is computed as

$$\lambda^{k+1} + a\lambda^k + b\lambda^{k-j} + c = 0, \tag{2.2}$$

where

$$a = r\alpha - 1, \tag{2.3a}$$

$$b = r\beta, \tag{2.3b}$$

$$c = r(1-\alpha-\beta). \tag{2.3c}$$

To see if all the roots of the polynomial equation (2.2) lie inside the unit circle in $\mathbb{C}$, one could apply the Schur-Cohn criterion [10]. The direct application however may require lengthy calculations to obtain concrete conditions in terms of parameters, which would be of interest in the context of applications to biological models.

The characteristic equation (2.2) has a quite general form including cases previously studied in the literature. For example, (2.2) with $b = 0$ (or $c = 0$) is studied in the famous paper [11]. Recently, in the paper [1] the authors formulated an explicit stability condition for (2.2) with $j = k-1$, applying the Schur–Cohn criterion. Proving contractivity of the solution for the *nonlinear* equation (2.1) involves a lot of computations and derives sufficient conditions for the global stability of the equilibrium, see [13, 16, 15].

Consider the polynomial equation (2.2) in the (a,b,c)-coefficient space (instead of the plane as in Section 1). We make drastic simplification by assuming that $j = 1$ holds, which avoids technical detail but shows interesting stability boundaries. Our first aim is to characterise the region

$$D := \left\{(a,b,c) \in \mathbb{R}^3 : \text{Every root of (2.2) locates inside the unit circle in } \mathbb{C}\right\}.$$

Define

$$L_1 := \left\{(a,b,c) \in \mathbb{R}^3 : \text{(2.2) has a root } \lambda\text{=1}\right\},$$

$$L_{-1} := \left\{(a,b,c) \in \mathbb{R}^3 : \text{(2.2) has a root } \lambda\text{=-1}\right\}.$$

One can immediately see that L_1 and L_{-1} respectively form planes:

$$L_1 = \left\{(a,b,c) \in \mathbb{R}^3 : 1+a+b+c = 0\right\},$$

$$L_{-1} = \left\{(a,b,c) \in \mathbb{R}^3 : (-1)^{k+1} + a(-1)^k + b(-1)^{k-1} + c = 0\right\}.$$

We are now interested in

$$C := \left\{(a,b,c) \in \mathbb{R}^3 : \text{(2.2) has a conjugate pair of complex roots with } |\lambda|\text{=1}\right\}.$$

Now let us introduce a sufficient condition for that every root locate "outside" the unit circle in $\mathbb{C}$, i.e., no roots in inside the unit circle in $\mathbb{C}$. The following result can be easily proven by using the fundamental theorem of algebra. We omit the proof.

Lemma 1. *Let us assume that $|c| \geq 1$ holds. Then equation (2.2) has a root with $|\lambda| \geq 1$ for any $(a,b) \in \mathbb{R}^2$.*

Thus, to find the stability region, we can restrict our attention to $c \in (-1,1)$. We introduce the following result.

Theorem 2. *Equation (2.2) has a conjugate pair of complex roots $\lambda = e^{\pm i\omega}$, $\omega \in (0,\pi)$ if and only if*

$$\begin{pmatrix} a \\ b \end{pmatrix} = -\frac{1}{\sin\omega}\begin{pmatrix} -c\sin((k-1)\omega)+\sin(2\omega) \\ c\sin(k\omega)-\sin\omega \end{pmatrix}, \quad \omega \in (0,\pi) \tag{2.4}$$

holds.

Proof. Let us substitute $\lambda = e^{i\omega}$, $\omega \in (0,\pi)$ into (2.2). We then get

$$\begin{aligned} 0 &= \cos((k+1)\omega) + a\cos(k\omega) + b\cos((k-1)\omega) + c, \\ 0 &= \sin((k+1)\omega) + a\sin(k\omega) + b\sin((k-1)\omega). \end{aligned}$$

Then one can get the conclusion by solving the two equations with respect to a and b. □

Then we have ingredients to plot the stability boundary for given k. We present L_1, L_{-1} and C for several $c \in [-1,1]$ fixing k, see Figure 2.1 where $k = 2$ and Figure 2.2 where $k = 4$. The colored region is the exact stability region, which can be detected by the application of the Rouché's theorem (we omit the detail, see again [5, 3] for the use of the Rouché's theorem to determine the stability region). In both Figures 2.1 and 2.2 one can see that the stability region disappears as c becomes ± 1, cf. Lemma 1. When $k = 4$, the characteristic equation has degree 5 and the stability boundary intersects itself, see the case that $c = \pm 0.8$ holds. The stability boundary that intersects itself seems to be one of the interesting properties of the difference equation with two delays, cf. [9]. One could of course visualise the stability region in the (a,b,c)-space as a three-dimensional object, which could also be informative to observe the full structure of the stability region.

What can we now say about stability of the equation (2.1)? To answer the question we interpret the obtained stability region in the (a,b,c)-space using the mapping (2.3) and form the stability region in the original parameter space: (r,α,β). Here we do not give a detailed analysis, but we would like to show that the stability threshold for r can be larger than the one obtained for the equation (2.1) in Section 1.

Let $k = 2$ and we simplify the equation (2.1) by assuming that

$$\beta = \frac{3}{4}(1-\alpha)$$

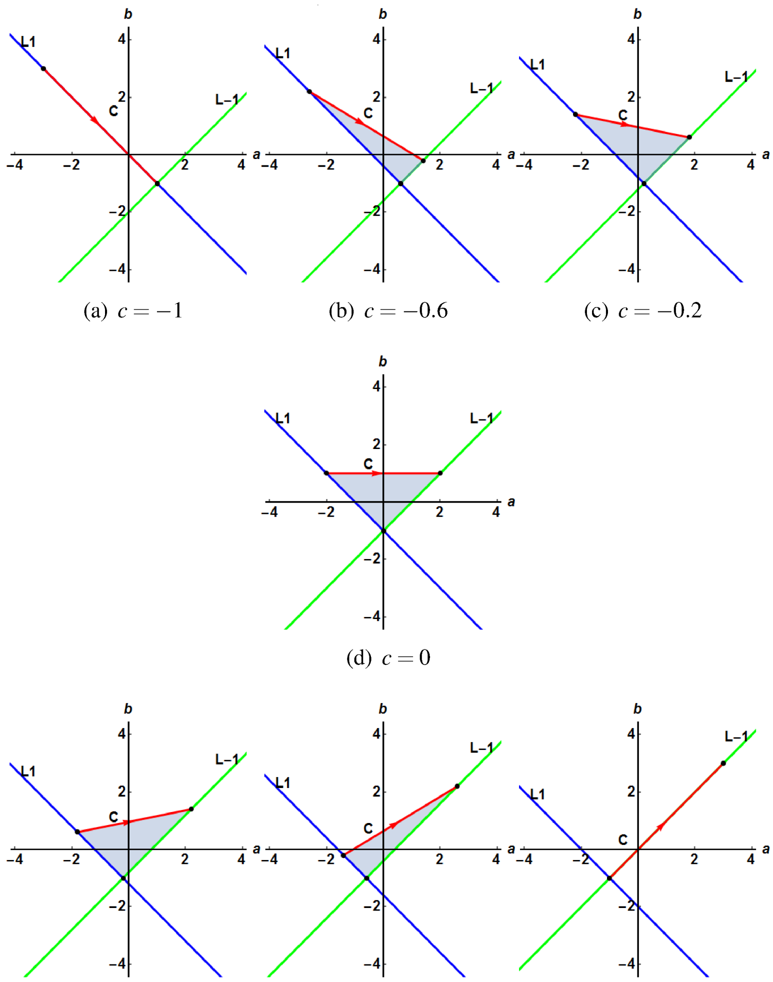
holds. Now equation (2.1) has two parameters: (r,α). The mapping (2.3) becomes

$$a = r\alpha - 1, \tag{2.5a}$$

$$b = \frac{3}{4}r(1-\alpha), \tag{2.5b}$$

$$c = \frac{1}{4}r(1-\alpha). \tag{2.5c}$$

Let us interpret Theorem 2 in the (α,r)-parameter plane.



(a) $c=-1$ (b) $c=-0.6$ (c) $c=-0.2$

(d) $c=0$

(e) $c=0.2$ (f) $c=0.6$ (g) $c=1$

FIGURE 2.1. Stability region for the characteristic equation (2.2) with $k=2$.

Corollary 3. *Equation (2.2) has a conjugate pair of complex roots $\lambda=e^{\pm i\omega}$, $\omega\in(0,\pi)$ if and only if*

$$\begin{pmatrix} r \\ \alpha \end{pmatrix}=\begin{pmatrix} 1-2\cos\omega+\frac{5}{3+2\cos\omega} \\ 1-\frac{4}{3+2\cos\omega}\left(1-2\cos\omega+\frac{5}{3+2\cos\omega}\right)^{-1} \end{pmatrix},\ \omega\in(0,\pi) \tag{2.6}$$

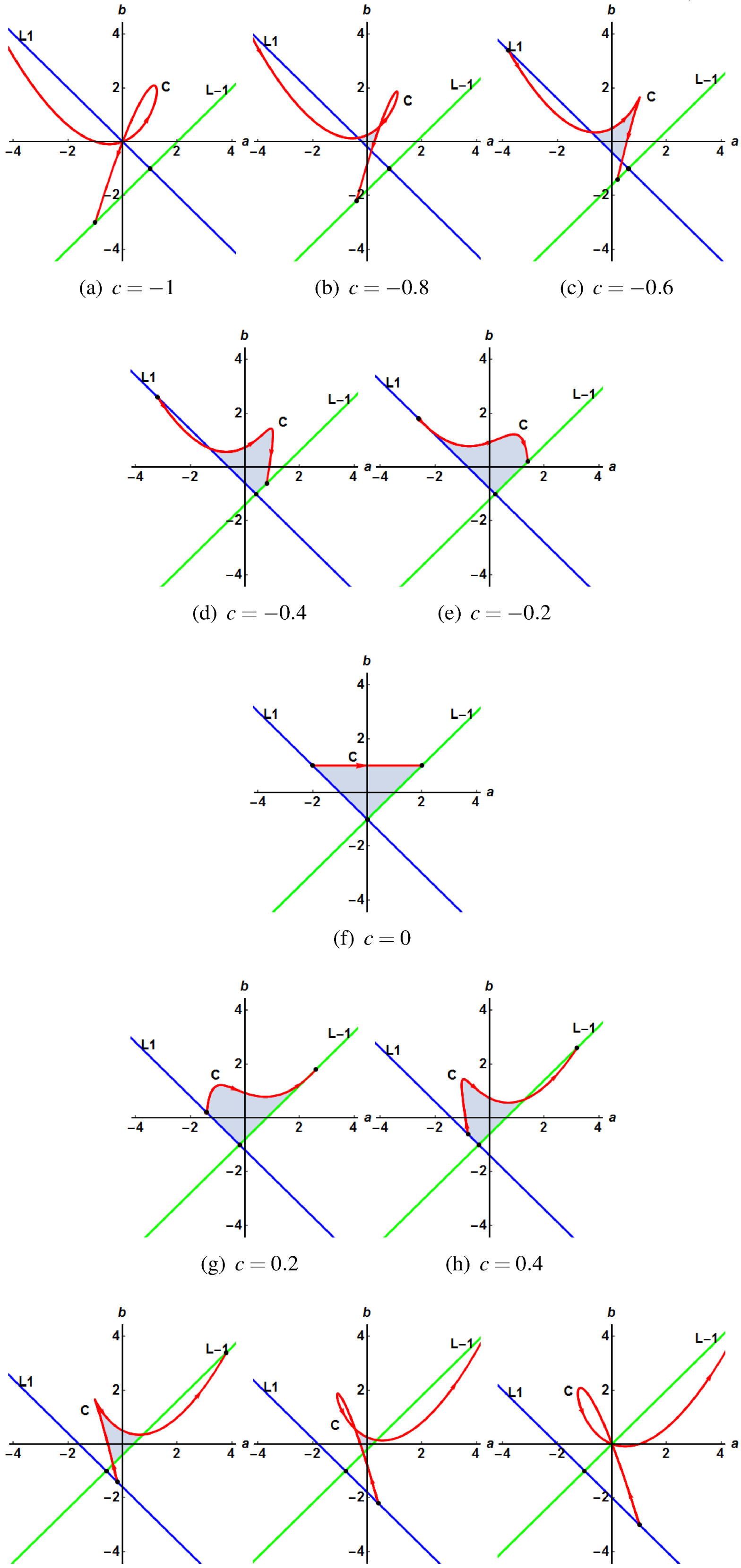
holds.

Proof. For $k=2$ the set C can be represented by

$$\begin{pmatrix} a \\ b \end{pmatrix}=\begin{pmatrix} c-2\cos\omega \\ -2c\cos\omega+1 \end{pmatrix},\ \omega\in(0,\pi).$$

Using (2.5) one has

$$\begin{pmatrix} r\alpha-1 \\ \frac{3}{4}r(1-\alpha) \end{pmatrix}=\begin{pmatrix} \frac{1}{4}r(1-\alpha)-2\cos\omega \\ -\frac{1}{2}r(1-\alpha)\cos\omega+1 \end{pmatrix},\ \omega\in(0,\pi).$$



(a) $c=-1$ (b) $c=-0.8$ (c) $c=-0.6$

(d) $c=-0.4$ (e) $c=-0.2$

(f) $c=0$

(g) $c=0.2$ (h) $c=0.4$

(i) $c=0.6$ (j) $c=0.8$ (k) $c=1$

FIGURE 2.2. Stability region for the characteristic equation (2.2) with $k=4$. When $c=\pm 0.8$ the curve that represents C intersects itself.

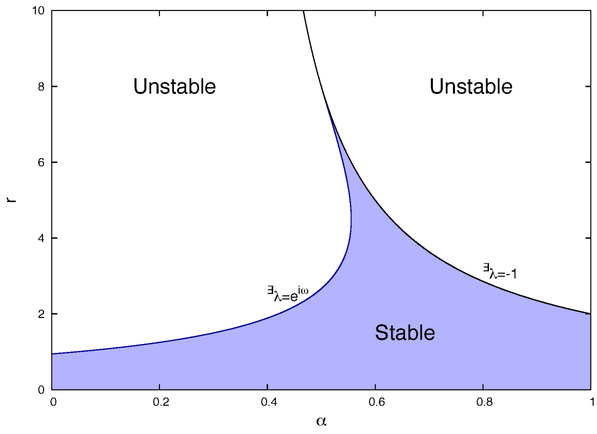


FIGURE 2.3. Stability region for the equilibrium of (2.1) in the (α, r)-parameter plane. Two curves meet at the point $(\frac{1}{2}, 8)$.

We now solve the two equations with respect to r and α. From the second equation we get

$$r(1-\alpha) = \frac{1}{\frac{3}{4}+\frac{1}{2}\cos\omega} = \frac{4}{3+2\cos\omega} \leftrightarrow r\alpha = r - \frac{4}{3+2\cos\omega}.$$

From the first equation it follows

$$r\alpha = \frac{4}{5} + \frac{1}{5}r - \frac{8}{5}\cos\omega.$$

From those equations one obtain the expression for r as in (2.6). Then one can get the expression for α. $\square$

It can be easily seen that equation (2.2) has a root $\lambda = -1$ when

$$r = \frac{4}{3\alpha - 1}.$$

In Figure 2.3 we plot those curves. One can compute that the parametric curve given by (2.6) starts at the point $\left(0, 4/\left(2+\sqrt{5}\right)\right)$ and ends at the point $(\frac{1}{2}, 8)$. From the shape of the curve one can increase r so that a stable equilibrium becomes unstable and then again becomes stable, for some $\alpha(< \frac{1}{2})$. As in Figure 1.2 (b) an unstable equilibrium can become stable as α decreases from 1 to around 0.6 (if r is less than 8). The shape of the stability boundary shows that delay in the nonlinear difference equation indeed can contribute to stability of the equilibrium, differently from what one could expect in the difference equation with one single delay in Section 1.

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