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Some remark on graph decomposition

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Abstract

Many important examples of generic structure have been constructed in model theory. Most of them are graph structures constructed by amalgamation property. In the meantime, there is the field of graph decomposition in graph theory. We consider the relation between them and try to characterize decomposable generic structures.

1. Amalgamation property and generic structure

We recall some definitions at first. In this note, we define graph structures as follows for simplicity.

Definition 1 Let the language $L=\{R(x, y)\}$ and $R(x, y)$ be a binary relation symbol.
An $R-$structure G is said to be a *graph* if
$R(x, y)$ is symmetric, $G\models\forall x\forall y[R(x, y)\longrightarrow R(y, x)]$,
$R(x, y)$ is irreflexive, $G\models\forall x[\neg R(x, x)]$.

We recall the definitions of amalgamation property and Fraïssé limit (generic structure).

Definition 2 Let L be a language and let $\mathbf{K}$ be a class of finite L-structures.
We say that $\mathbf{K}$ has *Amalgamation Property* if for any $A\subset B_{1}\in \mathbf{K}$ and $A\subset B_{2}\in \mathbf{K}$, there are $C\in \mathbf{K}$ and $B_{1}'\subset C$, and $B_{2}'\subset C$ such that $A\subset C$ and $B_{1}'\cong_{A}B_{1}$, and $B_{2}'\cong_{A}B_{2}$.

Theorem 3 *Let $L=\{R(x, y)\}$ and let $\mathbf{K}$ be a class of (isomorphism types of) finite L-structures.*
Suppose that $\emptyset\in \mathbf{K}$ and $\mathbf{K}$ is closed under substructures, and $\mathbf{K}$ has amalgamation property,
then there is a countable $L-$structure M with the following properties ;
1. *Any finite $X\subset M$ is a member of $\mathbf{K}$,*

2. *If* $A \subset B \in \mathbf{K}$ *and* $A \subset M$, *then there is a copy* $B' \subset M$ *such that* $B' \cong_A B$.

A countable L–strucute having the properties 1 *and* 2 *above is called a Fraïssé Limit* (*generic structure*) *of* $\mathbf{K}$.

2. Graph decomposition and some results

In this section, we recall some definitions around graph decomposition and remarkable results. First we recall the definition of tree-decomposition by N.Robertson and P.D.Seymour.

Definition 4 Let G be a graph and $V(G)$ be its vertex set. And let T be a tree, and $W = (W_t)_{t\in V(T)}$ be a family of vertex sets $W_t \subset V(G)$ indexed by the vertices t of $V(T)$.

The pair (T, W) is called a *tree – decomposition of* G if it satisfies the following three conditions :
(T1) $V(G) = \bigcup_{t\in V(T)} W_t$,
(T2) for every edge $e \in G$, there exists a $t \in V(T)$ such that both ends of e lie in W_t,
(T3) $W_{t_1} \cap W_{t_3} \subset W_{t_2}$ whenever $t_1 \le t_2 \le t_3$ in some path of T.

Definition 5 Let (T, W) be a tree-decomposition of G. And let κ be a cardinal.

We say (T, W) has *width* $< \kappa$ if $|W_t| < \kappa$ for every $t \in V(T)$

and $|\bigcup_{i=1}^{\infty} \bigcap_{i\le j} W_{t_j}| < \kappa$ for every infinite path $t_1, t_2, \cdots$ in T.

Next we recall the definition of simplicial tree-decomposition. This kind of decomposition was developed by R.Halin and R.Diestel. In this decomposition, attachment parts are complete graphs.

Definition 6 Let G be a graph, $\sigma > 0$ an ordinal, and let B_λ be an induced subgraph of G for every $\lambda < \sigma$.

The family $F = (B_\lambda)_{\lambda<\sigma}$ is called a *simplicial tree – decomposition of* G (*into primes*) if the following four conditions hold :
(S1) $G = \bigcup_{\lambda<\sigma} B_\lambda$,
(S2) $(\bigcup_{\lambda<\mu} B_\lambda) \cap B_\mu = S_\mu$ is a complete graph for each μ $(0 < \mu < \sigma)$,
(S3) no S_μ contains B_μ or any other B_λ $(0 \le \lambda < \mu < \sigma)$.
(S4) each S_μ is contained in B_λ for some $\lambda < \mu < \sigma$.
((S5) each B_λ is not separated by a simplex.)

I show some examples of simplicial tree-decomposition into primes.

Example 7 [2] [11]

1. Let $x_i\,(i < 3)$ be vertices of K_3. Consider the graph G whose vertices are $\{x_i,\, y_i : i < 3\}$ such that y_i is adjacent to x_i and x_{i+1} only. Let $B_0 = \{x_i : i < 3\}$, and $B_i = \{y_{i-1},\, x_{i-1},\, x_i\}$ and $S_i = \{x_{i-1},\, x_i\}$ for $1 \leq i \leq 3$ where $x_0 = x_3$ and $y_0 = y_3$.

2. Graph H_1 :

Let $\{x\}$ be a single vertex, $S = \{s_1,\, s_2,\, \cdots\}$ an infinite simplex, and $C = \{y_1,\, y_2,\, \cdots\}$ a one-way infinite path.

A graph H_1 is obtained from the disjoint union $\{x\} \cup S \cup C$ such that ;

x is joining to all the vertices of S, and there are edges $y_i s_j$ for any $j \leq i$.

For example, a decomposition of H_1 is $H_1 = (Y_1,\, Y_2,\, Y_3,\, \cdots,\, X)$ where $Y_i = H_1[\,y_i,\, y_{i+1},\, s_1,\, \cdots,\, s_i\,]$ and $X = H_1[\,x,\, s_1,\, s_2,\, \cdots]$. H_1 is not decomposed from $B_0 \ni x$.

The argument of graph decomposition are related to Graph Minor Theorem. And many characterizations of decomposition are obtained by means of the notions, subdivision and minor of graph. We recall the definitions of them.

Definition 8 A *subdivision* of a graph X, denoted by TX, is any graph arising from X by replacing its edges with independent paths of length ≥ 1.

Definition 9 Let G be a graph and $V(G)$ be its vertex set. And let X be another graph and $\{V_x : x \in V(X)\}$ is a partition of $V(G)$ into connected subsets such that ;

for any two vertices $x,\, y \in V(X)$, there is a $V_x - V_y$ edge in G if and only if x and y are adjacent in X.

In this situation, we say that there exists a *contractive homomorphism fro m G onto X* and denote $G = HX$.

And we call X is a *minor* of G if G has a subgraph G' such that $G' = HX$.

I show some results.

Theorem 10 *(N.Robertson, P.D.Seymour and R.Thomas* [1] *)*
Let κ be an infinite cardinal.
Then
A graph G contains no subgraph isomorphic to a subdivision of K_κ
if and only if
G admits a tree-decomposition of width $< \kappa$.

Definition 11 We say that two vertices of a graph *simplicially close* if they are not separated by any simplex.

And we call H a *simplicial minor of G* if H is obtained from G by

(1) taking a convex subgraph of G, and

(2) contracting connected parts of this convex subgraph satisfying that simplicially close vertices remain simplicially close.

Theorem 12 *(R.Diestel [4])*
A countable graph G admits a simplicial tree-decomposition into primes if and only if
neither H_1 nor H_2 is a simplicial minor of G (where H_2 is some variant of H_1).

3. Existence of universal graphs

In general, for a decomposable graph G, the way of decomposition is not unique. G has different decompositions by the choice of factors (parts) and the enumeration of factors. But for a generic graph G', if it is decomposable, then G' is decomposed uniformly, because generic graph G' has strong homogeneity. In reverse aspect, the decomposition of generic graph G' is a free amalgamation over some restricted bases.

There are characterizations of universal graphs by means of graph decomposition. We recall the definition of universal graph by some graph theorists.

Definition 13 Let $\mathcal{G}$ be a class of countable graphs.
A member G of $\mathcal{G}$ is called (*strongly*) *universal in* $\mathcal{G}$ if every $G' \in \mathcal{G}$ is isomorphic to some (induced) subgraph of G.

By this definition above, universal graphs may have no saturation.

Theorem 14 *(R.Diestel, R.Halin and W.Vogler [5])*
For Γ a class of countable graphs, we denote $\mathcal{G}(\Gamma)$ the class of all countable graphs that do not contain any subgraph isomorphic to a member of Γ.
Then $\mathcal{G}(TK_4) = \mathcal{G}(HK_4)$ has a strongly universal element, and for any n with $5 \le n \le \aleph_0$, $\mathcal{G}(TK_n) = \mathcal{G}(HK_n)$ has no universal element.

For the proof of theorem above, we recall some definitions and lemmas. In the next two lemmas, we denote by $\mathcal{G}$ the class $\mathcal{G}(TK_4) = \mathcal{G}(HK_4)$ and by $\mathcal{G}^2$ the class of all 2−connected graphs in $\mathcal{G}$.

Definition 15 Let G be a graph and $\mathcal{P}$ a set of finite paths in G. Call another set $L = L(\mathcal{P})$ of finite paths in G a *labelling* of $\mathcal{P}$ if each path in L is contained in some path of $\mathcal{P}$.
A labelling L is *admissible* if $T \subset T'$ or $T' \subset T$ whenever $T, T' \in L$ are not edge-disjoint.
Let H be a graph and $G \subset H$, and $\mathcal{P}$ an admissible labelled set of finite paths in G. We call H an *admissible extension* of G with respect to $\mathcal{P}$ if there exists an admissible labelled set $\mathcal{P}_{\mathcal{H}}$ of independent $G-G$ paths in H such that

$H = G \cup \bigcup_{P \in \mathcal{P}_{\mathcal{H}}} P$
and the endvertices of each $P \in \mathcal{P}_{\mathcal{H}}$ coincide with the endvertices of some $T \in L(\mathcal{P})$.

An admissible extension H is called *maximal* if the following hold for every $T \in L(\mathcal{P})$ with endvertices a, b ;

Let $\tilde{P}$ be an arbitrary admissible labelled path. Then $\mathcal{P}_{\mathcal{H}}$ contains infinitely many paths P with endvertices a, b such that an isomorphism $\Phi : \tilde{P} \longrightarrow P$ exists that maps the endvertices of $\tilde{P}$ onto a and b, and the labels of $\tilde{P}$ onto those of P.

Lemma 16 *Let $G \in \mathcal{G}$ and $\mathcal{P}$ a set of paths in G such that either $G \cong K_2$ with the admissible labelling $L(\mathcal{P}) = \mathcal{P} = \{G\}$ or $G' \subset G \in \mathcal{G}^2$ for some $G' \in \mathcal{G}$ where G is an admissible extension of G' and $\mathcal{P} = \mathcal{P}_G$. Then every admissible extension H of G with respect to $\mathcal{P}$ is contained in $\mathcal{G}^2$.*

Lemma 17 *Every $G \in \mathcal{G}^2$ can be expressed as $G = \bigcup_{i=1}^{\infty} G_i$ with $G_i \in \mathcal{G}^2$ for $i = 2, 3, \cdots$ in such a way that there exists a set $\mathcal{P}_0$ and $\mathcal{P}_i$ of independent $G_i - G_i$ paths in G for $i = 1, 2, \cdots$ such that*

1) $G_1 \cong K_2$,
2) $G_{i+1} = G_i \cup \bigcup_{P \in \mathcal{P}_i} P$,
3) *G_{i+1} is an admissible extension of G_i with respect to $\mathcal{P}_{i-1}$.*

It is known that 2−connected graphs are constructed from a cycle by successively adding paths. The lemmas above are some refined argument of it.

Theorem 18 *$\mathcal{G} = \mathcal{G}(TK_4) = \mathcal{G}(HK_4)$ has a strongly universal element.*

Sketch of proof :

We construct an elementary chain $\{G_i^2 : 1 \le i < \aleph_0\}$ inductively. Let $G_1^2 = K_2$ and $L(\mathcal{P}_0^2) = \mathcal{P}_0^2 = \{G_1^2\}$. Having defined $G_1^2, \cdots, G_i^2$ and $\mathcal{P}_0^2, \cdots, \mathcal{P}_{i-1}^2$, we define G_{i+1}^2 as any maximal admissible extension of G_i^2 with respect to $\mathcal{P}_{i-1}^2$ and put $\mathcal{P}_i^2 = \mathcal{P}_{G_{i+1}^2}$. By Lemma 16, $G_i^2 \in \mathcal{G}^2$. Let $G^2 = \bigcup_{1 \le i < \aleph_0} G_i^2$. So $G^2 \in \mathcal{G}^2$ and it is easily checked that G^2 is strongly universal in $\mathcal{G}^2$. For any $G \in \mathcal{G}^2$, by Lemma 17, we retake $G = \bigcup_{i<\aleph_0} G_i$ and we can take strong embeddings $\varphi_i : G_i \longrightarrow G^2$ inductively.

Let $G_1^* = G^2$. Having defined $G_1^*, \cdots, G_i^*$, we obtain G_{i+1}^* by attaching disjoint copies of G^2 to G_i^* in the following way ; for each vertex v of G_i^* and every $j \in \mathcal{N}$, we join infinitely many copies of G^2 to G_i^* identifying their $j-th$ vertex with v. Then any given $G \in \mathcal{G}$ can be embedded in $G^* = \bigcup_{1 \le i < \aleph_0} G_i^*$ inductvely along its block-cutvertex tree. ∎

When $5 \le n < \aleph_0$, in this case, we need more definitions and lemmas.

Definition 19 Let Γ be a set of finite graphs. Then every graph in $\mathcal{G} = \mathcal{G}(\Gamma)$ can be extended to a maximal element of $\mathcal{G}$ (after adding any further edge, it will no longer be in $\mathcal{G}$) by adding edges.

The *homomorphism base* $\mathcal{B}(H\Gamma)$ of the class $\mathcal{G}(H\Gamma)$ (and the *subdivision base* $\mathcal{B}(T\Gamma)$ of $\mathcal{G}(T\Gamma)$) is the class of all graphs that occur as a member of a prime decomposition of some maximal element of $\mathcal{G}$.

For the following two lemmas, let $\mathcal{G} = \mathcal{G}(H\Gamma)$ or $\mathcal{G} = \mathcal{G}(T\Gamma)$, and let $\mathcal{B}$ be the base of $\mathcal{G}$.

Lemma 20 *Let $G \subset B \in \mathcal{B}$, and suppose that G is maximal in $\mathcal{G}$. Then $G = B$ or G is a simplex.*

Lemma 21 *If $\mathcal{B}$ contains uncountably many pairwise non-isomorphic graphs that are maximal in $\mathcal{G}$, then $\mathcal{G}$ has no universal element.*

Sketch of proof :

Suppose that $\mathcal{G}$ has a maximal universal element G. So G contains uncountably many maximal elements B of $\mathcal{B}$ but no simplex. Now B is prime and maximal induced subgraph of G. As B is contained in some member B_τ of the prime decomposition of G, the decomposition of G has uncountably many members, a contradiction. ∎

And they construct an example which forms an uncountable class of non-isomorphic maximal planar prime graphs. Thus $\mathcal{G}(TK_5) = \mathcal{G}(HK_5)$ has no universal element.

If a graph B is prime and maximal in $\mathcal{G}(TG)$ (or $\mathcal{G}(HG)$) for some other graph G, then $B * 1$ is prime and maximal in $\mathcal{G}(TG * 1)$ (or $\mathcal{G}(HG * 1)$) where $G * 1$ denotes the graph arising from G by adding a new vertex and join it to all vertices of G.

For when $n = \aleph_0$, they show the next theorem.

Theorem 22 *Let Γ be a class of countable graphs, each containing an infinite path.*

Then $\mathcal{G} = \mathcal{G}(\Gamma)$ has no universal element.

Sketch of proof :

First by transfinite induction, for each countable ordinal λ, we take a graph $G_\lambda \in \mathcal{G}$. Let G_μ be defined for all $\mu < \lambda$. To obtain G_λ, take the disjoint union of all G_μ ($\mu < \lambda$), add a vertex w_λ, and join it to all other vertices. Let G^* be unuversal in $\mathcal{G}$. After some argument of combinatorics, we can show that there is $K_{\aleph_0}$ in G^*. ∎

Then they deduce the next theorem.

Theorem 23 *For any n with $5 \leq n \leq \aleph_0$, $\mathcal{G}(TK_n) = \mathcal{G}(HK_n)$ has no universal element.*

According to the previous argument, we can easily deduce the next corollary.

Corollary 24 *Let $\mathcal{G}$ be the class of all finite graphs of $\mathcal{G}(TK_4) = \mathcal{G}(HK_4)$. Then $\mathcal{G}$ has amalgamation property over $2-$connected bases.*

Sketch of proof :

Let A, B_1, $B_2 \in \mathcal{G}$ satisfying $A \subset B_1$ and $A \subset B_2$, and $A \in \mathcal{G}^2$. We may assume that B_1 and B_2 are $2-$connected blocks. We take graphs G_i and $\mathcal{P}_i$ of independent $G_i - G_i$ paths satisfying that G_{i+1} is an admissible extension of G_i with respect to $\mathcal{P}_{i-1}$ for some $i < n < \aleph_0$ inductively. Let $G_1 \cong K_2$ in A and $\mathcal{P}_0 = G_1$. At $i-$th stage, we take independent $G_i - G_i$ paths in A at first, and take disjoint paths in each side B_i for $i < 2$. It is easily checked that there is $C \in \mathcal{G}^2$ such that $A \subset B_i' \subset C$ and $B_i \cong_A B_i'$ for $i < 2$.

∎

4. Further problems

In [5], they show other classes $\mathcal{G}(TG)$ or $\mathcal{G}(HG)$ for some graphs G which have a universal graph.

Problem 25 *Are there other subclasses of $\mathcal{G}(TK_n)$ or $\mathcal{G}(HK_n)$ for $5 \leq n < \aleph_0$ which have a universal graph? Can we have some local argument?*

In reverse aspect, graph decomposition of universal graphs are free amalgamations of their subgraphs.

Problem 26 *Can we characterize decomposable graphs by predimension or dimension of generic structures?*

More generally,

Problem 27 *Can we classify decomposable graphs by stability theoretic notions?*

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