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<th>Title</th>
<th>A brief survey of recent results on $NTP_2$ and dense codense predicate expansions (Model theoretic aspects of the notion of independence and dimension)</th>
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</thead>
<tbody>
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A brief survey of recent results on NTP$_2$ and dense codense predicate expansions

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1 Introduction

The aim of this article is to review/summarize some of the recent results on NTP$_2$ theories and dense codense predicate expansions due to Chernikov [5], Berenstein and Vassiliev [4], [3], and Berenstein and H. Kim [2]. We do not include all the proofs of these results, and the interested reader is referred to the aforementioned papers for the complete proofs. However, we will sketch the proof of Theorem 3.14, a main result from [2]. This article is organized as follows: In Section 2, we review the notions of burden, NTP$_2$ and $\kappa^n_{\text{inp}}(T)$ as well as some of the key facts about them due to several authors (notably the "submultiplicativity of burden" by Chernikov [5]). In Section 3, we review the notion of dense codense predicate expansions and some of the key results due to [4], [3], [2]. We will conclude by sketching the proof of Theorem 3.14, a main result from [2].

2 NTP$_2$, burden and $\kappa^n_{\text{inp}}$

Throughout this section, we work inside a fixed, sufficiently saturated model of an arbitrary theory $T$.

The following definition is due to Adler [1].

Definition 2.1. For a partial type $\overline{p}(\overline{x})$, an inp-pattern in $\overline{p}(\overline{x})$ is a set of formulas $\{\phi_i(\overline{x}, \overline{a}_{i,j}) \mid i < \kappa, j < \omega\}$ where $\kappa$ is a cardinal, satisfying the following:

1. For each $i < \kappa, \{\phi_i(\overline{x}, \overline{a}_{i,j}) \mid j < \omega\}$ is $k_i$-inconsistent for some integer $k_i \geq 2$.

2. For every function $f : \kappa \to \omega, \overline{p}(\overline{x}) \cup \{\phi_i(\overline{x}, \overline{a}_{i,f(i)}) \mid i < \kappa\}$ is consistent.

The cardinal $\kappa$ is called the depth of the inp-pattern. The supremum of depths of all inp-patterns in $\overline{p}(\overline{x})$ is called the burden of $\overline{p}(\overline{x}),$ denoted by $\text{bdn}(\overline{p}(\overline{x}))$. $\text{bdn}(\text{tp}(\overline{a}/A))$ is abbreviated as $\text{bdn}(\overline{a}/A)$.

Remark 2.2. The notion of inp-patterns (where "inp" stands for "independent partitions") was introduced by Shelah in [7] where he considered inp-patterns in trivial type $\overline{x} = \overline{a}$ in order to define certain cardinals $\kappa^n_{\text{inp}}$ associated with a given theory. (We shall recall the definition of $\kappa^n_{\text{inp}}$ later.)

Observation 2.3. The following is clear from the definition of burden.

1. $\text{bdn}(\overline{p}(\overline{x})) = 0$ iff $\overline{p}(\overline{x})$ is an algebraic type.

2. For any inp-pattern in $\text{tp}(\overline{a}/\overline{b})$, there exists an inp-pattern of the same depth in $\text{tp}(\overline{a})$.

3. For any inp-pattern in $\text{tp}(\overline{a})$ and any tuple $\overline{b}$, there exists an inp-pattern of the same depth in $\text{tp}(\overline{a}\overline{b})$. 
4. $\text{bdn}(\overline{a}/\overline{b}) \leq \text{bdn}(\overline{a}) \leq \text{bdn}(\overline{a}\overline{b})$ for any tuples $\overline{a}$ and $\overline{b}$.

It has been observed by many researchers (for example, [8], [5], [1]) that the parameters $\{\overline{a}_{i,j}\}_{i,j}$ in an inp-pattern may be assumed to be 'indiscernible' in a certain sense. More precisely:

**Proposition 2.4** ([8], [1], [5]). If there exists an inp-pattern $\{\varphi_{i}(\overline{x}, \overline{a}_{i,j}) | i < \kappa, j < \omega\}$ in a type $p(\overline{x})$, we may assume that $\{\overline{a}_{i,j} | i < \kappa, j < \omega\}$ is a mutually indiscernible array, that is, for each $i < \kappa$, $\{\overline{a}_{i,j} | j < \omega\}$ is an indiscernible sequence over $\{\overline{a}_{k,j} | k \neq i, j < \omega\}$.


**Theorem 2.5** (Chernikov [5]). If there exists an inp-pattern of depth $\kappa_{1} \times \kappa_{2}$ in $\text{tp}(\overline{a}\overline{b})$, then either there exists an inp-pattern of depth $\kappa_{1}$ in $\text{tp}(\overline{a})$ or there exists an inp-pattern of depth $\kappa_{2}$ in $\text{tp}(\overline{b}/\overline{a})$.

**Corollary 2.6** ([5], the "submultiplicativity" of burden). For any tuples $\overline{a}_{1}, \ldots, \overline{a}_{n}$ and any cardinals $\kappa_{1}, \ldots, \kappa_{n}$,

\[ \text{bdn}(\overline{a}_{i}) < \kappa_{i} \text{ for each } i \implies \text{bdn}(\overline{a}_{1} \cdots \overline{a}_{n}) < \prod_{i=1}^{n} \kappa_{i}. \]

**Proof.** We may assume $n = 2$ and that all the $\kappa_{i}$'s are nonzero cardinals. We will prove the contrapositive. Assume $\text{bdn}(\overline{a}_{1}\overline{a}_{2}) = \kappa_{1} \times \kappa_{2}$. If $\kappa_{1} \times \kappa_{2}$ is a successor cardinal, then clearly there exists an inp-pattern of depth $\kappa_{1} \times \kappa_{2}$ in $\text{tp}(\overline{a}_{1}\overline{a}_{2})$, so either $\text{bdn}(\overline{a}_{1}) \geq \kappa_{1}$ or $\text{bdn}(\overline{a}_{2}) \geq \kappa_{2}$ by Theorem 2.5 and Observation 2.3(2), and we are done. On the other hand, if $\kappa_{1} \times \kappa_{2}$ is a limit cardinal, then Theorem 2.5 implies that either $\text{bdn}(\overline{a}_{1}) \geq \kappa_{1} \times \kappa_{2}$ or $\text{bdn}(\overline{a}_{2}) \geq \kappa_{1} \times \kappa_{2}$. Hence, either $\text{bdn}(\overline{a}_{1}) \geq \kappa_{1}$ or $\text{bdn}(\overline{a}_{2}) \geq \kappa_{2}$.

Next, we recall the notion of $\kappa_{\text{inp}}^{n}$ introduced by Shelah [7, Section III.7].

**Definition 2.7.** For any theory $T$ and any integer $n \geq 1$, $\kappa_{\text{inp}}^{n}(T)$ denotes the least cardinal $\tau$ such that there does not exist any inp-pattern of depth $\tau$ in the type $\{\overline{x} = \overline{x}\}$ where $\overline{x}$ has arity $n$. And $\kappa_{\text{inp}}^{n}(T) := \infty$ if such $\tau$ does not exist.

**Remark 2.8.** We use the convention that $\kappa < \infty$ for every cardinal $\kappa$.

**Observation 2.9.** The following is clear:

1. $n < \kappa_{\text{inp}}^{m}(T)$ (due to the equality symbol in every language).
2. $n \leq m \implies \kappa_{\text{inp}}^{n}(T) \leq \kappa_{\text{inp}}^{m}(T)$.
3. $\sup_{1 \leq n \leq \omega} \kappa_{\text{inp}}^{n}(T) \geq \aleph_{0}$.

Another important consequence of Theorem 2.5 is the following:

**Theorem 2.10** (Chernikov [5]). For any theory $T$, either $\kappa_{\text{inp}}^{n}(T) < \aleph_{0}$ for all $n$, or $\kappa_{\text{inp}}^{n}(T) = \kappa_{\text{inp}}^{n}(T)$ for all $n$.

**Proof.** Suppose that $\kappa_{\text{inp}}^{n}(T) < \aleph_{0}$ for some $n$. Let $N$ be the least such $n$.


**Proof of Claim.** Suppose $N > 1$. Then $C := \kappa_{\text{inp}}^{N-1}(T)$ is a natural number, and $\kappa_{\text{inp}}^{n}(T) \leq C$ for all $1 \leq i < N$. Since $C \times C < \kappa_{\text{inp}}^{N}(T)$, there exists an inp-pattern $\{\varphi_{i}(\overline{x}, \overline{a}_{i,j}) | i < C \times C, j < \omega\}$ in $\overline{x} = \overline{x}$ where $|\overline{x}| = N$. By Proposition 2.4, we may assume $\{\overline{a}_{i,j}\}_{i,j}$ is a mutually indiscernible array. Hence, if $\overline{b}$ is any tuple realizing $\bigwedge_{i < C \times C} \varphi_{i}(\overline{x}, \overline{a}_{i,j})$, then $\{\varphi_{i}(\overline{x}, \overline{a}_{i,j}) | i < C \times C, j < \omega\}$ is an inp-pattern in $\text{tp}(\overline{b})$. Let $\overline{b}$ be in the form $\overline{c}\overline{d}$ where $\overline{c}$ and $\overline{d}$ have nonzero arities. Then, by Theorem 2.5, either there
exists an inp-pattern of depth $C$ in $\text{tp}(\vec{c})$, or there exists an inp-pattern of depth $C$ in $\vec{d}$. This contradicts that $\kappa_{\text{inp}}(T) \leq C$ for all $1 \leq i < N$. This completes the proof of Claim.

Now we are ready to prove that $\kappa_{\text{inp}}^1(T) = \kappa_{\text{inp}}^n(T)$ for all $n$. In the case $\kappa_{\text{inp}}^1(T) = \infty$, the assertion is clear. So assume $\kappa_{\text{inp}}^1(T) < \infty$. Then $\kappa_{\text{inp}}^1(T)$ is an infinite cardinal by Claim above. Let $\tau := \kappa_{\text{inp}}^1(T)$. Suppose $\kappa_{\text{inp}}^n(T) \neq \tau$ for some $n$. Let $N$ be the least such $n$. Then $\kappa_{\text{inp}}^n(T) = \tau$ for all $1 \leq i < N$. But, since $\tau = \tau \times \tau \neq \kappa_{\text{inp}}^N(T)$, there exists an inp-pattern $\{\varphi_i(\vec{x},\vec{a}_{i,j}) \mid i < \tau \times \tau, j < \omega\}$ in $\vec{x} = \vec{x}$ where $|\vec{x}| = N$. Now we may repeat the same argument in the latter part of the proof of Claim, and derive a contradiction.

\begin{definition}[\cite{1}] A theory $T$ is said to be strong if all inp-patterns in $\vec{x} = \vec{x}$ have finite depths, for all arities $|\vec{x}|$.
\end{definition}

\begin{remark}
Clearly, a theory $T$ is strong iff $\kappa_{\text{inp}}^n(T) \leq \aleph_0$ for all $n < \omega$. In fact, by Theorem 2.10, $T$ is strong iff $\kappa_{\text{inp}}^1(T) \leq \aleph_0$.
\end{remark}

Next, we recall the definition of NTP$_2$ theories.

\begin{definition}
A theory $T$ is said to have $k$-TP$_2$ (where $k \geq 2$ is an integer) if there exists a formula $\varphi(\vec{x},\vec{y})$ witnessing it, i.e., there exists a set of tuples $\{\vec{a}_{i,j} \mid i, j < \omega\}$ such that:

1. $\{\varphi(\vec{x},\vec{a}_{i,j}) \mid j < \omega\}$ is $k$-inconsistent for every $i < \omega$,

2. $\{\varphi(\vec{x},\vec{a}_{i,f(i)}) \mid i < \omega\}$ is consistent for every function $f : \omega \to \omega$.

We say $T$ has NTP$_2$ if it does not have $k$-TP$_2$ for any $k \geq 2$.
\end{definition}

\begin{remark}
The notion of (2-)TP$_2$ (called the ‘tree property of the second kind’) was introduced by Shelah \cite{7}.
\end{remark}

\begin{observation}
A theory $T$ has $k$-TP$_2$ for some $k$ iff $\kappa_{\text{inp}}^n(T) = \infty$ for all $n$. In particular, every strong theory has NTP$_2$.
\end{observation}

\begin{proof}
$(\Rightarrow)$ is by compactness and Theorem 2.10, and $(\Leftarrow)$ is by the pigeon hole principle.
\end{proof}

Suppose that there exists $\{\varphi(\vec{x},\vec{a}_{i,j}) \mid i, j < \omega\}$ witnessing $k$-TP$_2$. Since this is a form of inp-pattern, we know that $\{\vec{a}_{i,j} \mid i, j < \omega\}$ can be chosen to be mutually indiscernible (by Proposition 2.4). In fact, we can even require that the ‘rows’ of $\{\vec{a}_{i,j} \mid i, j < \omega\}$ form an indiscernible sequence. More precisely: if we let $\vec{\beta}_i := \{\vec{a}_{i,j} \mid j < \omega\}$ for each $i < \omega$, then $\{\vec{\beta}_i \mid i < \omega\}$ is an indiscernible sequence. This can be proved by a routine application of Ramsey’s theorem together with compactness. Alternatively, this can be derived from a more general ‘tree-indiscernibility’ theorem. The interested reader is referred to \cite[Lemma 5.6]{9} for details.

We call $\{\vec{a}_{i,j} \mid i, j < \omega\}$ with the indiscernibility condition described above an indiscernible array.

\begin{proposition}
If a formula $\varphi(\vec{x},\vec{y})$ witnesses $k$-TP$_2$ for some $k \geq 2$, then some finite conjunction $\psi(\vec{x},\vec{y}_1, \cdots, \vec{y}_N) := \bigwedge_{i=1}^N \varphi(\vec{x},\vec{y}_i)$ witnesses 2-TP$_2$. Hence, a theory $T$ has NTP$_2$ iff it does not have 2-TP$_2$.
\end{proposition}

\begin{proof}
Use the fact that such $\varphi(\vec{x},\vec{y})$ can witness $k$-TP$_2$ with an indiscernible array. See \cite{6} or \cite{1} for details.
\end{proof}

\begin{remark}
Hence, there is no ambiguity in saying that a theory $T$ has $\text{TP}_2$ (without specifying ‘$k$’ in $k$-TP$_2$). However, by convention, $\text{TP}_2$ usually refers to 2-TP$_2$.
\end{remark}

\begin{theorem}[Chernikov \cite{5}]
If a theory $T$ has $k$-TP$_2$, then it can be witnessed by a formula $\varphi(x,\vec{y})$ where $x$ is a single variable.
\end{theorem}

\begin{proof}
An immediate consequence of Observation 2.15 and Proposition 2.16.
\end{proof}
3 Dense codense expansions

First, let us recall the definition of geometric theories.

**Definition 3.1.** Let $T$ be a theory.

1. $T$ is said to eliminate $\exists^\infty$ if, for any formula $\phi(x, \bar{y})$, there exists some natural number $n$ such that, for any parameters $\bar{a}$, $\phi(x, \bar{a})$ has infinitely many realizations iif it has more than $n$ realizations.

2. $T$ is said to satisfy the exchange property if, for any model $M \models T$, any subset $A \subseteq M$ and any $b, c \in M \setminus acl(A)$, $b \in acl(Ac)$ iif $c \in acl(Ab)$.

3. $T$ is said to be geometric if it eliminates $\exists^\infty$ and satisfies the exchange property.

**Definition 3.2.** Let $T$ be a geometric complete theory in a language $\mathcal{L}$, and let $\mathcal{L}_H := \mathcal{L} \cup \{H\}$ be the language obtained by adding a new unary predicate symbol $H$. Given any $\mathcal{L}$-model $M \models T$, let $(M, H(M))$ denote an expansion of $M$ to $\mathcal{L}_H$, where $H(M) := \{x \in M \mid H(x)\}$. $(M, H(M))$ is called a dense codense expansion of $M$ if every non-algebraic $\mathcal{L}$-formula $\phi(x, \bar{a})$ (where $x$ is a single variable) has realizations both in $H(M)$ and in $M \setminus acl(\bar{a} \cup H(M))$. A dense codense expansion $(M, H(M))$ is called:

1. a lovely pair if $H(M) \prec M$ (as $\mathcal{L}$-models),

2. an $H$-structure if $H(M)$ is an $\mathcal{L}$-algebraically independent subset of $M$.

**Remark 3.3.** The condition that every non-algebraic $\mathcal{L}$-formula $\phi(x, \bar{a})$ has realizations in $H(M)$ is called the density condition of $H$, and the condition that it always has realizations in $M \setminus acl(\bar{a} \cup H(M))$ is called the codensity condition of $H$.

**Theorem 3.4** ([3], [4]). Given any geometric complete theory $T$, all the lovely pairs (resp. $H$-structures) associated with $T$ are elementarily equivalent to one another.

**Notation.** Throughout the rest of this section, we fix a geometric complete theory $T$ in a language $\mathcal{L}$, and let $T^*$ denote the complete $\mathcal{L}_H$-theory of either the lovely pairs or the $H$-structures associated with $T$. And we will work inside an arbitrary $\bar{a}$-saturated model $(M, H(M)) \models T^*$ for some sufficiently large cardinal $\bar{a}$. When we say $A$ is a subset of $M$, we shall mean (unless explicitly stated otherwise) that $A$ is a subset of size $< \bar{a}$. $H$-subscripts in $acl_H(\bar{a})$, $tp_H(\bar{a})$ indicate that they are defined in the language $\mathcal{L}_H$. (On the other hand, $T$-subscripts will indicate that they are defined in the language $\mathcal{L}$.)

**Remark 3.5.** 1. It remains open whether $T^*$ eliminates $\exists^\infty$ although [4] provides a partial answer by proving that every $\mathcal{L}_H$-formula in the form $\phi(x, \bar{y}) \land H(\bar{y})$ does eliminate $\exists^\infty$.

2. Not all models of $T^*$ may be $H$-structures (resp. lovely pairs) although sufficiently saturated ones are. (See [4, Examples 2.11, 2.12].)

**Definition 3.6** ([4], [3]). A subset $A \subseteq M$ is called $H$-independent if $A \downarrow_{H(A)} H(M)$ (where $\downarrow$ denotes the algebraic independence relation).

**Observation 3.7.** If $\bar{a} \in M$ is $H$-independent, so is $\bar{a}h$ for any $h \in H(M)$.

**Proposition 3.8** ([2]). If there exists $\{\psi(\bar{x}, \bar{a}_{i,j}) \mid i, j < \omega\}$ witnessing $k$-TP$_2$, we may assume that $\{\bar{a}_{i,j}\}_{i,j<\omega}$ is an indiscernible array and that every $\bar{a}_{i,j}$ is $H$-independent.

**Lemma 3.9** ([3], [4]). For any $H$-independent tuples $\bar{a}$ and $\bar{b}$,

$$tp_H(\bar{a}) = tp_H(\bar{b}) \iff tp_T(\bar{a}H(\bar{a})) = tp_T(\bar{b}H(\bar{b})).$$
Definition 3.10. For any subset $A \subseteq M$, $\text{scl}_T(A \cup H(M))$ is called the small closure of $A$, and is denoted by $\text{scl}(A)$. Any subset $B \subseteq \text{scl}(A)$ is said to be $A$-small.

Remark 3.11. $M \setminus \text{scl}(A)$ is $\mathcal{L}_H$-type definable over $A$.

Proposition 3.12 ([2]). For any $\mathcal{L}_H$-formula $\varphi(x, \overline{a})$ where $\overline{a}$ is $H$-independent, there exists some $\mathcal{L}$-formula $\psi(x, \overline{a})$ such that

$$\models \varphi(x, \overline{a}) \land H(x) \leftrightarrow \psi(x, \overline{a}) \land H(x).$$

Proof. (Essentially a compactness argument.) Let $X \subseteq M^n$ be the set defined by $\varphi(x, \overline{a})$. We may assume that $H(M)^n \cap X$ and $H(M)^n \setminus X$ are both nonempty.

Claim. For any $\overline{h}_1 \in H(M)^n \cap X$ and any $\overline{h}_2 \in H(M)^n \setminus X$, there exists some $\mathcal{L}$-formula $\theta_{\overline{h}_1\overline{h}_2}(x, \overline{a})$ such that $\overline{h}_1 \models \theta(x, \overline{a})$ and $\overline{h}_2 \models \neg\theta(x, \overline{a})$.

Proof of Claim. Given such $\overline{h}_1$ and $\overline{h}_2$, clearly $\text{tp}_H(\overline{h}_1\overline{a}) \neq \text{tp}_H(\overline{h}_2\overline{a})$. Moreover, $\overline{h}_1\overline{a}$ and $\overline{h}_2\overline{a}$ are both $H$-independent by Observation 3.7. Hence Lemma 3.9 implies $\text{tp}_T(\overline{h}_1\overline{a}) \neq \text{tp}_T(\overline{h}_2\overline{a})$. This completes the proof of Claim.

For each $\overline{h}_2 \in H(M)^n \setminus X$, consider the following $\mathcal{L}_H$-type over $\overline{a}$:

$$\Sigma_{\overline{h}_2}(\overline{x}) := \{H(\overline{x}) \land \varphi(\overline{x}, \overline{a})\} \cup \{\neg\theta_{\overline{h}_1\overline{h}_2}(\overline{x}, \overline{a}) \mid \overline{h}_1 \in H(M)^n \cap X\}$$

which is clearly inconsistent. Since $(M, H(M))$ is saturated, there exist finitely many tuples $\overline{h}_1, \ldots, \overline{h}_k$ in $H(M)^n \cap X$ such that the $\mathcal{L}$-formula

$$\psi_{\overline{h}_2}(\overline{x}, \overline{a}) := \bigvee_{i=1}^k \theta_{\overline{h}_1\overline{h}_i}(\overline{x}, \overline{a})$$

is satisfied by every tuple in $H(M)^n \cap X$. Note $\overline{h}_2 \not\models \psi_{\overline{h}_2}(\overline{x}, \overline{a})$. Next, consider the following $\mathcal{L}_H$-type over $\overline{a}$:

$$\Sigma(\overline{a}) := \{H(\overline{a}) \land \neg\varphi(\overline{x}, \overline{a})\} \cup \{\psi_{\overline{h}_2}(\overline{x}, \overline{a}) \mid \overline{h}_2 \in H(M)^n \setminus X\}$$

which is clearly inconsistent. Again, since $(M, H(M))$ is saturated, there exist finitely many tuples $\overline{h}_1, \ldots, \overline{h}_m$ in $H(M)^n \setminus X$ such that the $\mathcal{L}$-formula

$$\psi(\overline{x}, \overline{a}) := \bigwedge_{i=1}^m \psi_{\overline{h}_i}(\overline{x}, \overline{a})$$

is not satisfied by any tuple in $H(M)^n \setminus X$. But $\psi(\overline{x}, \overline{a})$ is satisfied by every tuple in $H(M)^n \cap X$, and hence $\psi(\overline{x}, \overline{a})$ is a desired $\mathcal{L}$-formula.

Proposition 3.13 ([2]). For any $\mathcal{L}_H$-formula $\varphi(x, \overline{a})$ where $x$ is a single variable and $\overline{a}$ is $H$-independent, there exists some $\mathcal{L}$-formula $\psi(x, \overline{a})$ such that the symmetric difference $\varphi(x, \overline{a}) \Delta \psi(x, \overline{a})$ defines an $\overline{a}$-small set.

Proof. (Essentially a compactness argument.) Let $X \subseteq M$ be the set defined by $\varphi(x, \overline{a})$. Consider

$$Y_1 := \{x \in X \mid x \not\in \text{scl}(\overline{a})\} \quad \text{and} \quad Y_2 := \{x \in M \setminus X \mid x \not\in \text{scl}(\overline{a})\}.$$
Note that $Y_1$ and $Y_2$ are both $\mathcal{L}_H$-type definable over $\bar{a}$ (by Remark 3.11). Let $\Sigma_1(x)$ and $\Sigma_2(x)$ be $\mathcal{L}_H$-types over $\bar{a}$ defining $Y_1$ and $Y_2$, respectively. For each $c_2 \in Y_2$, consider the following $\mathcal{L}_H$-type over $\bar{a}$:

$$\Sigma_1(x) \cup \{\neg \theta_{c_1c_2}(x, \bar{a}) \mid c_1 \in Y_1\}$$

which is clearly inconsistent. Since $(M, H(M))$ is saturated, there exist finitely many $c_1^1, \cdots, c_1^k \in Y_1$ such that the $L$-formula

$$\psi_{c_2}(x, \bar{a}) := \bigvee_{i=1}^{k} \theta_{c_1^i c_2}(x, \bar{a})$$

is satisfied by every element in $Y_1$. Note $c_2 \not\in \psi_{c_2}(x, \bar{a})$. Next, consider the following $\mathcal{L}_H$-type over $\bar{a}$:

$$\Sigma_2(x) \cup \{\psi_{c_2}(x, \bar{a}) \mid c_2 \in Y_2\}$$

which is clearly inconsistent. Again, since $(M, H(M))$ is saturated, there exist finitely many $c_2^1, \cdots, c_2^m \in Y_2$ such that the $L$-formula

$$\psi(x, \bar{a}) := \bigwedge_{i=1}^{m} \psi_{c_2^i}(x, \bar{a})$$

is not satisfied by any element of $Y_2$. But $\psi(x, \bar{a})$ is satisfied by every element of $Y_1$. We conclude that $\varphi(x, \bar{a}) \triangle \psi(x, \bar{a})$ defines an $\bar{a}$-small set.

\[\square\]

**Theorem 3.14** (Main result in [2]). Let $T$ be any geometric complete theory.

1. If $T$ has NTP$_2$, so does $T^*$.

2. If $T$ is strong, so is $T^*$.

**Proof.** (Sketch) The proof of 2 is largely a generalization of that of 1, so we only sketch the proof of 1. Propositions 2.16 is routinely used throughout the proof. We start by assuming that $T^*$ has TP$_2$ witnessed by some $L_H$-formula $\varphi(x, \bar{y})$ (where $x$ is a single variable due to Theorem 2.18) and an indiscernible array $A := \{a_{i,j}\}_{i,j \leq w}$ where each $a_{i,j}$ is $H$-independent (due to Proposition 3.8). Next, we consider two cases. First consider the case where all the realizations of $\bigwedge_{i \leq w} \varphi(x, a_{i,0})$ are in $M \setminus \text{scl}(A)$. In this case, it is relatively straightforward to show (by applying Proposition 3.13 together with the codensity condition of $H$) that $T$ has TP$_2$. The other case is more complicated. Basically we reduce it to the case where some $L_H$-formula in the form $\phi(x, \bar{w}) \land H(\bar{x})$ witnesses TP$_2$. In fact, we may assume that $\bar{x}$ here is a single variable due to the submultiplicativity of burden (Corollary 2.6). Finally we apply Proposition 3.12 together with the density condition of $H$ to show that $T$ has TP$_2$. The interested reader is referred to [2] for full details.

\[\square\]

**References**


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