Equivariant definable Tietze extension theorem

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Abstract

Let G be a definably compact definable group, X a definable G set and A a G invariant definably compact definable subset of X. We prove that every G invariant definable function $f : A \to R$ is extensible to a G invariant definable function $F : X \to R$ with $F|A = f$.

1 Introduction

In this paper we consider equivariant definable Tietze extension theorem in an o-minimal expansion $\mathcal{N} = (R, +, \cdot, <, ...)$ of a real closed field R. It is known that there exist uncountably many o-minimal expansions of the field $\mathbb{R}$ of real numbers([7]).

Definable set and definable maps are studied in [2], [3], and see also [8]. Everything is considered in $\mathcal{N} = (R, +, \cdot, <, ...)$ and definable maps are assumed to be continuous unless otherwise stated.

Theorem 1.1 ([5]). *Let G be a definably compact definable group, X a definable G set and A a G invariant definably compact definable subset of X. Every G invariant definable function $f : A \to R$ is extensible to a G invariant definable function $F : X \to R$ with $F|A = f$.*

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2 Preliminaries

A subset X of R^n is *definable* (in $\mathcal{N}$) if it is defined by a formula (with parameters). Namely, there exist a formula $\phi(x_1, \dots, x_n, y_1, \dots, y_m)$ and elements $b_1, \dots, b_m \in R$ such that $X = \{(a_1, \dots, a_n) \in R^n | \phi(a_1, \dots, a_n, b_1, \dots, b_m)$ is true in $\mathcal{N}\}$.

For any $-\infty \leqq a < b \leqq \infty$, an open interval $(a,b)_R$ means $\{x \in R | a < x < b\}$, for any $a, b \in R$ with $a < b$, a closed interval $[a,b]_R$ means $\{x \in R | a \leqq x \leqq b\}$. We call $\mathcal{N}$ *o-minimal* (*order-minimal*) if every definable subset of R is a finite union of points and open intervals.

A real closed field $(R, +, \cdot, <)$ is an o-minimal structure and every definable set is a semialgebraic set [9], and a definable map is a semialgebraic map [9]. In particular, the semialgebraic category is a special case of a definable one.

The topology of R is the interval topology and the topology of R^n is the product topology. Note that R^n is a Hausdorff space.

The field $\mathbb{R}$ of real nubmers, $\mathbb{R}_{alg} = \{x \in \mathbb{R} | x$ is algeraic over $\mathbb{Q}\}$ are Archimedean real closed fields.

The Puiseux series $\mathbb{R}[X]^\wedge$, namely $\sum_{i=k}^{\infty} a_i X^{\frac{i}{q}}, k \in \mathbb{Z}, q \in \mathbb{N}, a_i \in \mathbb{R}$ is a non-Archimedean real closed field.

Fact 2.1. *(1) The characteristic of a real closed field is* 0.

(2) For any cardinality $\kappa \geqq \aleph_0$, *there exist* 2^κ *many non-isomorphic real closed fields whose cardinality are* κ.

(3) In a general real closed field, even for a C^∞ *function, the intermediate value theorem, existence theorem of maximum and minimum, Rolle's theorem, the mean value theorem do not hold. Even for a* C^∞*function* f *in one varianble, the result that* $f' > 0$ *implies* f *is increasing does not hold.*

Definition 2.2. Let $X \subset R^n$, $Y \subset R^m$ be definable sets.

(1) A continuous map $f : X \to Y$ is a *definable map* if the graph of f ($\subset R^n \times R^m$) is definable.

(2) A definable map $f : X \to Y$ is a *definable homeomorphism* if there exists a definable map $f' : Y \to X$ such that $f \circ f' = id_Y, f' \circ f = id_X$.

Definition 2.3. A group G is a *definable group* if G is definable and the group operations $G \times G \to G, G \to G$ are definable.

Let G be a definable group. A pair (X, ϕ) consisting a definable set X and a G action $\phi : G \times X \to X$ is a *definable G set* if ϕ is definable. We simply write X instead of (X, ϕ).

Definition 2.4. Let X, Y be definable G sets.

(1) A definable map $f : X \to Y$ is a *definable G map* if for any $x \in X, g \in G$, $f(gx) = gf(x)$.

(2) A definable G map $f : X \to Y$ is a *definable G homeomorphism* if there exists a definable G map $h : Y \to X$ such that $f \circ h = id_Y$, $h \circ f = id_X$.

Definition 2.5. (1) A definable set $X \subset R^n$ is *definably compact* if for any definable map $f : (a,b)_R \to X$, there exist the limits $\lim_{x \to a+0} f(x), \lim_{x \to b-0} f(x)$ in X.

(2) A definable set $X \subset R^n$ is *definably connected* if there exist no definable open subsets U, V of X such that $X = U \cup V, U \cap V = \emptyset, U \neq \emptyset, V \neq \emptyset$.

A compact (resp. A connected) definable set is definably compact (resp. definably connected). But a definably compact (resp. a definably connected) definable set is not always compact (resp. connected). For example, if $R = \mathbb{R}_{alg}$, then $[0,1]_{\mathbb{R}_{alg}} = \{x \in \mathbb{R}_{alg} | 0 \leqq x \leqq 1\}$ is definably compact and definably connected, but it is neither compact nor connected.

Theorem 2.6 ([6]). *For a definable set $X \subset R^n$, X is definably compact if and only if X is closed and bounded.*

The following is a definable version of the fact that the image of a compact (resp. a connected) set by a continuous map is compact (resp. connected).

Proposition 2.7. *Let $X \subset R^n$, $Y \subset R^m$ be definable set and $f : X \to Y$ a definable map. If X is definably compact (resp. definably connected), then $f(X)$ is definably compact (resp. definably connected).*

Theorem 2.8. *(1) (The intermediate value theorem) For a definable function f on a definably connected set X, if $a, b \in X$, $f(a) \neq f(b)$ then f takes all values between $f(a)$ and $f(b)$.*

(2) (Existence theorem of maximum and minimum) Every definable function on a definably compact definable set attains maximum and minimum.

(3) (Rolle's theorem) Let $f : [a,b]_R \to R$ be a definable function such that f is differentiable on $(a,b)_R$ and $f(a) = f(b)$. Then there exists c between a and c with $f'(c) = 0$.

(4) (The mean value theorem) Let $f : [a,b]_R \to R$ be a definable function which is differentiable on $(a,b)_R$. Then there exists c between a and c with $f'(c) = \frac{f(b)-f(a)}{b-a}$.

(5) Let $f : (a,b)_R \to R$ be a differentiable definable function. If $f' > 0$ on $(a,b)_R$, then f is increasing.

Example 2.9. *(1) Let $\mathcal{N}$ be $(\mathbb{R}_{alg}, +, \cdot, <)$. Then $f : \mathbb{R}_{alg} \to \mathbb{R}_{alg}, f(x) = 2^x$ is not defined([10]).*

(2) Let $\mathcal{N}$ be $(\mathbb{R}, +, \cdot, <)$. Then $f : \mathbb{R} \to \mathbb{R}, f(x) = 2^x$ is defined but not definable in $\mathcal{N}$, and $h : \mathbb{R} \to \mathbb{R}, h(x) = \sin x$ is defined but not definable in $\mathcal{N}$.

Definition 2.10. A definable map $f : X \to Y$ is *definably proper* if for any definably compact subset C of Y, $f^{-1}(C)$ is definably compact.

Theorem 2.11 (Existence of definable quotient). *Let G be a definably compact definable group and X a definable G set. Then the orbit space X/G exists as a definable set, and the orbit map $\pi : X \to X/G$ is definable, surjective and definably proper.*

The following theorem is the topological case of Tietze extension theorem.

Theorem 2.12 (Tietze extension theorem). *Let X be a normal space and A a closed subset of X. Then every continuous map $f : A \to \mathbb{R}$ is extensible to a continuous map $F : X \to \mathbb{R}$ with $F|A = f$.*

The following theorem is the definable case of Tietze extension theorem.

Theorem 2.13 (Definable Tietze extension theorem, [1]). *Let A be a definable closed subset of R^n. Then every definable map $f : A \to R$ is extensible to a definable map $F : R^n \to R$ with $F|A = f$.*

3 Idea of proof of Theorem 1.1

A definable map $f : X \to Y$ is *definably closed* if for any definable closed subset A of X, $f(A)$ is a definable closed subset of Y.

Theorem 3.1 ([4]). *Let $f : X \to Y$ be a definable map. Then f is definably proper if and only if f is definably closed and has definably compact fibers.*

Idea of Proof of Theorem 1.1.
Using Theorem 2.11, 2.13, 3.1, we have the result. ∎

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