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| 論文 | 該研究は、トーティゼ拡張定理の対称性とモデル理論的独立性および次元についての観点 
| | を中心に、数理解析研究所講究録で発表された。
| 担当者 | 川上智博 |
| 出版 | 数理解析研究所講究録 |
Equivariant definable Tietze extension theorem

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Abstract
Let $G$ be a definably compact definable group, $X$ a definable $G$ set and $A$ a $G$ invariant definably compact definable subset of $X$. We prove that every $G$ invariant definable function $f : A \to R$ is extensible to a $G$ invariant definable function $F : X \to R$ with $F|A = f$.

1 Introduction

In this paper we consider equivariant definable Tietze extension theorem in an o-minimal expansion $\mathcal{N} = (R, +, \cdot, <, ...)$ of a real closed field $R$. It is known that there exist uncountably many o-minimal expansions of the field $\mathbb{R}$ of real numbers([7]).

Definable set and definable maps are studied in [2], [3], and see also [8]. Everything is considered in $\mathcal{N} = (R, +, \cdot, <, ...)$ and definable maps are assumed to be continuous unless otherwise stated.

Theorem 1.1 ([5]). Let $G$ be a definably compact definable group, $X$ a definable $G$ set and $A$ a $G$ invariant definably compact definable subset of $X$. Every $G$ invariant definable function $f : A \to R$ is extensible to a $G$ invariant definable function $F : X \to R$ with $F|A = f$.

2010 Mathematics Subject Classification. 14P10, 57S10, 03C64.
Key Words and Phrases. Tietze extension theorem, o-minimal, real closed fields.
2 Preliminaries

A subset $X$ of $R^n$ is definable (in $\mathcal{N}$) if it is defined by a formula (with parameters). Namely, there exist a formula $\phi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ and elements $b_1, \ldots, b_m \in R$ such that $X = \{(a_1, \ldots, a_n) \in R^n | \phi(a_1, \ldots, a_n, b_1, \ldots, b_m) \}$ is true in $\mathcal{N}$.

For any $-\infty \leq a < b \leq \infty$, an open interval $(a, b)_R$ means $\{x \in R | a < x < b\}$, for any $a, b \in R$ with $a < b$, a closed interval $[a, b]_R$ means $\{x \in R | a \leq x \leq b\}$. We call $\mathcal{N}$ o-minimal (order-minimal) if every definable subset of $R$ is a finite union of points and open intervals.

A real closed field $(R, +, \cdot, <)$ is an o-minimal structure and every definable set is a semialgebraic set [9], and a definable map is a semialgebraic map [9]. In particular, the semialgebraic category is a special case of a definable one.

The topology of $R$ is the interval topology and the topology of $R^n$ is the product topology. Note that $R^n$ is a Hausdorff space.

The field $\mathbb{R}$ of real numbers, $\mathbb{R}_{alg} = \{x \in \mathbb{R} | x$ is algebraic over $\mathbb{Q}\}$ are Archimedean real closed fields.

The Puiseux series $\mathbb{R}[X]^\wedge$, namely $\displaystyle \sum_{i=k}^{\infty} a_i X^\frac{i}{q}, k \in \mathbb{Z}, q \in \mathbb{N}, a_i \in \mathbb{R}$ is a non-Archimedean real closed field.

Fact 2.1. (1) The characteristic of a real closed field is 0.

(2) For any cardinality $\kappa \geq \aleph_0$, there exist $2^\kappa$ many non-isomorphic real closed fields whose cardinality is $\kappa$.

(3) In a general real closed field, even for a $C^\infty$ function, the intermediate value theorem, existence theorem of maximum and minimum, Rolle's theorem, the mean value theorem do not hold. Even for a $C^\infty$ function $f$ in one variable, the result that $f' > 0$ implies $f$ is increasing does not hold.

Definition 2.2. Let $X \subset R^n, Y \subset R^m$ be definable sets.

(1) A continuous map $f : X \rightarrow Y$ is a definable map if the graph of $f (\subset R^n \times R^m)$ is definable.

(2) A definable map $f : X \rightarrow Y$ is a definable homeomorphism if there exists a definable map $f' : Y \rightarrow X$ such that $f \circ f' = id_Y, f' \circ f = id_X$.

Definition 2.3. A group $G$ is a definable group if $G$ is definable and the group operations $G \times G \rightarrow G, G \rightarrow G$ are definable.

Let $G$ be a definable group. A pair $(X, \phi)$ consisting a definable set $X$ and a $G$ action $\phi : G \times X \rightarrow X$ is a definable $G$ set if $\phi$ is definable. We simply write $X$ instead of $(X, \phi)$.
Definition 2.4. Let $X,Y$ be definable $G$ sets.

1. A definable map $f : X \to Y$ is a definable $G$ map if for any $x \in X, g \in G$, $f(gx) = gf(x)$.

2. A definable $G$ map $f : X \to Y$ is a definable $G$ homeomorphism if there exists a definable $G$ map $h : Y \to X$ such that $f \circ h = id_Y$, $h \circ f = id_X$.

Definition 2.5. (1) A definable set $X \subset \mathbb{R}^n$ is definably compact if for any definable map $f : (a,b)_\mathbb{R} \to X$, there exist the limits $\lim_{x \to a+0} f(x), \lim_{x \to b-0} f(x)$ in $X$.

(2) A definable set $X \subset \mathbb{R}^n$ is definably connected if there exist no definable open subsets $U, V$ of $X$ such that $X = U \cup V, U \cap V = \emptyset, U \neq \emptyset, V \neq \emptyset$.

A compact (resp. A connected) definable set is definably compact (resp. definably connected). But a definably compact (resp. a definably connected) definable set is not always compact (resp. connected). For example, if $R = \mathbb{R}_{alg}$, then $[0,1]_{\mathbb{R}_{alg}} = \{x \in \mathbb{R}_{alg} | 0 \leq x \leq 1\}$ is definably compact and definably connected, but it is neither compact nor connected.

Theorem 2.6 ([6]). For a definable set $X \subset \mathbb{R}^n$, $X$ is definably compact if and only if $X$ is closed and bounded.

The following is a definable version of the fact that the image of a compact (resp. a connected) set by a continuous map is compact (resp. connected).

Proposition 2.7. Let $X \subset \mathbb{R}^n, Y \subset \mathbb{R}^m$ be definable set and $f : X \to Y$ a definable map. If $X$ is definably compact (resp. definably connected), then $f(X)$ is definably compact (resp. definably connected).

Theorem 2.8. (1) (The intermediate value theorem) For a definable function $f$ on a definably connected set $X$, if $a,b \in X, f(a) \neq f(b)$ then $f$ takes all values between $f(a)$ and $f(b)$.

(2) (Existence theorem of maximum and minimum) Every definable function on a definably compact definable set attains maximum and minimum.

(3) (Rolle's theorem) Let $f : [a,b]_\mathbb{R} \to \mathbb{R}$ be a definable function such that $f$ is differentiable on $(a,b)_\mathbb{R}$ and $f(a) = f(b)$. Then there exists $c$ between $a$ and $c$ with $f'(c) = 0$.

(4) (The mean value theorem) Let $f : [a,b]_\mathbb{R} \to \mathbb{R}$ be a definable function which is differentiable on $(a,b)_\mathbb{R}$. Then there exists $c$ between $a$ and $c$ with $f'(c) = \frac{f(b)-f(a)}{b-a}$.
(5) Let \( f : (a, b)_R \rightarrow R \) be a differentiable definable function. If \( f' > 0 \) on \( (a, b)_R \), then \( f \) is increasing.

**Example 2.9.** (1) Let \( \mathcal{N} = (\mathbb{R}_{alg}, +, \cdot, <) \). Then \( f : \mathbb{R}_{alg} \rightarrow \mathbb{R}_{alg}, f(x) = 2^x \) is not defined ([10]).

(2) Let \( \mathcal{N} = (\mathbb{R}, +, \cdot, <) \). Then \( f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = 2^x \) is defined but not definable in \( \mathcal{N} \), and \( h : \mathbb{R} \rightarrow \mathbb{R}, h(x) = \sin x \) is defined but not definable in \( \mathcal{N} \).

**Definition 2.10.** A definable map \( f : X \rightarrow Y \) is *definably proper* if for any definably compact subset \( C \) of \( Y \), \( f^{-1}(C) \) is definably compact.

**Theorem 2.11** (Existence of definable quotient). Let \( G \) be a definably compact definable group and \( X \) a definable \( G \) set. Then the orbit space \( X/G \) exists as a definable set, and the orbit map \( \pi : X \rightarrow X/G \) is definable, surjective and definably proper.

The following theorem is the topological case of Tietze extension theorem.

**Theorem 2.12** (Tietze extension theorem). Let \( X \) be a normal space and \( A \) a closed subset of \( X \). Then every continuous map \( f : A \rightarrow \mathbb{R} \) is extensible to a continuous map \( F : X \rightarrow \mathbb{R} \) with \( F|A = f \).

The following theorem is the definable case of Tietze extension theorem.

**Theorem 2.13** (Definable Tietze extension theorem, [1]). Let \( A \) be a definable closed subset of \( \mathbb{R}^n \). Then every definable map \( f : A \rightarrow \mathbb{R} \) is extensible to a definable map \( F : \mathbb{R}^n \rightarrow \mathbb{R} \) with \( F|A = f \).

### 3 Idea of proof of Theorem 1.1

A definable map \( f : X \rightarrow Y \) is *definably closed* if for any definable closed subset \( A \) of \( X \), \( f(A) \) is a definable closed subset of \( Y \).

**Theorem 3.1** ([4]). Let \( f : X \rightarrow Y \) be a definable map. Then \( f \) is definably proper if and only if \( f \) is definably closed and has definably compact fibers.

Idea of Proof of Theorem 1.1.

Using Theorem 2.11, 2.13, 3.1, we have the result. \( \blacksquare \)
References


