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# Elementary equivalence of seperably uniruled fields of transcendental degree one

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**Abstract**

Let $K/k$ be a function field over a number field or a finite field $\kappa$ and let $\mathrm{trdeg}(K/\kappa)=1$. Let $L$ be a function field over a number field or a finite field. We prove that $K\equiv L$ implies $K\cong L$.

## 1 Introduction

Pop(2002) raised the following question: *for finitely generated fields, elementary equivalence is the same as isomorphism?* Among others he showed the following:

*Let $K$ and $L$ be function fields over prime fields with $K\equiv L$. Then they have the common constant field $\kappa$ and*

1. *there are embeddings $K\to L$ and $L\to K$.*

2. *Furthermore, if one of them is of general type over $\kappa$ then they are $\kappa$-isomorphic.*

We say that $K/\kappa$ is of general type if it is the function field of a projective smooth variety over $\kappa$ of general type. It is known that smooth hypersurfaces of dimension $n$ with degree $d>n+2$ are of general type. Roughly speaking, almost all varieties are of general type. We note that rational function fields and elliptic function fields are not of general type.

Non-general case remained an open question. However Scanlon (2008) announced the affirmative answer by using biinterpretability of such fields., which turned out to be faulty (2011).

On the other hand, Pop(2009?) proved : *If $K$ is a function field of a curve over a constant number field $k$ (that is,* $\mathrm{trdeg}(K/k)=1$*), then elementary equivalence implies isomorphism.* Pop gave a recipe which describes uniformly the $k-$valuations of function fields $K/k$ in one variable over number fields $k$. This allows us to give sentences $\varphi_K$ in the language of rings which describe the isomorphy type of $K$ among finitely generated fields.

In this note we add a non-general instance of "elementary equivalence implies isomorphism".

## 2 Function fields over prime fields

Let $K$ be a finitely generated fields (over a prime field). We define the constant field $\kappa$ of $K$ to be algebraic closure of its prime field in $K$. By function fields over $\kappa$, we mean finitely generated fields over $\kappa$ of transcendence degree $> 0$ over $\kappa$.

Suppose $K \equiv L$. We know that

1. $K$ and $L$ have the same prime field $k$,
2. their constant fields are isomorhic,
3. $\mathrm{trdeg}(K/k) = \mathrm{trdeg}(L/k)$, and
4. there are field embeddings $\iota : K \to L$ and $\iota' : L \to K$.

## 3 Uniruled fields

**Definition 1** *Let $K$ be a function field over $k$.*

1. *$K$ is called ruled over $k$ if there is a subfield $\Delta$ of $K$ containing $k$ so that $K = \Delta(t)$ for some element $t \in K$.*
2. *$K$ is called (separably) uniruled over $k$ if there is a (separable) finite extension $L$ of $K$ so that $L$ is ruled over $k$.*

For seperably uniruled fields $K/k$ of $\mathrm{trdeg}(K/k) = 1$, we have

**Theorem 2** *Let $K/k$ be an extension of transcendental degree 1 with $k$ algebraically closed in $K$. Then*

*$K/k$ is separably uniruled iff there exists $x, y \in K$ and $a, b \in k$ such that $K = k(x, y)$ and*

$$x^2 - ay^2 = b \text{ if } char(k) \neq 2$$
$$x^2 + xy - ay^2 = b \text{ if } char(k) = 2$$

*Furthermore there is an element $c$ which is separably algebraic over $k$ and an element $t$ transcendental over $\kappa$ such that $K(c) = k(c, t)$.*

For the proof, see [Ohm].

# 4 Uniruled fields of $\mathrm{trdeg}(K/k) = 1$

Let $K$ be a finitely generated field over its prime field $k$ and let $\kappa$ be its constant field. Let $L$ be a finitely generated field over a prime field.

We first consider the case that $K$ is a rational field $\kappa(t)$.

**Proposition 3** *$\kappa(t) \equiv L$ implies $\kappa(t) \cong L$.*

*Proof.* We note that $\mathrm{trdeg}(K/k) = \mathrm{trdeg}(L/k)$ and the constant field $\kappa'$ of $L$ is isomorphic to $\kappa$.

There is a field embedding $\iota : L \to K$. Clearly, $\iota$ maps $\kappa'$ isomorphically onto $\kappa$. Thus $\iota(L)$ is a subfield of $\kappa(t)$ containing $\kappa$. By Lüoth's Theorem, $\iota(L)$ is isomorphic to $\kappa(t)$, hence so is $L$. □

**Theorem 4** *Let $\kappa$ be a finite field or a number field. Let $K$ be a separably uniruled field over $\kappa$ of $\mathrm{trdeg}(K/\kappa) = 1$. Suppose $K \equiv L$. Then $K \cong L$.*

*Proof.* There is an element $c$ which is separably algebraic over $\kappa$ and an element $t$ transcendental over $\kappa$ such that $K(c) = \kappa(c, t)$. Since $c$ is algebraic over $\kappa$, $K(c)$ is interpretable in $K$ and $L(c)$ is interpretable in $L$.

Therefore we have $K(c) \equiv L(c)$, where $K(c)$ is a rational field over $\kappa(c)$ which is a number field or a finite field. Hence we have $K(c) \cong L(c)$ by the previous proposition. Since $K$ and $\kappa(c)$ are linearly disjoint over $\kappa$ and so are $L$ and $\kappa(c)$, we have $K \cong L$. □

# References

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