数理解析研究所講究録
第 1947 巻 2015 年 205-222

Large time behavior of solutions toward a multiwave pattern for the Cauchy problem of the scalar conservation law with degenerate flux and viscosity

Natsumi Yoshida (Ritsumeikan University/Osaka City University)

1 Introduction and main theorem

We consider the asymptotic behavior in time of solutions to Cauchy problem for scalar viscous conservation law with nonlinearly degenerate viscosity

$$\begin{cases} \partial_t u+\partial_x(f(u))=\mu\,\partial_x\left(|\partial_x u|^{p-1}\partial_x u\right) & (t>0, x\in\mathbb{R}),\\ u(0,x)=u_0(x) & (x\in\mathbb{R}),\\ \lim_{x\to\pm\infty}u(t,x)=u_\pm & (t\geq 0)\end{cases} \tag{1.1}$$

for $p>1$. Here $u=u(t,x)$ is the unknown function of $t>0$ and $x\in\mathbb{R}$, so-called the conserved quantity, $f=f(u)$ is the flux function depending only on u, μ is the viscosity coefficient, u_0 is the given initial data, and $u_\pm\in\mathbb{R}$ are the prescribed far field states. We assume the flux $f=f(u)$ is a given C^1-function satisfying $f(0)=f'(0)=0$, μ is a positive constant and far field states $u_\pm$ satisfy $u_-<u_+$ without loss of generality.

We are interested in the asymptotic behavior of solutions to the problem (1.1). It is known that the large time behavior is closely related to the weak solution ("Riemann solution") of the corresponding Riemann problem (cf. [13], [27]) for the non-viscous hyperbolic part of (1.1):

$$\begin{cases} \partial_t u+\partial_x(f(u))=0 & (t>0, x\in\mathbb{R}),\\ u(0,x)=u_0^{\mathrm{R}}(x) & (x\in\mathbb{R}),\end{cases} \tag{1.2}$$

where u_0^{R} is the Riemann data defined by

$$u_0^{\mathrm{R}}(x)=u_0^{\mathrm{R}}(x\,;\,u_-,u_+):=\begin{cases} u_- & (x<0),\\ u_+ & (x>0).\end{cases}$$

In fact, for the usual linear viscosity ($p=1$) case

$$\begin{cases} \partial_t u+\partial_x(f(u))=\mu\,\partial_x^2 u & (t>0, x\in\mathbb{R}),\\ u(0,x)=u_0(x) & (x\in\mathbb{R}),\\ \lim_{x\to\pm\infty}u(t,x)=u_\pm & (t\geq 0),\end{cases} \tag{1.3}$$

when the smooth flux function f is genuinely nonlinear on the whole space $\mathbb{R}$, i.e., $f''(u)\neq 0$ $(u\in\mathbb{R})$, Il'in-Oleĭnik [9] showed the following: if $f''(u)>0$ $(u\in\mathbb{R})$, that is, the Riemann solution consists of a single rarefaction wave solution, the global solution in time of the Cauchy problem (1.3) tends toward the rarefaction wave; if $f''(u)<0$ $(u\in\mathbb{R})$, that is, the Riemann solution consists of a single shock wave solution, the global solution of the Cauchy problem (1.3) does the corresponding smooth traveling wave solution ("viscous shock wave") of (1.3) with a spacial shift (cf. [8]). More generally, in the case of the flux functions which are not uniformly genuinely nonlinear, when the Riemann solution consists of a single shock wave satisfying Oleĭnik's shock condition, Matsumura-Nishihara [18] showed the asymptotic stability of the corresponding viscous shock wave. However, when we consider the circumstances where the Riemann solution generically forms a pattern of multiple nonlinear waves which consists of rarefaction waves, shock waves and waves of contact discontinuity (refer to [13]), there had been no results about the asymptotics toward the multiwave pattern. Recently, Matsumura-Yoshida [19] proved the asymptotics toward a multiwave pattern of the superposition of the rarefaction waves and the wave of the contact discontinuity. Namely, they investigated the case where the flux function f is smooth and genuinely nonlinear (that is, f is convex function or concave function) on the whole $\mathbb{R}$ except a finite interval $I := (a,b) \subset \mathbb{R}$, and linearly degenerate on I, that is,

$$\begin{cases} f''(u) > 0 & \big(u \in (-\infty, a\,] \cup [\,b, +\infty)\big), \\ f''(u) = 0 & \big(u \in (a,b)\big). \end{cases} \tag{1.4}$$

For the flux function satisfying (1.4), the corresponding Riemann solution does form multiwave pattern which consists of the contact discontinuity with the jump from $u = a$ to $u = b$ and the rarefaction waves, depending on the choice of a, b, u_- and u_+. Thanks to that the cases in which the interval (a,b) is disjoint from the interval (u_-, u_+) are similar as in the case the flux function f is genuinely nonlinear on the whole space $\mathbb{R}$, and the case $u_- < a < u_+ < b$ is the same as that for $a < u_- < b < u_+$, we may only consider the typical cases

$$a < u_- < b < u_+ \quad \text{or} \quad u_- < a < b < u_+. \tag{1.5}$$

Under the conditions (1.4) and (1.5), they have shown the unique global solution in time to the Cauchy problem (1.3) tends uniformly in space toward the multiwave pattern of the combination of the viscous contact wave and the rarefaction waves as the time goes to infinity. It should be noted that the rarefaction wave which connects the far field states u_- and u_+ $\big(u_\pm \in (-\infty, a\,]$ or $u_\pm \in [\,b, \infty)\big)$ is explicitly given by

$$u = u^r\left(\frac{x}{t}\,;\, u_-, u_+\right) := \begin{cases} u_- & \big(x \le \lambda(u_-)\,t\big), \\ (\lambda)^{-1}\left(\dfrac{x}{t}\right) & \big(\lambda(u_-)\,t \le x \le \lambda(u_+)\,t\big), \\ u_+ & \big(x \ge \lambda(u_+)\,t\big), \end{cases} \tag{1.6}$$

where $\lambda(u) := f'(u)$, and the viscous contact wave which connects u_- and u_+ $(u_\pm \in [\,a, b\,])$ is given by an exact solution of the linear convective heat equation

$$\partial_t u + \tilde{\lambda}\,\partial_x u = \mu\,\partial_x^2 u \qquad \left(\tilde{\lambda} := \frac{f(b) - f(a)}{b - a},\ t > 0, x \in \mathbb{R}\right) \tag{1.7}$$

which has the form

$$u = U\left(\frac{x - \tilde{\lambda}\,t}{\sqrt{t}}\,;\, u_-, u_+\right)$$

where $U\left(\frac{x}{\sqrt{t}}\,;\, u_-, u_+\right)$ is explicitly defined by

$$U\left(\frac{x}{\sqrt{t}}\,;\, u_-, u_+\right) := u_- + \frac{u_+ - u_-}{\sqrt{\pi}} \int_{-\infty}^{\frac{x}{\sqrt{4\mu t}}} \mathrm{e}^{-\xi^2}\, \mathrm{d}\xi \quad (t > 0, x \in \mathbb{R}). \tag{1.8}$$

Yoshida [27] also obtained the precise decay properties for the asymptotics. In the proof of them, the *a priori* energy estimates acquired by an L^2-energy method and careful estimates for the terms of nonlinear interactions of the viscous contact wave and the rarefaction waves.

In this article, we shall extend the results in the previous study in [19] to the case where the viscosity is the p-Laplacian type one (the related problems are studied in [4], [20], [21] and so on). For this case, a main difficulty arises from the fact that when $u_\pm \in [\,a, b\,]$, the asymptotic state is expected to be a self-similar type solution of a nonlinearly degenerate convective heat equation which may need the more subtle treatment than the Gaussian kernel type one (1.8) of the equation (1.7). There is only one result for the asymptotic behavior for the problem (1.1) in the case where the flux function is genuinely nonlinear on the whole space $\mathbb{R}$. Namely, Matsumura-Nishihara [17] proved the asymptotics which tends toward a single rarefaction wave by using the L^2 and L^p-energy estimates. We then consider the case where the flux function is given as (1.4) and the far field states as (1.5). We expect the asymptotic behavior of solutions to the Cauchy problem (1.1) to be similar as in [19]. In more detail, under the conditions (1.4) and (1.5), if the farfield states $u_\pm$ satisfy $u_\pm \in (-\infty, a\,]$ or $u_\pm \in [\,b, \infty)$, the asymptotic state of the solutions to the Cauchy problem (1.1) should be the rarefaction wave (1.6) which connects u_- and u_+, and if the far field states $u_\pm$ satisfy $u_\pm \in [\,a, b\,]$, the one should be the "contact wave for p-Laplacian type viscosity" which connects u_- and u_+, which is given by an exact solution

$$U\left(\frac{x-\tilde{\lambda}\,t}{t^{\frac{1}{p+1}}}\,;\,u_-,u_+\right) = u_- + \int_{-\infty}^{\frac{x-\tilde{\lambda}\,t}{t^{\frac{1}{p+1}}}} \Big(\big(A - B\xi^2\big)\vee 0\Big)^{\frac{1}{p-1}}\,\mathrm{d}\xi \tag{1.9}$$

$$\left(A, B > 0\,;\ \int_{-\infty}^{\infty}\big((A - B\,\xi^2)\vee 0\big)^{\frac{1}{p-1}}\,\mathrm{d}\xi = u_+ - u_-,\quad \tilde{\lambda} := \frac{f(b)-f(a)}{b-a}\right)$$

of the following p-Laplacian evolution equation

$$\partial_t u + \tilde{\lambda}\,\partial_x u = \mu\,\partial_x\Big(\,|\,\partial_x u\,|^{p-1}\,\partial_x u\Big) \qquad \left(\tilde{\lambda} := \frac{f(b)-f(a)}{b-a},\ t>0, x\in\mathbb{R}\right). \tag{1.10}$$

Here, the viscous contact wave U is constructed by the Barenblatt-Kompanceec-Zel'dovič solution ([1], [28], [24])

$$v(t,x) := \frac{1}{(1+t)^{\frac{1}{p+1}}}\left(\left(A - B\left(\frac{x}{(1+t)^{\frac{1}{p+1}}}\right)^2\right)\vee 0\right)^{\frac{1}{p-1}} \tag{1.11}$$

of the following Cauchy problem of the porous medium equation (cf. [2], [7], [10])

$$\begin{cases} \partial_t v = \mu\,\partial_x^2\Big(\,|\,v\,|^{p-1}\,v\Big) & (t>-1, x\in\mathbb{R}),\\ v(-1,x) = (u_+ - u_-)\,\delta(x) & (x\in\mathbb{R}\,;\,u_- < u_+),\\ \lim\limits_{x\to\pm\infty} v(t,x) = 0 & (t\geq -1) \end{cases} \tag{1.12}$$

where $\delta(x)$ is the Dirac δ-distribution.

Theorem 1.1. *Let the flux function f satisfy* (1.4) *and the far field states $u_\pm$* (1.5). *Assume that the initial data satisfies $u_0 - u_0^{\mathrm{R}} \in L^2$ and $\partial_x u_0 \in L^{p+1}$. Then the Cauchy problem* (1.1) *with $p>1$ has a unique global solution in time $u = u(t,x)$ satisfying*

$$\begin{cases} u - u_0^{\mathrm{R}} \in C^0\big([\,0,\infty)\,;L^2\big)\cap L^\infty\big(\mathbb{R}^+\,;L^2\big),\\ \partial_x u \in L^\infty\big(\mathbb{R}^+\,;L^{p+1}\big),\\ \partial_t u \in L^\infty\big(\mathbb{R}^+\,;L^{p+1}\big),\\ \partial_x\Big(\,|\,\partial_x u\,|^{p-1}\,\partial_x u\Big) \in L^2\big(\mathbb{R}_t^+ \times \mathbb{R}_x\big), \end{cases}$$

and the asymptotic behavior

$$\lim_{t\to\infty}\sup_{x\in\mathbb{R}}|\,u(t,x) - U_{multi}(\,t,x\,;\,u_-,u_+)\,| = 0,$$

where $U_{multi}(t,x) = U_{multi}(\,t,x;\,u_-,u_+)$ is defined as follows: in the case $a < u_- < b < u_+$,

$$U_{multi}(t,x) := U\left(\frac{x-\tilde{\lambda}\,t}{t^{\frac{1}{p+1}}}\,;\,u_-,b\right) + u^r\left(\frac{x}{t}\,;\,b,u_+\right) - b$$

and, in the case $u_- < a < b < u_+$,

$$U_{multi}(t,x) := u^r\left(\frac{x}{t}\,;\,u_-,a\right) - a + U\left(\frac{x-\tilde{\lambda}t}{t^{\frac{1}{p+1}}}\,;\,a,b\right) + u^r\left(\frac{x}{t}\,;\,b,u_+\right) - b.$$

The proof is given by a technical energy methods and the careful estimates for the interactions between the nonlinear waves. Important are the *a priori* uniform estimates (cf. [17], [19], [27]).

2 Preliminaries

In this section, we shall arrange the several lemmas concerning with the basic properties of the rarefaction wave and the viscous contact wave for accomplishing the proof of the main theorem. Since the rarefaction wave u^r is not smooth enough, we need some smooth approximated one U^r (cf. [5], [15], [16], [19]). In fact, we have the following results on U^r.

Lemma 2.1. *Assume that the far field states satisfy $u_- < u_+$, and the flux fanction $f \in C^3(\mathbb{R})$, $f''(u) > 0$ $\big(u \in [u_-, u_+]\big)$. Then we have the following properties:*
(1) *$U^r(t,x)$ is the unique C^2-global solution in space-time of the Cauchy problem*

$$\begin{cases} \partial_t U^r + \partial_x\big(f(U^r)\big) = 0 & (t>0, x\in\mathbb{R}),\\ U^r(0,x) = (\lambda)^{-1}\left(\dfrac{\lambda_- + \lambda_+}{2} + \dfrac{\lambda_+ - \lambda_-}{2}\tanh x\right) & (x\in\mathbb{R}),\\ \lim\limits_{x\to\pm\infty} U^r(t,x) = u_\pm & (t\ge 0). \end{cases}$$

(2) $u_- < U^r(t,x) < u_+$ *and* $\partial_x U^r(t,x) > 0 \quad (t>0, x\in\mathbb{R})$.
(3) *For any $1\le q\le\infty$, there exists a positive constant C_q such that*

$$\begin{aligned} &\|\,\partial_x U^r(t)\,\|_{L^q} \le C_q(1+t)^{-1+\frac{1}{q}} \quad (t\ge 0),\\ &\|\,\partial_x^2 U^r(t)\,\|_{L^q} \le C_q(1+t)^{-1} \qquad (t\ge 0). \end{aligned}$$

(4) $\lim\limits_{t\to\infty}\sup\limits_{x\in\mathbb{R}}\left|U^r(t,x) - u^r\left(\frac{x}{t}\right)\right| = 0.$
(5) *For any $\epsilon\in(0,1)$, there exists a positive constant C_ϵ such that*

$$|\,U^r(t,x) - u_+\,| \le C_\epsilon(1+t)^{-1+\epsilon}e^{-\epsilon|x-\lambda_+ t|} \quad (t\ge 0, x\ge\lambda_+ t).$$

(6) *For any $\epsilon\in(0,1)$, there exists a positive constant C_ϵ such that*

$$|\,U^r(t,x) - u_-\,| \le C_\epsilon(1+t)^{-1+\epsilon}e^{-\epsilon|x-\lambda_- t|} \quad (t\ge 0, x\le\lambda_- t).$$

(7) *For any $\epsilon\in(0,1)$, there exists a positive constant C_ϵ such that*

$$\left|U^r(t,x) - u^r\left(\frac{x}{t}\right)\right| \le C_\epsilon(1+t)^{-1+\epsilon} \qquad (t\ge 1, \lambda_- t\le x\le\lambda_+ t).$$

(8) *For any $(\epsilon,q)\in(0,1)\times[1,\infty]$, there exists a positive constant $C_{\epsilon,q}$ such that*

$$\left\|U^r(t,\cdot) - u^r\left(\frac{\cdot}{t}\right)\right\|_{L^q} \le C_{\epsilon,q}(1+t)^{-1+\frac{1}{q}+\epsilon} \qquad (t\ge 0).$$

We also prepare the next lemma for the properties of the contact wave for p-Laplacian type viscosity $U\left(\frac{x}{t^{\frac{1}{p+1}}}\,;\,u_-,u_+\right)$ defined by (1.11) (in the following, we abbreviate "contact wave for p-Laplacian type viscosity" to "viscous contact wave").

Lemma 2.2. *For any $p > 1$ and $u_\pm \in \mathbb{R}$, we have the following:*
(i) *U defined by* (1.11) *satisfies*

$$U \in \mathscr{B}^1\big((0,\infty)\times\mathbb{R}\big) \Big\backslash C^2\left(\left\{ (t,x)\in\mathbb{R}^+\times\mathbb{R} \,\middle|\, x = \pm\sqrt{\frac{A}{B}}\, t^{\frac{1}{p+1}} \right\}\right),$$

and is a self-similar type strong solution of the Cauchy problem

$$\begin{cases} \partial_t U - \mu\,\partial_x\left(|\,\partial_x U\,|^{p-1}\,\partial_x U\right) = 0 & (t>0, x\in\mathbb{R}),\\ U(0,x) = u_0^{\mathrm{R}}(x\,;\,u_-,u_+) = \begin{cases} u_- \\ u_+ \end{cases} & \begin{matrix}(x<0),\\ (x>0),\end{matrix} \\ \lim\limits_{x\to\pm\infty} U(t,x) = u_\pm & (t\geq 0). \end{cases}$$

(ii) *For $t > 0$ and $x \in \mathbb{R}$,*

$$\begin{cases} U(t,x) = u_-, & \left(x \leq -\sqrt{\frac{A}{B}}\, t^{\frac{1}{p+1}}\right), \\ u_- < U(t,x) < u_+,\ \partial_x U(t,x) > 0 & \left(-\sqrt{\frac{A}{B}}\, t^{\frac{1}{p+1}} < x < \sqrt{\frac{A}{B}}\, t^{\frac{1}{p+1}}\right), \\ U(t,x) = u_+, & \left(x \geq \sqrt{\frac{A}{B}}\, t^{\frac{1}{p+1}}\right). \end{cases}$$

(iii) *It holds that for any $1 \leq q < \infty$,*

$$\|\,\partial_x U(t)\,\|_{L^q} = C_1(\,A,B\,;\,p,q\,)\,t^{-\frac{q-1}{(p+1)q}} \qquad (t>0)$$

where

$$C_1(\,A,B\,;\,p,q\,) := \left(2A^{\frac{p+2q-1}{2(p-1)}}\,B^{-\frac{1}{2}} \int_0^{\frac{\pi}{2}} (\sin\theta)^{\frac{q}{p-1}}\,\mathrm{d}\theta \right)^{\frac{1}{q}}.$$

If $q = \infty$, we have

$$\|\,\partial_x U(t)\,\|_{L^\infty} = (2A)^{\frac{1}{p-1}}\ t^{-\frac{1}{p+1}} \qquad (t>0).$$

(iv) *It holds that for any $1 \leq q < \frac{p-1}{p-2}$ with $p > 2$, or any $1 \leq q < \infty$ with $1 < p \leq 2$,*

$$\|\,\partial_x^2 U(t)\,\|_{L^q} = C_2(\,A,B\,;\,p,q\,)\,t^{-\frac{2q-1}{(p+1)q}} \qquad (t>0)$$

where

$$C_2(\,A,B\,;\,p,q\,) := \left(2\left(\frac{2A^{-\frac{p-2}{p-1}}B}{p-1}\right)^q \left(\frac{B}{A}\right)^{-\frac{q+1}{2}} \int_0^{\frac{\pi}{2}} (\sin\theta)^{-\frac{2(p-2)q}{p-1}+1} (\cos\theta)^q\,\mathrm{d}\theta \right)^{\frac{1}{q}}.$$

If $1 < p \leq 2$, for $q = \infty$, we have

$$\|\,\partial_x^2 U(t)\,\|_{L^\infty} = \frac{2A^{\frac{|p-2|}{p-1}}B}{p-1}\left(\frac{B}{A}\right)^{-\frac{1}{2}}\ t^{-\frac{2}{p+1}} \qquad (t>0).$$

(v) *It holds that*

$$\left\|\partial_x\left(|\,\partial_x U\,|^{p-1}\,\partial_x U\right)(t)\right\|_{L^2} = C_3(\,A,B\,;\,p\,)\,t^{-\frac{2p+1}{2(p+1)}} \qquad (t>0)$$

where

$$C_3(\,A,B\,;\,p\,) := \left(2\left(\frac{2B^p}{p-1}\right)^2 \left(\frac{B}{A}\right)^{-\frac{3p-7}{2(p-1)}} \int_0^{\frac{\pi}{2}} (\sin\theta)^{\frac{p+3}{p-1}} (\cos\theta)^2\,\mathrm{d}\theta \right)^{\frac{1}{2}}.$$

(vi) $\lim\limits_{t\to\infty}\sup\limits_{x\in\mathbb{R}} \big|\,U(1+t,x) - U(t,x)\,\big| = 0.$

3 Reformulation of the problem

In this section, we reduce our Cauchy problem (1.1) to a simpler case and reformulate the problem in terms of the deviation from the asymptotic state (the same as in [19], [27]). At first, without loss of generality, we shall consider the case where $a<0$, $b=0$ and the flux function $f(u)$ satisfies

$$\begin{cases} f''(u)>0 & (u\in(-\infty,a]\cup[0,+\infty)), \\ f(u)=0 & (u\in(a,0)), \end{cases} \tag{3.1}$$

under changing the variables and constant as $x-\tilde{\lambda}\,t\mapsto x$, $u-b\mapsto u$, $f(u+b)-f'(b)\,u-f(a)\mapsto f(u)$ and $a-b\mapsto a$ in this order. For the far field states $u_\pm\in\mathbb{R}$, we only deal with the typical case $a<u_-<0<u_+$ for simplicity, since the case $u_-<a<0<u_+$ can be treated technically in the same way of the proof as $a<u_-<0<u_+$. Indeed, in the case $u_-<a<0<u_+$, as we shall see in Section 4 and Section 5, there appears the extra nonlinear interaction terms between two rarefaction waves $u^r(\frac{x}{t}\,;\,u_-,a)$ and $u^r(\frac{x}{t}\,;\,0,u_+)$ with $\lambda(a)=\lambda(0)=0$ in the remainder term of the viscous conservation law for the asymptotics U_{multi} (see the right-hand side of (3.5)). These terms can be handled in much easier way by Lemma 2.1 than that for other essential nonlinear interaction terms between the rarefaction and the viscous contact waves. Furthermore, we should point out that the problem under the assumptions for the flux function (3.1) and the far field states $a<u_-<0<u_+$ is essentially the same as that for $a=-\infty$, because obtaining the a priori and the uniform energy estimates for the former one can be given in almost the same way as the latter one. Therefore, it is quite natural for us to treat only a simple case

$$\begin{cases} f''(u)>0 & (u\in[0,\infty)), \\ f(u)=0 & (u\in(-\infty,0)), \end{cases} \tag{3.2}$$

and assume $u_-<0<u_+$.

We first should note by Lemma 2.1 and Lemma 2.2, the asymptotic state U_{multi} can be replaced by a following approximated one

$$\tilde{U}(t,x):=U(1+t,x)+U^r(t,x) \tag{3.3}$$

where

$$U(1+t,x)=U\left(\frac{x}{(1+t)^{\frac{1}{p+1}}}\,;\,u_-,0\right),\quad U^r(t,x)=U^r(\,t,x\,;\,0,u_+).$$

This is because, from Lemma 2.1 and Lemma 2.2,

$$\begin{aligned}\sup_{x\in\mathbb{R}}\left|\,\tilde{U}(t,x)-U_{multi}(t,x)\,\right| &\le \sup_{x\in\mathbb{R}}\left|\,U(1+t,x)-U(t,x)\,\right| \\ &+\sup_{x\in\mathbb{R}}\left|\,U^r(t,x)-u^r\left(\frac{x}{t}\right)\,\right|\to 0\quad(t\to\infty).\end{aligned}$$

In the following, we write $U(1+t,x)$ again $U(t,x)$ for simplicity. Then it is noted that $\tilde{U}$ approximately satisfies the equation of (1.1) as

$$\partial_t\tilde{U}+\partial_x(f(\tilde{U}))-\mu\,\partial_x\left(\,|\,\partial_x\tilde{U}\,|^{p-1}\partial_x\tilde{U}\,\right)=-F_p(U,U^r), \tag{3.4}$$

where the remainder term $F_p(U,U^r)$ is explicitly given by

$$\begin{aligned} F_p(U,U^r):=&\widetilde{F_p}(U,U^r) \\ &+\mu\,\partial_x\left(\,|\,\partial_xU+\partial_xU^r\,|^{p-1}(\,\partial_xU+\partial_xU^r\,)-|\,\partial_xU\,|^{p-1}\,\partial_xU\right) \\ :=&-\left(\,f'(U+U^r)-f'(U^r)\,\right)\partial_xU^r-f'(U+U^r)\,\partial_xU \\ &+\mu\,\partial_x\left(\,|\,\partial_xU+\partial_xU^r\,|^{p-1}(\,\partial_xU+\partial_xU^r\,)-|\,\partial_xU\,|^{p-1}\,\partial_xU\right)\end{aligned} \tag{3.5}$$

which consists of the interaction terms of the viscous contact wave U and the approximation of the rarefaction wave U^r, and the approximation error of U^r as solution to the conservation law for the p-Laplacian type viscosity. Here we should note that U is monotonically nondecreasing and U^r is monotonically increasing, that is, $\partial_x \tilde{U}(t,x) > 0$ $\big(t \geq 0, x \in \mathbb{R}\big)$ which is frequently used hereinafter. Now putting

$$u(t,x) = \tilde{U}(t,x) + \phi(t,x) \tag{3.6}$$

and using (3.5), we can reformulate the problem (1.1) in terms of the deviation ϕ from $\tilde{U}$ as

$$\begin{cases} \partial_t \phi + \partial_x \Big(f(\tilde{U}+\phi) - f(\tilde{U}) \Big) \\ \quad -\mu\, \partial_x \Big(\,|\, \partial_x \tilde{U} + \partial_x \phi \,|^{p-1} (\partial_x \tilde{U} + \partial_x \phi) - |\, \partial_x \tilde{U} \,|^{p-1} \partial_x \tilde{U} \Big) \\ \qquad\qquad\qquad\qquad = F_p(U, U^r) \qquad (t > 0, x \in \mathbb{R}), \\ \phi(0,x) = \phi_0(x) := u_0(x) - \tilde{U}(0,x) \qquad\qquad (x \in \mathbb{R}). \end{cases} \tag{3.7}$$

Then we look for the unique global solution in time ϕ which has the asymptotic behavior

$$\sup_{x \in \mathbb{R}} |\, \phi(t,x) \,| \xrightarrow[t \to \infty]{} 0.$$

Here we note the fact $\phi_0 \in L^2$ and $\partial_x \phi_0 \in L^{p+1}$ by the assumptions on u_0 and the fact

$$\partial_x \tilde{U}(0,\cdot\,) = \partial_x U(0,\cdot\,) + \partial_x U^r(0,\cdot\,) \in L^{p+1}.$$

In the following, we always assume that the flux function $f \in C^1(\mathbb{R}) \cap C^3(\,[0,\infty))$ satisfies (3.2), and the far field states satisfy $u_- < 0 < u_+$. Then the corresponding our main theorem for ϕ we should prove is as follows.

Theorem 3.1. *Suppose $\phi_0 \in L^2$ and $\partial_x \phi_0 \in L^{p+1}$. Then there exists the unique global solution in time $\phi = \phi(t,x)$ of the Cauchy problem* (3.7) *satisfying*

$$\begin{cases} \phi \in C^0(\,[0,\infty)\,; L^2) \cap L^\infty(\mathbb{R}^+\,; L^2), \\ \partial_x \phi \in L^\infty(\mathbb{R}^+\,; L^{p+1}) \cap L^{p+1}(\mathbb{R}_t^+ \times \mathbb{R}_x), \\ \partial_x(\tilde{U}+\phi) \in L^\infty(\mathbb{R}^+\,; L^{p+1}) \cap L^{p+2}(\mathbb{R}_t^+ \times \{x \in \mathbb{R} \,|\, u > 0\}), \\ \partial_t(\tilde{U}+\phi) \in L^\infty(\mathbb{R}^+\,; L^{p+1}), \\ \partial_x \Big(\,|\, \partial_x(\tilde{U}+\phi) \,|^{p-1} \partial_x(\tilde{U}+\phi) \Big) \in L^2(\mathbb{R}_t^+ \times \mathbb{R}_x), \end{cases}$$

and the asymptotic behavior

$$\lim_{t \to \infty} \sup_{x \in \mathbb{R}} |\, \phi(t,x) \,| = 0.$$

In order to show the desired asymptotics, we show the following *a priori* estimates which are independent of T in the next sections.

Proposition 3.1 (uniform estimates I). *For any initial data $\phi_0 \in L^2$ and $\partial_x \phi_0 \in L^{p+1}$, there exists a positive constant*

$$C_p(\phi_0) = C\big(\,\|\phi_0\|_{L^2} \big)$$

such that

$$\begin{aligned} &\| \,\phi(t)\, \|_{L^2}^2 + \int_0^t G(\tau)\, \mathrm{d}\tau \\ &+ \int_0^t \int_{-\infty}^{\infty} (\partial_x \phi)^2 \left(|\partial_x \phi|^{p-1} + |\partial_x U|^{p-1} + |\partial_x U^r|^{p-1} \right) \mathrm{d}x \mathrm{d}\tau \leq C_p(\phi_0, \partial_x \phi_0) \end{aligned} \tag{3.8}$$

for $t \geq 0$, where

$$G(t) := \left(\int_{\tilde{U} \geq 0} \phi^2 \, \partial_x \tilde{U} \, \mathrm{d}x \right)(t) + \left(\int_{\tilde{U}+\phi \geq 0, \tilde{U} < 0} \left(\tilde{U} + \phi \right)^2 \partial_x \tilde{U} \, \mathrm{d}x \right)(t)$$
$$+ \left(\int_{\tilde{U}+\phi < 0, \tilde{U} \geq 0} \left(\tilde{U} + |\, \phi \,| \right)^2 \partial_x \tilde{U} \, \mathrm{d}x \right)(t).$$

Furthermore, we have the L^{p+1}-energy estimate for $\partial_x u$ as follows.

Proposition 3.2 (uniform estimates II). *For any initial data $\phi_0 \in L^2$ and $\partial_x \phi_0 \in L^{p+1}$, there exists a positive constant*

$$C_p(\phi_0, \partial_x u_0) = C\big(\|\phi_0\|_{L^2}, \ \|\partial_x u_0\|_{L^{p+1}} \big)$$

such that for $t \geq 0$,

$$\begin{aligned} &\|\, \partial_x u(t) \,\|_{L^{p+1}}^{p+1} + \int_0^t \int_{-\infty}^{\infty} |\, \partial_x u \,|^{2(p-1)} \left(\partial_x^2 u \right)^2 \, \mathrm{d}x \mathrm{d}\tau \\ &\qquad + \int_0^t \|\, \partial_x u(\tau) \,\|_{L^{p+2}(\{x \in \mathbb{R} \,|\, u > 0\})}^{p+2} \, \mathrm{d}\tau \leq C_p(\phi_0, \partial_x u_0). \end{aligned} \tag{3.9}$$

4 Uniform estimates I

In this section, we show the basic uniform energy estimates with $p > 1$ which is not depending on T, that is, Proposition 3.1. Now let us rewrite the basic L^2-energy inequality, that is Proposition 3.1 (uniform estimates I):

$$\begin{aligned} &\|\, \phi(t) \,\|_{L^2}^2 + \int_0^t G(\tau) \, \mathrm{d}\tau \\ &+ \int_0^t \int_{-\infty}^{\infty} \left(\partial_x \phi \right)^2 \left(|\partial_x \phi|^{p-1} + |\partial_x U|^{p-1} + |\partial_x U^r|^{p-1} \right) \mathrm{d}x \mathrm{d}\tau \leq C_p(\phi_0) \end{aligned} \tag{4.1}$$

for $t \geq 0$, where $G(t)$ is defined as in Proposition 3.1. The proof of (4.1) is given by the following two lemmas.

Lemma 4.1. *It holds that for $t \geq 0$,*

$$\begin{aligned} &\|\, \phi(t) \,\|_{L^2}^2 + \int_0^t G(\tau) \, \mathrm{d}\tau \\ &\quad + \int_0^t \int_{-\infty}^{\infty} \left(\partial_x \phi \right)^2 \left(|\partial_x \phi|^{p-1} + |\partial_x U|^{p-1} + |\partial_x U^r|^{p-1} \right) \mathrm{d}x \mathrm{d}\tau \\ &\leq C_p \|\, \phi_0 \,\|_{L^2}^2 + C_p \int_0^t \left(\|\, \phi(\tau) \,\|_{L^2}^2 + 1 \right) \left| \int_{-\infty}^{\infty} \left| \widetilde{F_p}(U, U^r) \right| \mathrm{d}x \right|^{\frac{3p+1}{3p}} (\tau) \, \mathrm{d}\tau \\ &\quad + C_p \int_0^t \left| \int_{-\infty}^{\infty} \left(\partial_x U + \partial_x U^r \right)^{p-1} \left(\partial_x U^r \right)^2 \mathrm{d}x \right| (\tau) \, \mathrm{d}\tau. \end{aligned}$$

Lemma 4.2. *It holds that*

$$\int_0^{\infty} \left| \int_{-\infty}^{\infty} \left| \widetilde{F_p}(U, U^r) \right| \mathrm{d}x \right|^{\frac{3p+1}{3p}} (t) \, \mathrm{d}t < \infty,$$

$$\int_0^\infty \left| \int_{-\infty}^\infty (\,\partial_x U + \partial_x U^r\,)^{p-1} (\,\partial_x U^r\,)^2 \,\mathrm{d}x \right| (t)\,\mathrm{d}t < \infty.$$

Once Lemma 4.1 and Lemma 4.2 are proved, by Gronwall's inequality, we have the uniform boundedness

$$\begin{aligned} \|\,\phi(t)\,\|_{L^2}^2 \leq & C_p(\,\|\,\phi_0\,\|_{L^2}^2 + 1\,) \\ & \times \exp\left\{ \int_0^\infty \left| \int_{-\infty}^\infty \left| \widetilde{F_p}(U, U^r) \right| \mathrm{d}x \right|^{\frac{3p+1}{3p}} \mathrm{d}t \right\} < \infty \end{aligned}$$

which easily implies (4.2), that is, Proposition 3.1.

Proof of Lemma 4.1. For $p > 1$, multiplying the equation in (3.7) by ϕ and integrating it with respect to x and t, we have

$$\begin{aligned} & \frac{1}{2}\,\|\,\phi(t)\,\|_{L^2}^2 + \int_0^t \int_{-\infty}^\infty \Big(f(\tilde{U} + \phi) - f(\tilde{U}) - f'(\tilde{U})\,\phi \Big)\, \partial_x \tilde{U} \,\mathrm{d}x\mathrm{d}\tau \\ & + \mu \int_0^t \int_{-\infty}^\infty (\,\partial_x \phi\,) \Big(\,|\,\partial_x \tilde{U} + \partial_x \phi\,|^{p-1} (\,\partial_x \tilde{U} + \partial_x \phi\,) - |\,\partial_x \tilde{U}\,|^{p-1} \partial_x \tilde{U} \Big) \,\mathrm{d}x\mathrm{d}\tau \qquad (4.2) \\ & = \frac{1}{2}\,\|\,\phi_0\,\|_{L^2}^2 + \int_0^t \int_{-\infty}^\infty \phi\ F_p(U, U^r)\,\mathrm{d}x\mathrm{d}\tau. \end{aligned}$$

To estimate the second term in the left-hand side of (4.2), noting the shape of the flux function f, we divide the integral domain of x depending on the signs of $\tilde{U} + \phi$, $\tilde{U}$ and ϕ as

$$\begin{aligned} & \int_{-\infty}^\infty \Big(f(\tilde{U} + \phi) - f(\tilde{U}) - f'(\tilde{U})\,\phi \Big)\, \partial_x \tilde{U} \,\mathrm{d}x \\ & = \int_{-\infty}^\infty \left(\int_0^\phi \Big(\lambda(\tilde{U} + \eta) - \lambda(\tilde{U}) \Big)\, \mathrm{d}\eta \right) (\,\partial_x \tilde{U}\,)\,\mathrm{d}x \qquad (4.3) \\ & = \int_{\tilde{U}+\phi \geq 0, \tilde{U} \geq 0, \phi \geq 0} + \int_{\tilde{U}+\phi \geq 0, \tilde{U} \geq 0, \phi \leq 0} + \int_{\tilde{U}+\phi \geq 0, \tilde{U} < 0} + \int_{\tilde{U}+\phi < 0, \tilde{U} \geq 0} \end{aligned}$$

where we used the fact that the integral is clearly zero on the domain $\tilde{U} + \phi \leq 0$ and $\tilde{U} \leq 0$. By Lagrange's mean-value theorem, we easily get as

$$\left(\int_{-\infty}^\infty \left(\int_0^\phi \Big(\lambda(\tilde{U} + \eta) - \lambda(\tilde{U}) \Big)\, \mathrm{d}\eta \right) (\,\partial_x \tilde{U}\,)\,\mathrm{d}x \right) (t) \sim G(t) \qquad (4.4)$$

where $G = G(t)$ is defined in Proposition 3.1 (cf. [19], [27]). Next, we also estimate the third term in the left-hand side of (4.2) as

$$\begin{aligned} & \int_{-\infty}^\infty (\,\partial_x \phi\,) \Big(\,|\,\partial_x \tilde{U} + \partial_x \phi\,|^{p-1} (\,\partial_x \tilde{U} + \partial_x \phi\,) - |\,\partial_x \tilde{U}\,|^{p-1} \partial_x \tilde{U} \Big) \,\mathrm{d}x \\ & \geq \nu_p^{-1} \int_{-\infty}^\infty (\,\partial_x \phi\,)^2 \Big(|\partial_x \phi|^{p-1} + |\partial_x U|^{p-1} + |\partial_x U^r|^{p-1} \Big) \,\mathrm{d}x \end{aligned} \qquad (4.5)$$

for some constant $\nu_p > 0$ which is depend only on p. Furthermore, we should note

$$\begin{aligned} \left| \int_{-\infty}^\infty \phi\, F_p(U, U^r)\,\mathrm{d}x \right| \leq & \int_{-\infty}^\infty |\,\phi\,| \left| \widetilde{F_p}(U, U^r) \right| \mathrm{d}x \\ & + \mu \int_{-\infty}^\infty |\,\partial_x \phi\,| \left((\,\partial_x U + \partial_x U^r\,)^p - (\,\partial_x U\,)^p \right) \mathrm{d}x. \end{aligned} \qquad (4.6)$$

Substituting (4.4), (4.5) and (4.6) into (4.2), we get the energy inequality

$$
\begin{aligned}
&\frac{1}{2}\|\phi(t)\|_{L^2}^2 + C_p^{-1}\int_0^t G(\tau)\,\mathrm{d}\tau \\
&\quad + \mu\nu_p^{-1}\int_0^t\int_{-\infty}^{\infty}(\partial_x\phi)^2\left(|\partial_x\phi|^{p-1} + |\partial_x U|^{p-1} + |\partial_x U^r|^{p-1}\right)\mathrm{d}x\mathrm{d}\tau \\
&\leq \frac{1}{2}\|\phi_0\|_{L^2}^2 + \int_0^t\int_{-\infty}^{\infty}|\phi|\left|\widetilde{F_p}(U,U^r)\right|\mathrm{d}x\mathrm{d}\tau \\
&\quad + \mu\int_0^t\int_{-\infty}^{\infty}|\partial_x\phi|\left((\partial_x U + \partial_x U^r)^p - (\partial_x U)^p\right)\mathrm{d}x\mathrm{d}\tau.
\end{aligned}
\tag{4.7}
$$

We estimate the second term in the right-hand side of (4.7) as follows:

$$
\begin{aligned}
&\int_{-\infty}^{\infty}|\phi|\left|\widetilde{F_p}(U,U^r)\right|\mathrm{d}x \\
&\leq C_p\|\phi\|_{L^2}^{\frac{2p}{3p+1}}\|\partial_x\phi\|_{L^{p+1}}^{\frac{p+1}{3p+1}}\int_{-\infty}^{\infty}\left|\widetilde{F_p}(U,U^r)\right|\mathrm{d}x \\
&\leq \frac{\mu}{4\nu_p}\|\partial_x\phi\|_{L^{p+1}}^{p+1} + C_p\|\phi\|_{L^2}^{\frac{2}{3}}\left(\int_{-\infty}^{\infty}\left|\widetilde{F_p}(U,U^r)\right|\mathrm{d}x\right)^{\frac{3p+1}{3p}} \\
&\leq \frac{\mu}{4\nu_p}\|\partial_x\phi\|_{L^{p+1}}^{p+1} + C_p(\|\phi\|_{L^2}^2+1)\left|\int_{-\infty}^{\infty}\left|\widetilde{F_p}(U,U^r)\right|\mathrm{d}x\right|^{\frac{3p+1}{3p}},
\end{aligned}
\tag{4.8}
$$

where we used Young's inequality and the following Sobolev inequality (cf. [27]):

$$
\|\phi\|_{L^\infty} \leq \left(\frac{3p+1}{p+1}\right)^{\frac{p+1}{3p+1}}\|\phi\|_{L^2}^{\frac{2p}{3p+1}}\|\partial_x\phi\|_{L^{p+1}}^{\frac{p+1}{3p+1}}.
\tag{4.9}
$$

By the Cauchy-Schwarz inequality and Young's inequality, we also estimate the third term in the right-hand side of (4.7) as follows:

$$
\begin{aligned}
&\mu\int_{-\infty}^{\infty}|\partial_x\phi|\left((\partial_x U + \partial_x U^r)^p - (\partial_x U)^p\right)\mathrm{d}x \\
&= \mu p\int_{-\infty}^{\infty}|\partial_x\phi|\left((\partial_x U + \theta\partial_x U^r)^{p-1}\partial_x U^r\right)\mathrm{d}x \\
&\qquad\qquad (\exists\theta = \theta(t,x)\in(0,1)) \\
&\leq \frac{\mu}{4\nu_p}\int_{-\infty}^{\infty}(\partial_x\phi)^2(\partial_x U + \partial_x U^r)^{p-1}\mathrm{d}x \\
&\quad + C_p\left|\int_{-\infty}^{\infty}(\partial_x U^r)^2(\partial_x U + \partial_x U^r)^{p-1}\mathrm{d}x\right|.
\end{aligned}
\tag{4.10}
$$

Thus, substituting (4.8) and (4.10) into (4.7), we complete the proof of Lemma 4.1.

Proof of Lemma 4.2. Firstly, we have

$$
\begin{aligned}
&\left|\int_{-\infty}^{\infty}(\partial_x U^r)^2(\partial_x U + \partial_x U^r)^{p-1}\mathrm{d}x\right| \\
&\leq C_p\|\partial_x\tilde{U}(t)\|_{L^\infty}^{p-1}\|\partial_x U^r(t)\|_{L^\infty} \\
&\leq C_p(1+t)^{-1-\frac{p-1}{p+1}},
\end{aligned}
\tag{4.11}
$$

that is,

$$
\int_{-\infty}^{\infty}(\partial_x U^r)^2(\partial_x U + \partial_x U^r)^{p-1}\mathrm{d}x \in L^1_t(0,\infty)
\tag{4.12}
$$

where we used Lemma 2.1 and Lemma 2.2. Then, it suffices to show, by the definition of the remainder term $\widetilde{F_p}(U, U^r)$, that

$$\int_{-\infty}^{\infty} |\, f'(U+U^r) - f'(U^r)\,|\, \partial_x U^r \,\mathrm{d}x \in L_t^{\frac{3p+1}{3p}}(0,\infty), \tag{4.13}$$

$$\int_{-\infty}^{\infty} |\, f'(U+U^r)\,|\, \partial_x U \,\mathrm{d}x \in L_t^{\frac{3p+1}{3p}}(0,\infty). \tag{4.14}$$

To obtain (4.13) and (4.14), it is very natural to divide the integral region $\mathbb{R}$ depending on the sign of $\tilde{U} = U + U^r$. So, for any $t \geq 0$, we introduce

$$X : [\,0,\infty) \ni t \longmapsto X(t) \in \mathbb{R}$$

such that

$$\tilde{U}\big(t, X(t)\big) = U\big(t, X(t)\big) + U^r\big(t, X(t)\big) = 0 \quad \big(t \geq 0\big), \tag{4.15}$$

that is,

$$\begin{aligned} U^r\big(t, X(t)\big) &= -U\big(t, X(t)\big) \\ &= \int_{X(t)}^{\infty} \frac{1}{t^{\frac{1}{p+1}}} \left(\left(A - B\left(\frac{y}{t^{\frac{1}{p+1}}}\right)^2 \right) \vee 0 \right)^{\frac{1}{p-1}} \mathrm{d}y. \end{aligned} \tag{4.16}$$

Here we note that $X(t)$ uniquely exists because U^r is strictly monotonically increasing with respect to x on the whole $\mathbb{R}$ and U is also strictly monotonically increasing on $-\sqrt{\frac{A}{B}}\,(1+t)^{\frac{1}{p+1}} < x < \sqrt{\frac{A}{B}}\,(1+t)^{\frac{1}{p+1}}$. Furthermore, note that $\tilde{U}(t,-\infty) = u_- < 0 < u_+ = \tilde{U}(t,\infty)$. Therefore we can divide the integral region $\mathbb{R}$ into $\big(-\infty, X(t)\big)$ where $\tilde{U} < 0$ and $\big(X(t), \infty\big)$ where $\tilde{U} > 0$. As a basic behavior of $X(t)$, we can show by Lemma 2.1 and Lemma 2.2 that there exists a positive time T_0 such that for some $\delta \in \left(0, \sqrt{\frac{A}{B}}\right)$,

$$\left(\sqrt{\frac{A}{B}} - \delta\right)(1+t)^{\frac{1}{p+1}} < X(t) < \sqrt{\frac{A}{B}}\,(1+t)^{\frac{1}{p+1}} \qquad \big(t \geq T_0\big). \tag{4.17}$$

Indeed, by an easy fact

$$\sup_{x\in\mathbb{R}} \left| u^r\left(\frac{x}{1+t}\right) - u^r\left(\frac{x}{t}\right) \right| \leq C(1+t)^{-1}, \tag{4.18}$$

and Lemma 2.1, it follows that

$$\sup_{x\in\mathbb{R}} \left| U^r(t,x) - u^r\left(\frac{x}{1+t}\right) \right| \leq C_\epsilon (1+t)^{-1+\epsilon} \quad \big(\epsilon \in (0,1)\big), \tag{4.19}$$

which implies

$$\begin{aligned} &\lim_{t\to\infty} \tilde{U}\left(t, \left(\sqrt{\frac{A}{B}} - \delta\right)(1+t)^{\frac{1}{p+1}}\right) \\ &\quad = -\int_{\sqrt{\frac{A}{B}}-\delta}^{\infty} \left((A - B\xi^2) \vee 0 \right)^{\frac{1}{p-1}} \mathrm{d}\xi < 0, \end{aligned} \tag{4.20}$$

and

$$\tilde{U}\left(t, \sqrt{\frac{A}{B}}\,(1+t)^{\frac{1}{p+1}}\right) = U^r\left(t, \sqrt{\frac{A}{B}}\,(1+t)^{\frac{1}{p+1}}\right) > 0 \quad (\forall t \geq 0\,). \tag{4.21}$$

So we have (4.17) by (4.20) and (4.21). Then, by (4.16), (4.18) and (4.19), we have for any $\epsilon \in (0,1)$, there exists a positive constant C_ϵ such that

$$\left| (\lambda)^{-1}\left(\frac{X(t)}{1+t}\right) - \int_{\frac{X(t)}{(1+t)^{\frac{1}{p+1}}}}^{\infty} \left(\left(A - B\xi^2\right) \vee 0\right)^{\frac{1}{p-1}} \mathrm{d}\xi \right| \leq C_\epsilon (1+t)^{-1+\epsilon}, \tag{4.22}$$

for $t \geq T_0$. Using (4.22), we can show more precise large time behavior of $X(t)$ as in the following lemma.

Lemma 4.3. *It holds that for each $p > 1$, there exists a positive constant C_p such that*

$$\left| \sqrt{\frac{A}{B}} - \frac{X(t)}{(1+t)^{\frac{1}{p+1}}} \right| \leq C_p (1+t)^{-\frac{p-1}{p+1}} \qquad \big(t \geq T_0\big).$$

Now we complete the proof of Lemma 4.2. Using Lemmas 2.1, 2.2 and 4.3, we first prove (4.13). Dividing the integral region as we mentioned above as

$$\int_{-\infty}^{\infty} \left| f'(U+U^r) - f'(U^r) \right| \partial_x U^r \,\mathrm{d}x = \int_{-\infty}^{X(t)} + \int_{X(t)}^{\infty} =: I_{11} + I_{12},$$

we estimate each integral as follows:

$$\begin{aligned}
I_{11}(t) &= \int_{-\infty}^{X(t)} \left| f'(U+U^r) - f'(U^r) \right| \partial_x U^r \,\mathrm{d}x \\
&= \int_{-\infty}^{X(t)} \partial_x \big(f(U^r)\big) \,\mathrm{d}x \\
&\leq C \left| U^r\big(t, X(t)\big) \right|^2 \\
&\leq C \left(\frac{X(t)}{1+t} + C_\epsilon (1+t)^{-1+\epsilon} \right)^2 \\
&\leq C_p \,(1+t)^{-1-\frac{p-1}{p+1}} + C_\epsilon (1+t)^{-1-(1-2\epsilon)} \qquad \big(\epsilon \in (0,1),\ t \geq 0\big),
\end{aligned} \tag{4.23}$$

$$\begin{aligned}
I_{12}(t) &= \int_{X(t)}^{\infty} \left| f'(U+U^r) - f'(U^r) \right| \partial_x U^r \,\mathrm{d}x \\
&\leq C \int_{X(t)}^{\infty} |\, U \,|\, \partial_x U^r \,\mathrm{d}x \\
&\leq C(1+t)^{-1} \int_{X(t)}^{\infty} \int_{\frac{x}{(1+t)^{\frac{1}{p+1}}}}^{\infty} \left(\left(A - B\xi^2\right) \vee 0\right)^{\frac{1}{p-1}} \mathrm{d}\xi \mathrm{d}x \\
&= C(1+t)^{-1} \int_{\frac{X(t)}{(1+t)^{\frac{1}{p+1}}}}^{\infty} \left(\left(A - B\xi^2\right) \vee 0\right)^{\frac{1}{p-1}} \left(\xi\,(1+t)^{\frac{1}{p+1}} - X(t)\right) \mathrm{d}\xi \\
&\leq C_p (1+t)^{-1+\frac{1}{p+1}} \int_{\frac{X(t)}{(1+t)^{\frac{1}{p+1}}}}^{\sqrt{\frac{A}{B}}} \left(A - B\,\xi^2\right)^{\frac{1}{p-1}} \mathrm{d}\xi \\
&= C_p (1+t)^{-1+\frac{1}{p+1}} \cdot \frac{p-1}{2Bp} \left(A - B\left(\frac{X(t)}{(1+t)^{\frac{1}{p+1}}}\right)^2 \right)^{\frac{p}{p-1}} \\
&\leq C_p (1+t)^{-1+\frac{1}{p+1}} \left| \sqrt{\frac{A}{B}} - \frac{X(t)}{(1+t)^{\frac{1}{p+1}}} \right|^{\frac{p}{p-1}} \\
&\leq C_p (1+t)^{-1-\frac{p-1}{p+1}} \qquad \big(t \geq T_0\big),
\end{aligned} \tag{4.24}$$

where we used the facts $\| \partial_x U^r(t) \|_{L^\infty} \le C(1+t)^{-1}$ in Lemma 2.1 and Lemma 4.3. Hence, choosing ϵ suitably small in (4.23), we can easily conclude $I_{11}, I_{12} \in L_t^{\frac{3p+1}{3p}}(0,\infty)$, which proves (4.13). Next, we similarly show (4.14). In this case, noting

$$\int_{-\infty}^{\infty} f'(U+U^r)\,\partial_x U\,\mathrm{d}x = \int_{X(t)}^{\infty} f'(U+U^r)\,\partial_x U\,\mathrm{d}x =: I_{21},$$

we estimate I_{21}, by integration by parts, as follows:

$$\begin{aligned}
I_{21}(t) &= \int_{X(t)}^{\infty} f'(U+U^r)\,\partial_x U\,\mathrm{d}x \\
&= -\int_{X(t)}^{\infty} U\, f''(U+U^r)\big(\partial_x U + \partial_x U^r\big)\,\mathrm{d}x \\
&\le C\int_{X(t)}^{\infty} -\frac{1}{2}\,\partial_x(U^2)\,\mathrm{d}x + C\int_{X(t)}^{\infty} |\,U\,|\,\partial_x U^r\,\mathrm{d}x \\
&\le C\,\big|\,U^r(t,X(t))\big|^2 + C_p(1+t)^{-1-\frac{p-1}{p+1}} \\
&\le C_\epsilon(1+t)^{-1-(1-2\epsilon)} + C_p(1+t)^{-1-\frac{p-1}{p+1}} \qquad \big(\epsilon\in(0,1),\ t\ge T_0\big).
\end{aligned}$$

Hence, choosing ϵ suitably small again, we easily have $I_{21} \in L^{\frac{3p+1}{3p}}(0,\infty)$. Thus, the proof of Lemma 4.2 is completed.

Thus, we do complete the proof of Proposition 3.1.

5 Uniform estimates II

In this section, In order to complete the uniform estimates for the asymptotics not depending on T, we show Proposition 3.2. To do that, we assume that the solution to our Cauchy problem (3.7) satisfies the same regularity as in Section 4. What we should prove is the following energy inequality:

$$\begin{aligned}
&\| \partial_x u(t) \|_{L^{p+1}}^{p+1} + \int_0^t \int_{-\infty}^{\infty} |\,\partial_x u\,|^{2(p-1)} \big(\partial_x^2 u\big)^2\,\mathrm{d}x\mathrm{d}\tau \\
&\qquad + \int_0^t \| \partial_x u(\tau) \|_{L^{p+2}(\{x\in\mathbb{R}\,|\,u>0\})}^{p+2}\,\mathrm{d}\tau \le C_p(\phi_0, \partial_x u_0) \quad \big(t\ge 0\big).
\end{aligned} \tag{5.1}$$

In order to obtain (5.1), we multiple the equation in (1.1) by

$$-\partial_x\Big(\,|\,\partial_x u\,|^{q-1}\,\partial_x u\Big)$$

with $q>1$, integrate it with respect to x, and get

$$\begin{aligned}
&\frac{1}{q+1}\frac{\mathrm{d}}{\mathrm{d}t}\,\| \partial_x u(t) \|_{L^{q+1}}^{q+1} + \mu\, p\, q\int_{-\infty}^{\infty} |\,\partial_x u\,|^{p+q-2}\big(\partial_x^2 u\big)^2\,\mathrm{d}x \\
&\qquad + \frac{q}{q+1}\int_{-\infty}^{\infty} f''(u)\,|\,\partial_x u\,|^{q+1}\,\partial_x u\,\mathrm{d}x = 0.
\end{aligned} \tag{5.2}$$

Now we separate the integral region to the third term in the left-hand side of (5.2) as

$$\begin{aligned}
&\int_{-\infty}^{\infty} f''(u)\,|\,\partial_x u\,|^{q+1}\,\partial_x u\,\mathrm{d}x \\
&= \int_{\partial_x u\ge 0} + \int_{\partial_x u<0} \\
&= \int_{\partial_x u\ge 0} f''(u)\,|\,\partial_x u\,|^{q+2}\,\mathrm{d}x - \int_{\partial_x u<0} f''(u)\,|\,\partial_x u\,|^{q+2}\,\mathrm{d}x.
\end{aligned} \tag{5.3}$$

Substituting (5.3) into (5.2), we get the following equality

$$
\begin{aligned}
&\frac{1}{q+1}\frac{\mathrm{d}}{\mathrm{d}t}\,\|\,\partial_x u(t)\,\|_{L^{q+1}}^{q+1} + \mu\, p\, q\int_{-\infty}^{\infty}|\,\partial_x u\,|^{p+q-2}\left(\partial_x^2 u\right)^2 \mathrm{d}x \\
&+\frac{q}{q+1}\int_{\partial_x u\geq 0} f''(u)\,|\,\partial_x u\,|^{q+2}\,\mathrm{d}x = \frac{q}{q+1}\int_{\partial_x u<0} f''(u)\,|\,\partial_x u\,|^{q+2}\,\mathrm{d}x.
\end{aligned}
\tag{5.4}
$$

We have the following result which plays the most important role in the proof of (5.1).

Lemma 5.1. *For each $q>1$, there exists a positive constant C_q such that*

$$
\int_{\partial_x u<0} f''(u)\,|\,\partial_x u\,|^{q+2}\,\mathrm{d}x \leq C_q\int_{\partial_x u<0}|\,\partial_x \phi\,|^{q+2}\,\mathrm{d}x. \tag{5.5}
$$

In fact, taking care of the relation

$$
\partial_x u = \partial_x \tilde{U} + \partial_x \phi < 0 \iff \partial_x \phi < 0,\ \partial_x \tilde{U} < |\,\partial_x \phi\,|, \tag{5.6}
$$

we immediately have

$$
\begin{aligned}
&\int_{\partial_x u<0} f''(u)\,|\,\partial_x u\,|^{q+2}\,\mathrm{d}x \\
&\leq 2^{q+2}\left(\sup_{0\leq u\leq \tilde{C}+1} f''(u)\right)\int_{\partial_x\phi<0,\partial_x\tilde{U}<|\partial_x\phi|}|\,\partial_x \phi\,|^{q+2}\,\mathrm{d}x.
\end{aligned}
\tag{5.7}
$$

Remark 5.1. Under the relation (5.6), noting

$$
\int_0^{\infty}\int_{-\infty}^{\infty}|\,\partial_x \phi\,|^{p+1}\,\mathrm{d}x\mathrm{d}t < \infty
$$

from (4.1) in Section 4 and taking $q=p-1$ to (5.4), we can easily show that for $p\geq \frac{3}{2}$,

$$
\begin{aligned}
&\frac{1}{p}\,\|\,\partial_x u(t)\,\|_{L^p}^{p} + \mu\, p\,(p-1)\int_0^{\infty}\int_{-\infty}^{\infty}|\,\partial_x u\,|^{2p-3}\left(\partial_x^2 u\right)^2 \mathrm{d}x\mathrm{d}t \\
&\qquad+\frac{p-1}{p}\int_0^{\infty}\int_{\partial_x u\geq 0} f''(u)\,|\,\partial_x u\,|^{p+1}\,\mathrm{d}x\mathrm{d}t < \infty,
\end{aligned}
\tag{5.8}
$$

which namely means that for $p\geq\frac{3}{2}$,

$$
\left\{
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\,\|\,\partial_x u\,\|_{L^p}^{p} \in L^1_t(0,\infty), \\
&\int_{-\infty}^{\infty}|\,\partial_x u\,|^{2p-3}\left(\partial_x^2 u\right)^2 \mathrm{d}x \in L^1_t(0,\infty), \\
&\int_{-\infty}^{\infty} f''(u)\,|\,\partial_x u\,|^{p+1}\,\mathrm{d}x \sim \int_{u>0}|\,\partial_x u\,|^{p+1}\,\mathrm{d}x \in L^1_t(0,\infty).
\end{aligned}
\right.
\tag{5.9}
$$

Integrating (5.4) with respect to t and taking $q=p$, we have the energy equality

$$
\begin{aligned}
&\frac{1}{p+1}\,\|\,\partial_x u(t)\,\|_{L^{p+1}}^{p+1} + \mu\, p^2\int_0^t\int_{-\infty}^{\infty}|\,\partial_x u\,|^{2(p-1)}\left(\partial_x^2 u\right)^2 \mathrm{d}x\mathrm{d}\tau \\
&+\frac{p}{p+1}\int_0^t\int_{\partial_x u\geq 0} f''(u)\,|\,\partial_x u\,|^{p+2}\,\mathrm{d}x\mathrm{d}\tau \\
&=\frac{1}{p+1}\,\|\,\partial_x u_0\,\|_{L^{p+1}}^{p+1} + \frac{p}{p+1}\int_0^t\int_{\partial_x u<0} f''(u)\,|\,\partial_x u\,|^{p+2}\,\mathrm{d}x\mathrm{d}\tau.
\end{aligned}
\tag{5.10}
$$

The most difficult term to stimate is the second term in the right-hand side. We prepare the following "boundary zero condition type" interpolation inequality to overcome the difficulty.

Lemma 5.2. *It holds that*

$$\begin{aligned}&\int_{\partial_x u<0} |\,\partial_x u\,|^{p+2}\,\mathrm{d}x\\ &\le C_p\left(\int_{\partial_x u<0} |\,\partial_x u\,|^{2(p-1)}\left(\partial_x^2 u\right)^2\mathrm{d}x\right)^{\frac{1}{3p+1}}\left(\int_{\partial_x u<0} |\,\partial_x u\,|^{p+1}\,\mathrm{d}x\right)^{\frac{3p+2}{3p+1}}.\end{aligned}\tag{5.11}$$

Proof of Lemma 5.2. Since $\partial_x u$ is absolutely continuous, we first note that for any $x \in \{x \in \mathbb{R} \mid \partial_x u < 0\}$, there exsists $x_k \in \mathbb{R} \cup \{-\infty\}$ such that

$$\partial_x u(x_k) = 0,\ \ \partial_x u(y) < 0\ \big(y \in (x_k, x)\big).$$

Therefore by using the Cauchy-Schwarz inequality, it follows that for such x and x_k with $q \ge p\,(>1)$,

$$\begin{aligned}&|\,\partial_x u\,|^q = (\,-\partial_x u\,)^q = q\int_{x_k}^{x} (\,-\partial_x u\,)^{q-1}\left(\,-\partial_x^2 u\,\right)\mathrm{d}y\\ &\le q\int_{\partial_x u<0} (\,-\partial_x u\,)^{q-1}\left(\,-\partial_x^2 u\,\right)\mathrm{d}x\\ &\le q\left(\int_{\partial_x u<0} (\,-\partial_x u\,)^{2(p-1)}\left(\,-\partial_x^2 u\,\right)^2\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_{\partial_x u<0} (\,-\partial_x u\,)^{2(q-p)}\,\mathrm{d}x\right)^{\frac{1}{2}}.\end{aligned}\tag{5.12}$$

Hence

$$\begin{aligned}&\|\,\partial_x u(t)\,\|_{L^\infty_x(\{\partial_x u<0\})}\\ &\le q^{\frac{1}{q}}\left(\int_{\partial_x u<0} |\,\partial_x u\,|^{2(p-1)}\left(\partial_x^2 u\right)^2\mathrm{d}x\right)^{\frac{1}{2q}}\left(\int_{\partial_x u<0} |\,\partial_x u\,|^{2(q-p)}\,\mathrm{d}x\right)^{\frac{1}{2q}}.\end{aligned}\tag{5.13}$$

So we get

$$\begin{aligned}&\int_{\partial_x u<0} |\,\partial_x u\,|^{p+2}\,\mathrm{d}x\\ &\le \|\,\partial_x u\,\|_{L^\infty_x(\{\partial_x u<0\})}\int_{\partial_x u<0} |\,\partial_x u\,|^{p+1}\,\mathrm{d}x\\ &\le q^{\frac{1}{q}}\left(\int_{\partial_x u<0} |\,\partial_x u\,|^{2(p-1)}\left(\partial_x^2 u\right)^2\mathrm{d}x\right)^{\frac{1}{2q}}\\ &\qquad\times\left(\int_{\partial_x u<0} |\,\partial_x u\,|^{2(q-p)}\,\mathrm{d}x\right)^{\frac{1}{2q}}\left(\int_{\partial_x u<0} |\,\partial_x u\,|^{p+1}\,\mathrm{d}x\right).\end{aligned}\tag{5.14}$$

Taking $q = \frac{3p+2}{2}$ in (5.15), we have

$$\begin{aligned}\int_{\partial_x u<0} |\,\partial_x u\,|^{p+2}\,\mathrm{d}x \le \left(\frac{3p+2}{2}\right)^{\frac{2}{3p+2}}&\left(\int_{\partial_x u<0} |\,\partial_x u\,|^{p+1}\,\mathrm{d}x\right)^{\frac{3p+2}{3p+1}}\\ \times&\left(\int_{\partial_x u<0} |\,\partial_x u\,|^{2(p-1)}\left(\partial_x^2 u\right)^2\mathrm{d}x\right)^{\frac{1}{3p+1}}.\end{aligned}\tag{5.15}$$

Thus we complete the proof.

Using Young's inequality to Lemma 5.2, (5.11), we also have

Lemma 5.3. *It follows that for any $\epsilon > 0$, there exists a positive constant $C_p(\epsilon)$ such that,*

$$\begin{aligned}
&\int_{\partial_x u<0} |\,\partial_x u\,|^{p+2}\,\mathrm{d}x \\
&\leq \epsilon \int_{\partial_x u<0} |\,\partial_x u\,|^{2(p-1)}\left(\partial_x^2 u\right)^2 \mathrm{d}x + C_p(\epsilon)\left(\int_{\partial_x u<0} |\,\partial_x u\,|^{p+1}\,\mathrm{d}x\right)^{\frac{3p+2}{3p}}.
\end{aligned} \tag{5.16}$$

Substituting (5.16) with $\epsilon = \frac{\mu p^2}{2}$ into (5.10), we have

$$\begin{aligned}
&\frac{1}{p+1}\,\|\,\partial_x u(t)\,\|_{L^{p+1}}^{p+1} + \frac{\mu p^2}{2}\int_0^t\int_{-\infty}^{\infty} |\,\partial_x u\,|^{2(p-1)}\left(\partial_x^2 u\right)^2 \mathrm{d}x\mathrm{d}\tau \\
&+ \frac{p}{p+1}\int_0^t\int_{\partial_x u\geq 0} f''(u)\,|\,\partial_x u\,|^{p+2}\,\mathrm{d}x\mathrm{d}\tau \\
&\leq \frac{1}{p+1}\,\|\,\partial_x u_0\,\|_{L^{p+1}}^{p+1} + C_p\int_0^t\left(\int_{\partial_x u<0} |\,\partial_x u\,|^{p+1}\,\mathrm{d}x\right)^{\frac{2}{3p}+1}\mathrm{d}\tau.
\end{aligned} \tag{5.17}$$

Now recalling Lemma 5.1, we have

$$\int_{\partial_x u<0} |\,\partial_x u\,|^{p+1}\,\mathrm{d}x \leq C_p\int_{-\infty}^{\infty} |\,\partial_x \phi\,|^{p+1}\,\mathrm{d}x \in L_t^1(0,\infty). \tag{5.18}$$

We also note $\frac{2}{3p} < 1$ and focus on the fact

$$\left(\int_{\partial_x u<0} |\,\partial_x u\,|^{p+1}\,\mathrm{d}x\right)^{\frac{2}{3p}} \leq C_p\left(1 + \int_{\partial_x u<0} |\,\partial_x u\,|^{p+1}\,\mathrm{d}x\right) \tag{5.19}$$

for some posisive constant C_p. Hence, substituting (5.18) and (5.19) into (5.17), we have

$$\begin{aligned}
&\frac{1}{p+1}\,\|\,\partial_x u(t)\,\|_{L^{p+1}}^{p+1} + \frac{\mu p^2}{2}\int_0^t\int_{-\infty}^{\infty} |\,\partial_x u\,|^{2(p-1)}\left(\partial_x^2 u\right)^2 \mathrm{d}x\mathrm{d}\tau \\
&+ \frac{p}{p+1}\int_0^t\int_{\partial_x u\geq 0} f''(u)\,|\,\partial_x u\,|^{p+2}\,\mathrm{d}x\mathrm{d}\tau \\
&\leq \frac{1}{p+1}\,\|\,\partial_x u_0\,\|_{L^{p+1}}^{p+1} + C_p\int_0^t\int_{-\infty}^{\infty} |\,\partial_x \phi\,|^{p+1}\,\mathrm{d}x\mathrm{d}\tau \\
&\qquad + C_p\int_0^t \|\,\partial_x u(\tau)\,\|_{L^{p+1}}^{p+1}\left(\int_{-\infty}^{\infty} |\,\partial_x \phi\,|^{p+1}\,\mathrm{d}x\right)\mathrm{d}\tau.
\end{aligned} \tag{5.20}$$

By using Gronwall's inequality, we have

$$\begin{aligned}
\|\,\partial_x u(t)\,\|_{L^{p+1}}^{p+1} &\leq C_p\left(\|\,\partial_x u_0\,\|_{L^{p+1}}^{p+1} + \|\,\phi_0\,\|_{L^2}^2 + 1\right) \\
&\quad \times \exp\left\{C_p\int_0^{\infty}\int_{-\infty}^{\infty} |\,\partial_x \phi\,|^{p+1}\,\mathrm{d}x\mathrm{d}t\right\} < \infty.
\end{aligned} \tag{5.21}$$

Hence, substituting (5.22) into (5.21), it finally holds

$$\begin{aligned}
&\|\,\partial_x u(t)\,\|_{L^{p+1}}^{p+1} + \int_0^t\int_{-\infty}^{\infty} |\,\partial_x u\,|^{2(p-1)}\left(\partial_x^2 u\right)\mathrm{d}x\mathrm{d}\tau \\
&\qquad + \int_0^t \|\,\partial_x u(\tau)\,\|_{L^{p+2}(\{x\in\mathbb{R}\,|\,u>0\})}^{p+2}\,\mathrm{d}\tau \\
&\leq C\left(\|\,\phi_0\,\|_{L^2},\ \|\,\partial_x u_0\,\|_{L^{p+1}}\right).
\end{aligned} \tag{5.22}$$

Thus, we do complete the proof of Proposition 3.2.

References

[1] G. I. Barenblatt, *On the motion of suspended particles in a turbulent flow taking up a half-space or a plane open channel of finite depth*, Prikl. Mat. Meh. **19** (1955), pp. 61-88 (in Russian).

[2] J. A. Carrillo and G. Toscani, *Asymptotic L^1-decay of solutions of the porous medium equation to self-similarity*, Indiana Univ. Math. J. **49** (2000), pp. 113-142.

[3] E. Gagliardo, *Ulteriori proprietà di alcune classi di funzioni in più variabili*, Ricerche di Mat. Napoli **8** (1959), pp. 24-51 (in Italian).

[4] M. E. Gurtin and R. C. MacCamy, *On the diffusion of biological populations*, Math. Biosci. **33** (1979), pp. 35-49.

[5] I. Hashimoto and A. Matsumura, *Large time behavior of solutions to an initial boundary value problem on the half space for scalar viscous conservation law*, Methods Appl. Anal. **14** (2007), pp. 45-59.

[6] Y. Hattori and K. Nishihara, *A note on the stability of rarefaction wave of the Burgers equation*, Japan J. Indust. Appl. Math. **8** (1991), pp. 85-96.

[7] F. Huang, R. Pan and Z. Wang, *L^1 Convergence to the Barenblatt Solution for Compressible Euler Equations with Damping*, Arch. Rational Mech. Anal. **200** (2011), pp. 665-689.

[8] A. M. Il'in, A. S. Kalašnikov and O. A. Oleĭnik, *Second-order linear equations of parabolic type*, Uspekhi Math. Nauk SSSR **17** (1962), pp. 3-146 (in Russian); English translation in Russian Math. Surveys **17** (1962), pp. 1-143.

[9] A. M. Il'in and O. A. Oleĭnik, *Asymptotic behavior of the solutions of the Cauchy problem for some quasi-linear equations for large values of the time*, Mat. Sb. **51** (1960), pp. 191-216 (in Russian).

[10] S. Kamin, *Source-type solutions for equations of nonstationary filtration*, J. Math. Anal. Appl. **64** (1978), pp. 263-276.

[11] O. A. Ladyženskaja, V. A. Solonnikov and N. N. Ural'ceva, *Linear and quasilinear equations of parabolic type*, Transl. Math. Monographs, vol. 23, Amer. Math. Soc., Providence, Rhode. Island, 1968.

[12] P. D. Lax, *Hyperbolic systems of conservation laws II*, Comm. Pure Appl. Math. **10** (1957), pp. 537-566.

[13] H. W. Liepmann and A. Roshko, *Elements of Gasdynamics*, John Wiley & Sons, Inc., New York, 1957.

[14] J. L. Lions, *Quelques mèthodes de rèsolution des problèmes aux limites non linèaires*, Dunod Gauthier-Villars, Paris, 1969 (in French).

[15] T.-P. Liu, A. Matsumura and K. Nishihara, *Behaviors of solutions for the Burgers equation with boundary corresponding to rarefaction waves*, SIAM J. Math. Anal. **29** (1998), pp. 293-308.

[16] A. Matsumura and K. Nishihara, *Asymptotic toward the rarefaction wave of solutions of a one-dimensional model system for compressible viscous gas*, Japan J. Appl. Math. **3** (1986), pp. 1-13.

[17] A. Matsumura and K. Nishihara, *Asymptotics toward the rarefaction wave of the solutions of Burgers' equation with nonlinear degenerate viscosity* Nonlinear Anal. TMA **23** (1994), pp. 605-614.

[18] A. Matsumura and K. Nishihara, *Asymptotic stability of traveling waves for scalar viscous conservation laws with non-convex nonlinearity*, Comm. Math. Phys. **165** (1994), pp. 83-96.

[19] A. Matsumura and N. Yoshida, *Asymptotic behavior of solutions to the Cauchy problem for the scalar viscous conservation law with partially linearly degenerate flux*, SIAM J. Math. Anal. **44** (2012), pp. 2526-2544.

[20] T. Nagai and M. Mimura, *Some nonlinear degenerate diffusion equations related to population dynamics*, J. Math. Soc. Japan **35** (1983), pp. 539-562.

[21] T. Nagai and M. Mimura, *Asymptotic behavior for a nonlinear degenerate diffusion equation in population dynamics*, SIAM J. Appl. Math. **43** (1983), pp. 449-464.

[22] J. Nash, *Continuity of solutions of parabolic and elliptic equations*, Amer. J. Math. **80** (1958), pp. 931-954.

[23] L. Nirenberg, *On elliptic partial differential equations*, Ann. Scuola Norm. Sup. Pisa (Ser. III) **13** (1959), pp. 115-162.

[24] R. E. Pattle, *Diffusion from an instantaneous point source with a concentration-dependent coefficient*, Quart. J. Mech. Appl. Math. **12** (1959), pp. 407-409.

[25] M. H. Protter and H. F. Weinberger, *Maximum Principles in Differential Equations*, Springer-Verlag, New York-Berlin, 1984.

[26] J. Smoller, *Shock Waves and Reaction-diffusion Equations*, Springer-Verlag, New York-Berlin, 1983.

[27] N. Yoshida, *Decay properties of solutions toward a multiwave pattern for the scalar viscous conservation law with partially linearly degenerate flux*, Nonlinear Anal. TMA **96** (2014), pp. 189-210.

[28] Ya. B. Zel'dovič and A. S. Kompanceec, *On the theory of propagation of heat with the heat conductivity depending upon the temperature*, Collection in honor of the seventieth birthday of academician A. F. Ioffe, Izdat. Akad. Nauk SSSR, Moscow (1950), pp. 61-71 (in Russian).