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Transverse instability for nonlinear Schrödinger equation

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1 Introduction

In this report, we consider the stability for standing waves of nonlinear Schrödinger equation

$$i\partial_t u=-\Delta u-|u|^{p-1}u, \quad u(t,x,y):\mathbb{R}\times\mathbb{R}\times\mathbb{T}_L\to\mathbb{C}, \tag{1}$$

where $p>1$, $\mathbb{T}_L=\mathbb{R}/2\pi L\mathbb{Z}$ and u is an unknown complex-valued function. Cauchy problem of (1) is locally well-posed in H^1 (see [9, 14, 26, 27]). The equation (1) has mass and energy conservation:

$$Q(u)=\frac{1}{2}\|u\|_{L^2(\mathbb{R}\times\mathbb{T}_L)}^2, \quad E(u)=\frac{1}{2}\|\nabla u\|_{L^2(\mathbb{R}\times\mathbb{T}_L)}^2-\frac{1}{p+1}\|u\|_{L^{p+1}(\mathbb{R}\times\mathbb{T}_L)}^{p+1},$$

where $u\in H^1(\mathbb{R}\times\mathbb{T}_L)$. By a standing wave, we mean a non trivial solution of (1) with the form $u(t,x,y)=e^{i\omega t}\varphi(x,y)$, where $\omega>0$ and $\varphi\in H^1(\mathbb{R}\times\mathbb{T}_L)$. Then, a function $e^{i\omega t}\varphi$ is a standing wave if and only if φ is a solution of

$$-\Delta\varphi+\omega\varphi-|\varphi|^{p-1}\varphi=0, \quad \varphi(x,y):\mathbb{R}\times\mathbb{T}_L\to\mathbb{C}. \tag{2}$$

We define the stability of standing waves as follows.

Definition 1. We say that a standing wave $e^{i\omega t}\varphi$ is orbitally stable if for any $\varepsilon>0$ there exists $\delta>0$ such that for all $u_0\in H^1(\mathbb{R}\times\mathbb{T}_L)$ with $\|u_0-\varphi\|_{H^1}<\delta$, the solution $u(t)$ of (1) with the initial data $u(0)=u_0$ exists globally in time and satisfies

$$\sup_{t\geq 0}\inf_{\theta\in\mathbb{R},(x,y)\in\mathbb{R}\times\mathbb{T}_L}\left\|u(t,\cdot,\cdot)-e^{i\theta}\varphi(\cdot-x,\cdot-y)\right\|_{H^1}<\varepsilon.$$

Otherwise, we say the standing wave $e^{i\omega t}\varphi$ is orbitally unstable in H^1.

One dimensional nonlinear Schrödinger equation:

$$i\partial_t u=-\partial_x^2 u-|u|^{p-1}u, \quad u(t,x):\mathbb{R}\times\mathbb{R}\to\mathbb{C}, \tag{3}$$

has the standing wave solution $e^{i\omega t}\varphi_\omega$ of (3) for $\omega>0$, where φ_ω is the symmetric positive solution of

$$-\partial_x^2\varphi+\omega\varphi-|\varphi|^{p-1}\varphi=0, \quad \varphi(x):\mathbb{R}\to\mathbb{C}. \tag{4}$$

The orbital stability of the standing wave $e^{i\omega t}\varphi_\omega$ is well known. Showing the convergence of the minimizing sequence of the minimization problem which is solved the minimizer φ_ω, Cazenave and Lions [4] proved that the standing wave $e^{i\omega t}\varphi_\omega$ is stable for $1 < p < 5$. Using the variational characterization of the standing wave $e^{i\omega t}\varphi_\omega$, Berestycki and Cazenave [2] showed that the standing wave $e^{i\omega t}\varphi_\omega$ is unstable for $p > 5$. Constructing the sufficient condition for blow up solution by virial identity, Weinstein [30] proved that the standing wave $e^{i\omega t}\varphi_\omega$ is unstable for $p = 5$.

We define the line standing $e^{i\omega t}\tilde{\varphi}_\omega$ as

$$\tilde{\varphi}_\omega(x,y) = \varphi_\omega(x), \quad (x,y) \in \mathbb{R} \times \mathbb{T}_L.$$

Since the standing wave $e^{i\omega t}\varphi_\omega$ is unstable for $p \geq 5$ on $\mathbb{R}$, the line standing wave $e^{i\omega t}\tilde{\varphi}_\omega$ is also unstable on $\mathbb{R} \times \mathbb{T}_L$. On the other hand, for $1 < p < 5$ the standing wave $e^{i\omega t}\varphi_\omega$ is stable. However, for $1 < p < 5$ in some cases the line standing wave is unstable by a perturbation which is dependent on the transverse direction T_L. We say that this instability for line standing waves is the transverse instability.

There exist many papers treating the transverse instability for various equations (see [1, 3, 17, 18, 21, 22, 23, 24]). In [1], Alexander-Pego-Sachs showed the linear stability for line solitons of KP-I or KP-II equation. Deconinck-Plinovsky-Carter [3] studied the linear stability for line standing waves of a hyperbolic Schrödinger equation. In [18], Mizumachi-Tzvetkov proved the asymptotic stability for line solitons of KP-II equation on $\mathbb{R}\times\mathbb{T}_L$ for all $L > 0$. Mizumachi studied the stability for line solitons of KP-II equation on $\mathbb{R}^2$. In $\mathbb{R}^2$, the line soliton is unstable in the sense of the orbital stability with the modulation of the amplitude and the phase shift which is independent of the transverse direction. Modulating the local amplitude and the local phase shift which is dependent of the transverse direction, Mizumachi showed the asymptotic stability of the line soliton. In [23], Rousset-Tzvetkov showed the sufficient condition for the linear instability of line soliton. Rousset-Tzvetkov showed the transverse instability for line soliton of KP-I equation on $\mathbb{R}^2$ and $\mathbb{R} \times \mathbb{T}_L$ in [21, 22, 23].

For the equation (1), Rousset-Tzvetkov [22] proved the following stability result for the line standing wave $e^{i\omega t}\tilde{\varphi}_\omega$ for $p = 3$ and Y. [28] showed the stability for $p \neq 3$.

Theorem 2. *Let $1 < p < 5$ and $\omega > 0$.*

(i) *If $0 < L < L_{\omega,p}$, then the line standing wave $e^{i\omega t}\tilde{\varphi}_\omega$ is stable.*
(ii) *If $L_{\omega,p} < L$, then the line standing wave $e^{i\omega t}\tilde{\varphi}_\omega$ is unstable.*

Here,

$$L_{\omega,p} = \frac{2}{\sqrt{(p-1)(p+3)\omega}}.$$

The statement (i) of Theorem 2 follows the linear instability result by Rousset-Tzvetkov [23] and the method in [12]. Therefore, the main statement of Theorem 2 is (ii). In [21, 22], Rousset-Tzvetkov developed the argument by Grenier [11] for the incompressible Euler equation and applied the argument to the transverse instability of various equation. In Section 3, we show the outline of the proof in [22]. Since the nonlinear term $|u|^{p-1}u$ is not smooth in the sense of Fréchet differentiation for $1 < p < 5$ and $p \neq 3$, we can not apply the argument in [21, 22] to the stability of line standing waves for $p \neq 3$. In [28], using an estimate for high frequency parts of the solution which has unstable mode, the author showed the stability for line sanding wave for $L \neq L_{\omega,p}$. In Section 4, we show the outline of the proof in [28]. In the case $L = L_{\omega,p}$, the linearize operator around the line standing wave has an extra eigenfunction corresponding to eigenvalue 0 and no eigenvalues with non zero real part. In the case $L > L_{\omega,p}$, the instability for line standing waves comes from the linear instability of the linearized equation around the line standing wave. To prove the instability, Rousset-Tzvetkov and the author used the linear instability of line standing wave in [22, 28]. Therefore, we can not apply the spectral arguments in [7, 21, 22, 28]. By the degeneracy of the kernel of the linearized operator, the stability of the line standing wave does not follow the method in [12]. We control the orbit of solutions near the line standing wave by combing the bifurcation result and the argument in Maeda [16]. The following theorem is the stability result for the line standing wave in the case $L = L_{\omega,p}$ in [29].

Theorem 3. *Let $\omega > 0$, $1 < p < 5$ and $L = L_{\omega,p}$. Then, there exist $2 < p_1 < p_2 < 3$ satisfies the following properties.*

(i) *If $2 \le p < p_1$, then the line standing wave $e^{i\omega t}\tilde{\varphi}_\omega$ is stable.*
(ii) *If $p_2 < p < 5$, then the line standing wave $e^{i\omega t}\tilde{\varphi}_\omega$ is unstable.*

Since we can not obtain an explicit value related the high order term of the Fréchet derivative of the energy, we do not show the stability for the line standing wave $e^{i\omega t}\tilde{\varphi}_\omega$ for $p_1 \le p \le p_2$ in [29].

The rest of paper is organized follows. In Section 2, we introduce the properties of the linearized equation and define some notations. In Section 3, we show the outline of the proof of (ii) of Theorem 2 for $p = 3$. In Section 4, we explain the outline of the proof of (ii) of Theorem 2 for $p \neq 3$. In Section 5, we show the outline of the proof of Theorem 3.

2 Preliminaries

In this section, we consider the linearized equation and define some notations.

Let $u(t)$ be a solution of (1) and $v(t) = e^{-i\omega t}u(t) - \tilde{\varphi}_\omega$. Then, $v(t)$ is a solution of

$$J\partial_t \vec{v} = \mathcal{A}\vec{v} + F(\vec{v}), \tag{5}$$

where

$$\vec{v} = \begin{pmatrix} \mathrm{Re}\ v \\ \mathrm{Im}\ v \end{pmatrix}, \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} -\Delta + \omega - p|\tilde{\varphi}_\omega|^{p-1} & 0 \\ 0 & -\Delta + \omega - |\tilde{\varphi}_\omega|^{p-1} \end{pmatrix},$$

$$F(\vec{v}) = \begin{pmatrix} -|\tilde{\varphi}_\omega + v|^{p-1}(\tilde{\varphi}_\omega + \mathrm{Re}\ v) + |\tilde{\varphi}_\omega|^{p-1}\tilde{\varphi}_\omega + p|\tilde{\varphi}_\omega|^{p-1}\mathrm{Re}\ v \\ -|\tilde{\varphi}_\omega + v|^{p-1}\mathrm{Im}\ v + |\tilde{\varphi}_\omega|^{p-1}\mathrm{Im}\ v \end{pmatrix}.$$

Let

$$S(a) = \begin{pmatrix} -\partial_x^2 + a^2 + \omega - p|\tilde{\varphi}_\omega|^{p-1} & 0 \\ 0 & -\partial_x^2 + a^2 + \omega - |\tilde{\varphi}_\omega|^{p-1} \end{pmatrix}.$$

Then, by Fourier expansion, we have

$$\mathcal{A}\vec{u} = \sum_{n \in Z} S(n/L)\vec{u}_n,$$

where $u \in L^2(\mathbb{R} \times \mathbb{T}_L)$ and

$$\vec{u}(x,y) = \begin{pmatrix} \mathrm{Re}\ u(x,y) \\ \mathrm{Im}\ u(x,y) \end{pmatrix} = \sum_{n \in \mathbb{Z}} e^{\frac{iny}{L}} \begin{pmatrix} \mathrm{Re}\ u_{R,n}(x) \\ \mathrm{Im}\ u_{I,n}(x) \end{pmatrix} = \sum_{n \in \mathbb{Z}} e^{\frac{iny}{L}} \vec{u}_n(x).$$

In the following, we regard

$$\vec{u} = \begin{pmatrix} \mathrm{Re}\ u \\ \mathrm{Im}\ u \end{pmatrix} = u.$$

The following lemma shows the spectrum properties of $-J\mathcal{A}$.

Lemma 4. *Let $\omega > 0$. If $0 < a^{-1} \le L_{\omega,p}$, then $-JS(a)$ has no eigenvalues with the positive real part. If $a^{-1} > L_{\omega,p}$, then $-JS(a)$ has an eigenvalue with the positive real part and the dimension of the eigenspace of $-JS(a)$ corresponding to eigenvalues with the positive real part is finite dimension.*

The proof of this lemma follows the argument in [23](see [28]). By Lemma 4, if $L > L_{\omega,p}$ then $-J\mathcal{A}$ has an eigenvalue with positive real part and there exist $k_0 \in \mathbb{Z}$ and $\chi \in H^1(\mathbb{R} \times \mathbb{T}_L)$ such that $\|\chi\|_{L^2(\mathbb{R}\times\mathbb{T}_L)} = 1$, χ is eigenfunction of $-J\mathcal{A}$ corresponding to $\mu_* = \max\{\lambda > 0 | \lambda \in \sigma(-J\mathcal{A})\}$ and

$$\chi(x,y) = \chi_1(x) e^{\frac{ik_0 y}{L}} + \chi_2(x) e^{\frac{-ik_0 y}{L}}.$$

Let $u_\delta(t)$ be the solution of (1) with $u_\delta(0) = \delta\chi + \tilde{\varphi}_\omega$. We define $v_\delta(t)$ as the solution of (5) corresponding to $u_\delta(t)$. We investigate the growth of L^2-norm of $v_\delta(t)$.

3 Outline of the proof of (ii) of Theorem 2 for $p = 3$

In this section, we explain the outline of the argument in [22]. Let $p = 3$, $L > L_{\omega,p}$ and $v^0(t) = e^{\mu_* t}\chi$. To control the growth of v_δ, we construct an approximate solution with finite Fourier modes corresponding to the transverse direction. We consider the following problem

$$\begin{aligned} i\partial_t v^k - S(k/L)v^k &= -\sum_{j+l=k-1, j\ge 0, l\ge 0} (2\tilde{\varphi}_\omega v^j \bar{v}^l + \tilde{\varphi}_\omega v^j v^l) - \sum_{j+l+m=k-2, j\ge 0, l\ge 0, m\ge 0} v^j \bar{v}^l v^m, \\ v^k(0) &= 0. \end{aligned} \tag{6}$$

The right hand side of the first equation of (6) is a polynomial of $v^0, \dots, v^{k-1}$. Therefore, solving the linear equation with the external force, we obtain the solution v^k. Moreover, v^k consists of finite Fourier modes corresponding to the transverse direction $\mathbb{T}_L$. Thus, we have the following estimate for v^k.

Lemma 5. *For $k \geq 0$, there exists $C_k > 0$ such that*

$$\left\|v^k(t)\right\|_{H^2} \leq C_k e^{(k+1)\mu_* t}.$$

This lemma follows Proposition 16 in [22]. For $\delta > 0$ we define the approximate solution of v_δ as

$$v^{ap}_{M,\delta} = \sum_{n=0}^{M} \delta^{n+1} v^n.$$

Let $w_{M,\delta}(t) = v_\delta(t) - v^{ap}_{M,\delta}(t) = e^{-i\omega t}u_\delta(t) - \tilde{\varphi}_\omega - v^{ap}_{M,\delta}(t)$. Then, $w_{M,\delta}$ satisfies

$$\begin{aligned} i\partial_t w - \mathcal{A}w + 2\tilde{\varphi}_\omega v^{ap}_{M,\delta}\bar{w} + 2\tilde{\varphi}_\omega \bar{v}^{ap}_{M,\delta} w + 2\tilde{\varphi}_\omega v^{ap}_{M,\delta} w + 2|v^{ap}_{M,\delta}|^2 w \\ + (v^{ap}_{M,\delta})^2 \bar{w} + N(v^{ap}_{M,\delta}, w) + |w|^2 w = -G, \end{aligned}$$

where $N(v^{ap}_{M,\delta}, w)$ is higher order terms with respect to w and

$$G = i\partial_t v^{ap}_{M,\delta} - \mathcal{A}v^{ap}_{M,\delta} + 2\tilde{\varphi}_\omega |v^{ap}_{M,\delta}|^2 + \tilde{\varphi}_\omega (v^{ap}_{M,\delta})^2 + |v^{ap}_{M,\delta}|^2 v^{ap}_{M,\delta}.$$

Let

$$T_* = \sup\{T > 0 | \; \|w(t)\|_{H^2} \leq 1 \text{ for } t \in [0, T]\}.$$

By Lemma 5, the definition of v^k and the energy estimate for w, we have for $t \in [0, T_*]$

$$\frac{d}{dt}\|w(t)\|^2_{H^2} \leq C(1 + \left\|v^{ap}_{M,\delta}\right\|^2_{H^2})\|w(t)\|^2_{H^2} + C_M \delta^{2(M+2)} e^{2(M+2)\mu_* t}.$$

Therefore, for

$$0 \leq t \leq \min\{T_{\kappa,\delta}, T_*\},$$

we have

$$\frac{d}{dt}\|w(t)\|^2_{H^2} \leq (C + \kappa^2 C'_M)\|w(t)\|^2_{H^2} + C_M \delta^{2(M+2)} e^{2(M+2)\mu_* t},$$

where $T_{\kappa,\delta} = \frac{\log(\kappa/\delta)}{\mu_*}$. If we choose $\kappa > 0$ and $M > 0$ such that $2(M+2)\mu_* - (C + \kappa^2 C'_M) > 0$, then we have for $0 \leq t \leq \min\{T_{\kappa,\delta}, T_*\}$

$$\|w(t)\|_{H^2} \leq C_M \kappa^{M+2}.$$

For sufficiently small $\kappa > 0$ we have for $0 \leq t \leq \min\{T_{\kappa,\delta}, T_*\}$

$$\|w(t)\|_{H^2} \leq \frac{1}{2}.$$

Thus, $\min\{T_{\kappa,\delta}, T_*\} = T_{\kappa,\delta}$. Let

$$(P_{\leq k}u)(x,y) = \sum_{n=-k}^{k} u_n(x)e^{\frac{iny}{L}},$$

where

$$u(x,y) = \sum_{n=-\infty}^{\infty} u_n(x)e^{\frac{iny}{L}}.$$

Then, for $\theta \in \mathbb{R}$ and $(x,y) \in \mathbb{R} \times \mathbb{T}_L$

$$\begin{aligned}\left\|u_\delta(T_{\kappa,\delta},\cdot,\cdot) - e^{i\theta}\tilde{\varphi}_\omega(\cdot - x, \cdot - y)\right\|_{L^2} &\geq \left\|(I - P_{\leq 0})(u_\delta(T_{\kappa,\delta},\cdot,\cdot) - e^{i\theta}\tilde{\varphi}_\omega(\cdot - x, \cdot - y))\right\|_{L^2} \\ &= \left\|(I - P_{\leq 0})(u_\delta(T_{\kappa,\delta}) - e^{i\omega T_{\kappa,\delta}}\tilde{\varphi}_\omega)\right\|_{L^2} \\ &= \left\|(I - P_{\leq 0})(v^{ap}_{M,\delta}(T_{\kappa,\delta}) + w(T_{\kappa,\delta}))\right\|_{L^2} \\ &\geq c\left\|\delta e^{T_{\kappa,\delta}\mu_*}\chi\right\|_{L^2} - C\delta^2 e^{2(T_{\kappa,\delta}\mu_*)} \geq c\kappa - C\kappa^2.\end{aligned}$$

For sufficiently small $\kappa > 0$ we have

$$\left\|u_\delta(T_{\kappa,\delta},\cdot,\cdot) - e^{i\theta}\tilde{\varphi}_\omega(\cdot - x, \cdot - y)\right\|_{L^2} \geq \frac{c\kappa}{2}.$$

This inequality shows the instability for the line standing wave $e^{i\omega t}\tilde{\varphi}_\omega$.

4 Outline of the proof of (ii) of Theorem 2 for $p \neq 3$

In this section, we explain the outline of the proof of (ii) of Theorem 2 for $p \neq 3$. Let $\omega > 0$, $1 < p < 5$ and $L > L_{\omega,p}$. For $1 < p < 5$ with $p \neq 3$, the nonlinearity $|u|^{p-1}u$ is not smooth in the sense Fréchet differentiation. Therefore, we can not apply the argument in [22] to the case $p \neq 3$.

By Duhamel's principle, we have

$$v_\delta(t) = \delta e^{t\mu_*}\chi - J\int_0^t e^{-(t-s)JA}F(v_\delta(s))ds.$$

Then we have the following estimate for the semi group e^{-tJA}.

Lemma 6. *For $k > 0$ and $\varepsilon > 0$, there exists $C_{k,\varepsilon} > 0$ such that*

$$\left\|e^{-tJA}P_{\leq k}v\right\|_{L^2} \leq C_{k,\varepsilon}e^{(\mu_*+\varepsilon)t}\|P_{\leq k}v\|_{L^2}, \quad t \geq 0, v \in L^2(\mathbb{R} \times \mathbb{T}_L).$$

The proof of this lemma is similar to the proof of Lemma 3.3 in [28].

Remark 7. The estimate

$$\left\|e^{-tJA}v\right\|_{L^2} \leq C_\varepsilon e^{(\mu_*+\varepsilon)t}\|v\|_{L^2}, \quad t \geq 0, v \in L^2(\mathbb{R} \times \mathbb{T}_L). \tag{7}$$

does not follow the proof of Lemma 3.3 in [28]. The estimate of (7) corresponding to the linearized operator of the one dimensional nonlinear Schrödinger equation (3) around the standing wave $e^{i\omega t}\varphi_\omega$ follows the spectrum mapping theorem in [8]. In [8], to prove the spectrum mapping theorem, we use the decay of the resolvent $(-\partial_x^2+\alpha_1+i\alpha_2)^{-1}$ as $|\alpha_1|\to\infty$ on a weighted space. However, $(-\partial_x^2-\partial_y^2+\alpha_1+i\alpha_2)^{-1}$ does not decay as $|\alpha_1|\to\infty$. Therefore, we can not show the estimate (7) in the argument in [28].

To control high frequency parts of $v_\delta(t)$, we apply the following lemma.

Lemma 8. *There exist $K_0>0$ and $C>0$ such that for $\delta>0$ and $t>0$*

$$\|v_\delta(t)\|_{H^1}\le C\|P_{\le K_0}v_\delta(t)\|_{L^2}+o(\delta)+o(\|v_\delta(t)\|_{H^1}).$$

Using the conservation law, we estimate high frequency parts and prove Lemma 8 in [28]. By Lemma 6 and Lemma 8, we have

$$\begin{aligned}\|v_\delta(t)\|_{H^1}&\le C\delta e^{t\mu_*}+C\int_0^t\|e^{-(t-s)JA}P_{\le K_0}F(v_\delta(s))\|_{L^2}ds+o(\delta)+o(\|v_\delta(t)\|_{H^1})\\&\le C\delta e^{t\mu_*}+\int_0^t e^{\min\{2,p\}(t-s)\mu}(\|v_\delta(s)\|_{H^1}^2+\|v_\delta(s)\|_{H^1}^p)ds+o(\delta)+o(\|v_\delta(t)\|_{H^1}).\end{aligned}$$

Thus, there exists $C_0>0$ such that for sufficiently small $\delta>0$ and $\kappa>0$

$$\|v_\delta(t)\|_{H^1}\le C_0e^{\mu_* t},\quad \text{for } t\in[0,T_{\kappa,\delta}],$$

where

$$T_{\kappa,\delta}=\frac{\log(\kappa/\delta)}{\mu_*}.$$

Then,

$$\begin{aligned}|\langle\chi,v_\delta(T_{\kappa,\delta})\rangle_{L^2}|&=\left|\delta e^{\mu_*T_{\kappa,\delta}}+\int_0^{T_{\kappa,\delta}}\langle\chi,-Je^{(T_{\kappa,\delta}-s)JA}F(v_\delta(s))\rangle_{L^2}ds\right|\\&\le\kappa-C\int_0^{T_{\kappa,\delta}}e^{\min\{2,p\}(T_{\kappa,\delta}-s)\mu_*}(\|v_\delta(s)\|_{H^1}^2+\|v_\delta(s)\|_{H^1}^p)ds\\&\le\kappa-C\kappa^{\min\{2,p\}}.\end{aligned}$$

Since

$$\|(I-P_{\le0})v\|_{L^2}\ge|\langle\chi,v\rangle_{L^2}|,$$

we have for $(x,y)\in\mathbb{R}\times\mathbb{T}_L$ and $\theta\in\mathbb{R}$

$$\begin{aligned}\|u_\delta(T_{\kappa,\delta},\cdot,\cdot)-e^{i\theta}\tilde\varphi_\omega(\cdot-x,\cdot-y)\|_{L^2}&\ge\|(I-P_{\le0})(u(T_{\kappa,\delta})-e^{i\omega t}\tilde\varphi_\omega)\|_{L^2}\\&\ge\|(I-P_{\le0})v_\delta(T_{\kappa,\delta})\|_{L^2}\\&\ge\kappa-C\kappa^{\min\{2,p\}}.\end{aligned}$$

This shows the instability for the line standing wave $e^{i\omega t}\tilde\varphi_\omega$.

5 Outline of the proof of Theorem 3

In this section, we explain the outline of the proof of Theorem 3. Let $\omega_0 > 0$. We consider the case $L = L_{\omega_0,p}$. By Lemma 4, the linearized operator $-J\mathcal{A}$ of (1) around the line standing wave $e^{i\omega_0 t}\tilde{\varphi}_{\omega_0}$ does not have eigenvalues with the positive real part. Therefore, we can not apply the argument for the stability in [22, 28].

To prove the stability for the line standing wave $e^{i\omega_0 t}\tilde{\varphi}_{\omega_0}$, we consider the Lyapunov functional method. We define the action

$$S_\omega(u) = E(u) + \omega Q(u).$$

Then, $\tilde{\varphi}_{\omega_0}$ is a critical point of S_{ω_0} and $S''_{\omega_0}(\tilde{\varphi}_{\omega_0}) = \mathcal{A}$.

For $0 < \omega < \omega_0$, we have

$$\mathrm{Ker}(S''_\omega(\tilde{\varphi}_\omega)) = \mathrm{Span}\{i\tilde{\varphi}_\omega, \partial_x\tilde{\varphi}_\omega\},$$

where $\mathrm{Span}\{v_1, \dots, v_k\}$ means the $\mathbb{R}$-linear space spanned by $v_1, \dots, v_k$. Moreover, $S''_\omega(\tilde{\varphi}_\omega)$ has exactly one negative eigenvalue and the negative eigenvalue of $S''_\omega(\tilde{\varphi}_\omega)$ is simple. We introduce the distance and neighborhoods

$$d_\omega(u) = \inf_{\theta, x\in\mathbb{R},} \left\|u(\cdot,\cdot) - e^{i\theta}\tilde{\varphi}_\omega(\cdot - x, \cdot)\right\|_{H^1},$$

$$N_{\varepsilon,\omega} = \{u \in H^1 | d_\omega(u) < \varepsilon\},$$
$$N^0_{\varepsilon,\omega} = \{u \in N_{\varepsilon,\omega} | Q(u) = Q(\tilde{\varphi}_\omega)\}.$$

Using the gauge transform $e^{i\theta}$, the phase shift and the mass conservation, we control the kernel and the negative eigenvalue of $S''_\omega(\tilde{\varphi}_\omega)$ and obtain the following coerciveness lemma.

Lemma 9. *Let* $0 < \omega < \omega_0$. *Then there exist* $c, \varepsilon_0 > 0$, $\theta(u) : N^0_{\varepsilon_0,\omega} \to \mathbb{R}$ *and* $b(u) : N^0_{\varepsilon_0,\omega} \to \mathbb{R}$ *such that for* $u \in N^0_{\varepsilon_0,\omega}$

$$E(u) - E(\tilde{\varphi}_\omega) \geq c\left\|u(\cdot,\cdot) - e^{i\theta(u)t}\tilde{\varphi}_\omega(\cdot - b(u), \cdot)\right\|^2_{H^1}.$$

The proof of Lemma 9 follows the analysis of the linearized operator $S''_\omega(\tilde{\varphi}_\omega)$ in the proof of Theorem 3.4 of [12]. The stability of the line standing wave $e^{i\omega t}\tilde{\varphi}_\omega$ with $0 < \omega < \omega_0$ follows proof by contradiction. We assume there exist $\varepsilon_1 > 0$, a sequence $\{t_n\}_n$ and a sequence $\{u_n\}_n$ of solutions such that $t_n > 0$ and $u_n(0) \to \tilde{\varphi}_\omega$ in H^1 and

$$\inf_{\theta\in\mathbb{R}} \left\|u_n(t_n) - e^{i\theta}\tilde{\varphi}_\omega\right\|_{H^1} > \varepsilon_1. \tag{8}$$

Let

$$v_n = \sqrt{\frac{Q(\tilde{\varphi}_\omega)}{Q(u_n)}}u_n(t_n).$$

Since Q is the mass conservation law, we have $Q(v_n) = Q(\tilde{\varphi}_\omega)$. By the definition of v_n, $\|v_n - u_n(t_n)\|_{H^1} \to 0$ and $E(v_n) - E(u_n) \to 0$ as $n \to \infty$. Therefore, $E(v_n) - E(\tilde{\varphi}_\omega) \to 0$

as $n \to \infty$. By Lemma 9, we have $d_\omega(u_n(t_n)) \le C(E(v_n) - E(\tilde\varphi_\omega) + \|v_n - u_n(t_n)\|_{H^1}) \to 0$ as $n \to \infty$. This contradicts the assumption (8) and we obtain the stability of the line standing wave $e^{i\omega t}\tilde\varphi_\omega$.

In the case $\omega = \omega_0$, we have

$$\mathrm{Ker}(\mathcal{A}) = \mathrm{Span}\{i\tilde\varphi_{\omega_0}, \partial_x\tilde\varphi_{\omega_0}, \psi_{\omega_0}\cos(y/L), \psi_{\omega_0}\sin(y/L)\},$$

where ψ_ω is the eigenfunction of $-\partial_x^2 + \omega - p|\varphi_\omega|^{p-1}\varphi_\omega$ corresponding to the negative eigenvalue and satisfying

$$\psi_\omega = (\varphi_\omega)^{\frac{p+1}{2}}.$$

Then, the kernel of $\mathcal{A}$ has extra functions $\psi_{\omega_0}\cos(y/L), \psi_{\omega_0}\sin(y/L)$. Therefore, the analysis for the second derivative of the action S_{ω_0} or the energy E are not sufficient to prove the coerciveness lemma. In the following proposition, we show the bifurcation of standing waves.

Proposition 10. *Let $p \ge 2$. Then there exist an open interval I and $\varphi(a) \in C^2(I, H^2(\mathbb{R}\times\mathbb{T}_L))$ such that $0 \in I$, $\varphi(a) > 0$,*

$$-\Delta\varphi(a) + \omega(a)\varphi(a) - |\varphi(a)|^{p-1}\varphi(a) = 0,$$

$$\varphi(a) = \tilde\varphi_{\omega_0} + a\psi_{\omega_0}\cos(y/L) + r(a),$$

where $\|r(a)\|_{H^2} = O(a^2)$,

$$\omega(a) = \omega_0 + \frac{\omega''(0)}{2}a^2 + o(a^2).$$

The proof of Proposition 10 follows the proof of Theorem 4 in [15] (see [29]). Proposition 10 shows that extra functions $\psi_{\omega_0}\cos(y/L), \psi_{\omega_0}\sin(y/L)\}$ of the kernel of $\mathcal{A}$ come from the bifurcation of standing waves. Combining the argument in Maeda [16] and Proposition 10, we prove the following lemma.

Lemma 11. *Let $p \ge 2$. There exist $\varepsilon_0, C > 0$, $\theta(u) : N_{\varepsilon_0,\omega_0} \to \mathbb{R}$, $b(u) : N_{\varepsilon_0,\omega_0} \to \mathbb{R}$, $a(u) : N_{\varepsilon_0,\omega_0} \to \mathbb{R}$, $\alpha(u) : N_{\varepsilon_0,\omega_0} \to \mathbb{R}$ and $\rho(a) : \mathbb{R} \to \mathbb{R}$ such that for $u \in N^0_{\varepsilon_0,\omega_0}$*

$$\begin{aligned} S_{\omega_0}(u) - S_{\omega_0}(\tilde\varphi_{\omega_0}) &= \frac{1}{2}\langle Aw(u), w(u)\rangle_{H^{-1},H^1} + \eta(a(u)) + o(\|w(u)\|_{H^1}^2) + o(\eta(a(u))) \\ &= \frac{1}{2}\langle Aw(u), w(u)\rangle_{H^{-1},H^1} + C\frac{d^2\|\varphi(a)\|_{L^2}^2}{da^2}\Big|_{a=0}|a(u)|^4 \\ &\quad + o(\|w(u)\|_{H^1}^2) + o(|a(u)|^4), \end{aligned}$$

where $\rho(a) = O(a^2)$, $\alpha(u) = o(d_{\omega_0}(u))$,

$$\eta(a) = S_{\omega(a)}(\varphi(a)) - S_{\omega_0}(\tilde\varphi_{\omega_0}) + (\omega_0 - \omega(a))Q(\tilde\varphi_{\omega_0}),$$

$$w(u)(x,y) = e^{i\theta(u)}u(x - b(u), y) - (1 + \alpha(u))\varphi(a(u))(x,y) - \rho(a)\partial_\omega\tilde\varphi_\omega|_{\omega=\omega_0}(x,y).$$

The proof of Lemma 11 is in Section 3 in [29]. Lemma 11 shows that the sign of $\frac{d^2\|\varphi(a)\|_{L^2}^2}{da^2}|_{a=0}$ changes the structure of the action on $N^0_{\varepsilon_0,\omega_0}$. Applying the stability argument in [16], we obtain the following proposition (see [16, 29]).

Proposition 12. *Let* $p \geq 2$. *We have the following two.*

(i) *If* $\frac{d^2\|\varphi(a)\|_{L^2}^2}{da^2}|_{a=0} > 0$, *then the line standing wave* $e^{i\omega_0 t}\tilde{\varphi}_\omega$ *is stable.*

(ii) *If* $\frac{d^2\|\varphi(a)\|_{L^2}^2}{da^2}|_{a=0} < 0$, *then the line standing wave* $e^{i\omega_0 t}\tilde{\varphi}_\omega$ *is unstable.*

Estimating $\frac{d^2\|\varphi(a)\|_{L^2}^2}{da^2}|_{a=0}$, we obtain Theorem 3.

Remark 13. We can not obtain the exact value of $\frac{d^2\|\varphi(a)\|_{L^2}^2}{da^2}|_{a=0}$ in [29]. Therefore, we do not show the stability of the line standing wave $e^{i\omega_0 t}\tilde{\varphi}_{\omega_0}$ for $p_1 \leq p \leq p_2$. Moreover, in Proposition 10 to obtain the C^2 regularity of $\varphi(a)$ with respect to a we use $p \geq 2$. Thus, we do not show the stability of the line standing wave $e^{i\omega_0 t}\tilde{\varphi}_{\omega_0}$ for $1 < p < 2$.

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