数理解析研究所講究録
第 1948 巻 2015 年 77-84

A note on linear deformations of plane curve singularities

K.Inaba*, M.Ishikawa†, M.Kawashima‡ and T.T.Nguyen§

1. Introduction

It is well known that a complex linear deformation of a complex isolated singularity is generically a Morse function (cf. [4, 18]). Instead of complex linear deformations, we are interested in "real" linear deformations of complex singularities. In [7], the authors studied linear deformations of plane curve singularities of Brieskorn type. In this note, we shortly introduce our result including backgrounds.

Our main result is the following:

Theorem 1.1 ([7]). *Let $f : \mathbb{C}^2 \to \mathbb{C}$ be a polynomial map given by $f(z,w) = z^p + w^q$ with $p, q \geq 2$. For any generic choice of $a, b \in \mathbb{C}$, there exists a linear deformation $f_t(z,w)$ of f such that f_t is a generic map for any $t \in (0,1]$ and $f_1(z,w) = f(z,w) + a\bar{z} + b\bar{w}$.*

Here a deformation of f is called *linear* if it is given in the form $f_t(z,w) = f(z,w) + a_1 z + b_1 w + a_2\bar{z} + b_2\bar{w}$, where a_1, b_1, a_2, b_2 are analytic functions with variable $t \in \mathbb{R}$ which vanish at $t = 0$ and $\bar{z}$ and $\bar{w}$ are the complex conjugates of z and w respectively. See Section 3 for the definition of a generic map.

The singular set of a linear deformation of $f(z,w) = z^2 + w^2$ has three cusps and the image of the singular set is as shown in Figure 1. This example appears in a paper of Y. Lekili [9, Move 4 in p.292] as a move which modifies a Lefschetz fibration into a generic map. Thanks to Theorem 1.1, we are sure that the set of linear deformations of plane curve singularities of Brieskorn type into generic maps is non-empty. Therefore we may ask how many cusps do they have. The answer is the following:

Theorem 1.2 ([7]). *Let f_t be a linear deformation into generic maps in Theorem 1.1. Suppose that $p \geq q \geq 2$. Then the number $c(f_t)$ of cusps of f_t, $t \in (0,1]$, satisfies the inequalities $(p+1)(q-1) \leq c(f_t) \leq (p-1)(q+1)$.*

If we restrict the problem to the case where f is a Morse singularity, i.e., $f(z,w) = z^2 + w^2$, we can show that any linear deformation f_t of f is a generic map in general and the set of singular values of f_t, $t \in (0,1]$, in $\mathbb{R}^2$ is a scaling and rotation of the curve in Figure 1.

The authors would like to thank the organizers of the Workshop "Singularity theory of differential maps and its applications" held in RIMS, Kyoto, during 2 - 5th December 2014. The second and fourth authors are supported by the Japan

*Tohoku University, sb0d02@math.tohoku.ac.jp
†Tohoku University, ishikawa@m.tohoku.ac.jp
‡Tokyo University of Science, kawashima@ma.kagu.tus.ac.jp
§Tohoku University & Institute of Mathematics, VAST, ntthang@math.ac.vn

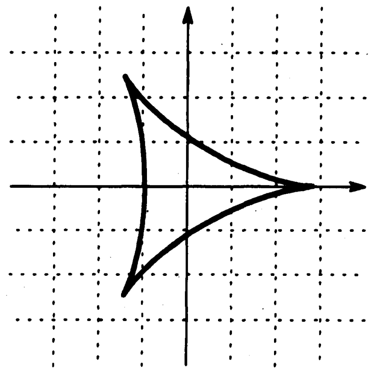

FIGURE 1. The image of singular set of a linear deformation of $f(z,w) = z^2 + w^2$. This curve is parametrized as $h(\theta) = e^{2i\theta} + 2e^{-i\theta}$, $\theta \in S^1$.

Society for the Promotion of Science (JSPS) Postdoctoral Fellowship for Foreign Researchers' Grant-in-Aid 25/03014.

2. Two backgrounds

In recent studies, there are two approaches in the study of real singularities from viewpoint of complex singularity theory; one is a *broken Lefschetz fibration* and the other is a *mixed polynomial.*

A broken Lefschetz fibration is a Lefschetz fibration which may have indefinite fold singularities. It is proved by O. Saeki in [17] that any continuous map from a closed manifold of dimension $n > 2$ into S^2 is homotopic to a C^∞ stable map without definite fold singularity. In particular, any closed manifold of dimension $n > 2$ admits a broken Lefschetz fibration. On the other hand, D. Auroux, S.K. Donaldson, L. Katzarkov revealed a relationship between broken Lefschetz fibrations and near-symplectic structures in closed 4-manifolds [1], which is a generalization of the correspondence between Lefschetz fibrations and symplectic structures due to S.K. Donaldson [3] and R. Gompf [5]. A closed 2-form ω on a closed 4-manifold X is called a *near-symplectic structure* if $\omega^2 \geq 0$, ω does not have rank 2 at any point and, at each point x where ω vanishes, the rank of the intrinsically defined derivative $\nabla\omega_x : TX_x \to \Lambda^2 T^* X_x$ is 3. Note that the set of points where ω vanishes is a 1-dimensional submanifold in X, which corresponds to the set of indefinite fold singularities. Y. Lekili then presented a set of moves which relates broken Lefschetz fibrations. The linear deformation shown in Figure 1 was introduced in his paper as a move which removes a Morse singularity in a broken Lefschetz fibration.

A mixed polynomial is a polynomial with complex and complex-conjugate variables. Since any real polynomial $f : \mathbb{R}^{2n} \to \mathbb{R}^2$ with even variables can be represented by a mixed polynomial as

$$f(x_1, y_1, \ldots, x_n, y_n) = f\left(\frac{z_1 - \bar{z}_1}{2}, \frac{z_1 + \bar{z}_1}{2i}, \ldots, \frac{z_n - \bar{z}_n}{2}, \frac{z_n + \bar{z}_n}{2i}\right)$$

the class of mixed polynomials coincides with the class of real polynomials $f : \mathbb{R}^{2n} \to \mathbb{R}^2$. The notion of mixed polynomial was introduced by M.A. Ruas, J. Seade and A. Verjovsky in [16] implicitly, and by J. Cisneros-Molina in [2]. It is natural to ask what kind of real singularities having properties similar to complex singularities. For example, a real singularity of type $f\bar{g}$, which is a product of complex and complex-conjugate polynomials, had been studied by A. Pichon and J. Seade [13, 14, 15]. These have nice properties similar to complex singularities. M. Oka studied the singularities of mixed polynomials from viewpoint of Newton polygons, which also have nice properties similar to complex ones. Remark that both of these singularities are very far from singularities of stable maps since these singularities are usually isolated.

A motivation of our paper [7] is to give a concrete discussion on the move of Lekili, i.e., linear deformations of Morse singularities, and generalize the result into the singularities of Brieskorn type. Remark that it is difficult to say that his move yields a stable map because the source manifold is not compact. Recently, the first author and the third author studied the same problem for singularities of type $f\bar{g}$ and for higher dimensional case respectively, see [6] and [8].

We close this section with one useful lemma.

Lemma 2.1 ([11]). *Let f be a mixed polynomial with variables $(z_1, \dots, z_n)$ and their conjugates. A point $p \in \mathbb{C}^n$ is a singular point of f if and only if there exists a complex number α with $|\alpha| = 1$ such that*

$$\overline{\frac{\partial f}{\partial z_i}(p)} = \alpha \frac{\partial f}{\partial \bar{z}_i}(p), \quad i = 1, \dots, n.$$

The linear deformations in Theorem 1.1 are given in the form of mixed polynomials and the set of singularities is determined by the above equations. Since the set of singularities is one-dimensional, the indeterminate value $\alpha \in S^1$ can be regarded as a parameter of the set of singularities.

3. Generic maps and Levine's criterion

Let X be a 4-manifold and Y be a 2-manifold.

Definition 3.1. A smooth map $f : X \to Y$ is called a *generic map* if for each point $p \in X$, there exist local coordinates (x_1, x_2, x_3, x_4) centered at p and those of Y at $f(p)$ such that f is locally described in one of the following form:

(1) $(x_1, x_2, x_3, x_4) \mapsto (x_1, x_2)$,
(2) $(x_1, x_2, x_3, x_4) \mapsto (x_1, x_2^2 + x_3^2 + x_4^2)$,
(3) $(x_1, x_2, x_3, x_4) \mapsto (x_1, x_2^2 + x_3^2 - x_4^2)$,
(4) $(x_1, x_2, x_3, x_4) \mapsto (x_1, x_2^2 \pm x_3^2 + x_1 x_4 + x_4^3)$.

The point in case (1) is a regular point. The point in case (2), (3) and (4) is called a *definite fold*, an *indefinite fold* and a *cusp*, respectively.

Note that generic maps actually exist "generically" in $C^\infty(X, Y)$.

We prepare a few notations. Let $f : X \to Y$ be a smooth map and df denote the induced map from TX to $f^{-1}(TY)$ and $df_p = df|T_pX$ for $p \in X$, where TX is the tangent bundle of X, T_pX is the tangent space of X at p and $f^{-1}(TY)$ is the vector bundle over X whose fiber at $p \in X$ is $T_{f(p)}Y$. Set

$$S_1(f) = \{p \in X \mid \operatorname{rank} df_p = 1\}.$$

Let U and V be coordinate neighborhoods of $p \in X$ and $f(p) \in Y$, respectively, such that $f(U) \subset V$. Since $TX|U$ and $TY|V$ are trivial we can choose bases $\{u_i\}$

and $\{v_j\}$ of the sections of these restricted bundles. Let $\{u_i^*\}$ and $\{v_j^*\}$ be the associated dual bases

$$\langle u_i, u_{i'}^* \rangle = \delta_{ii'}, \quad \langle v_j, v_{j'}^* \rangle = \delta_{jj'},$$

where $\langle\ ,\ \rangle$ is the pairing of a vector space with its dual. Let $w_j = v_j \circ f$ and $w_j^* = v_j^* \circ f$. Since df is linear on each fiber, there are smooth functions $a_{ij}, i = 1, \ldots, 4$, $j = 1, 2$, such that

$$df = \sum_{i,j} a_{ij} u_i^* \otimes w_j,$$

where

$$(df(u_i))_p = \sum_{j=1,2} a_{ij}(p) w_j(p).$$

To prove Theorem 1.1, we need to calculate the higher differentials of H. Levine in [10] by choosing suitable basis. We here explain a recipe how to determine if a given map is a generic map or not. For details of higher differentials, see [10].

Suppose $p \in S_1(f)$. We may choose local coordinates (x_1, x_2, x_3, x_4) of X at p and those of Y at $f(p)$ such that $f = (g, h) : \mathbb{R}^4 \to \mathbb{R}^2$ satisfies $\operatorname{grad} g(p) = (1, 0, 0, 0)$ and $\operatorname{grad} h(p) = (0, 0, 0, 0)$, where $\mathbb{R}^4$ and $\mathbb{R}^2$ are regarded as coordinate neighborhoods of X at p and Y at $f(p)$ respectively. Choosing these neighborhoods sufficiently small, we may assume that $\left\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}\right\}$ is a basis of sections of $T\mathbb{R}^4$ in the neighborhood of p. Set $E = TX|S_1(f)$ and $F = f^{-1}(TY)|S_1(f)$ and define L and G by the exactness of the sequence

$$0 \longrightarrow L \longrightarrow E \xrightarrow{df} F \xrightarrow{\pi_1} G \longrightarrow 0.$$

We denote the fibers of L and G at $p \in S_1(f)$ by L_p and G_p respectively. Define the map $\varphi^1 : E \to L^* \otimes F$, for each $p \in S_1(f)$, by

$$\varphi_p^1(v)(l) = \sum_{i,j} \langle v, da_{ij}(p) \rangle \langle l, u_i^*(p) \rangle w_j(p), \tag{3.1}$$

with $v \in T_pX$ and $l \in L_p$. Then the second differential $d^2 f : E \to L^* \otimes G$ of f is defined as $d^2 f_p(v)(l) = \pi_1(\varphi_p^1(v)(l))$. In our setting, $\dim L_p = 3, \dim G_p = 1$, and $d^2 f_p$ is represented by the matrix

$$M = \left(\frac{\partial^2 h}{\partial x_i \partial x_j}\right)_{i=1,2,3,4,\ j=2,3,4.}$$

Hence $d^2 f_p$ is surjective if and only if $\operatorname{rank} M = 3$. We can check that the restriction $d^2 f_p | L_p$ is represented by

$$H = \left(\frac{\partial^2 h}{\partial x_i \partial x_j}\right)_{i=2,3,4, j=2,3,4,}$$

which is exactly the Hessian of h with variables (x_2, x_3, x_4).

Lemma 3.2. *In the above setting, $p \in S_1(f)$ is a fold if and only if* $\operatorname{rank} H = 3$ *at* p.

Suppose $\operatorname{rank} H = 2$ at $p \in X$ and $d^2 f_p$ is surjective. By choosing suitable coordinates (x_1, x_2, x_3, x_4), we may further assume that $\frac{\partial^2 h}{\partial x_4 \partial x_j}(p) = 0$ for all $j = 2, 3, 4$. Then set

$$\xi_j = -\frac{\partial g}{\partial x_j}\frac{\partial}{\partial x_1} + \frac{\partial g}{\partial x_1}\frac{\partial}{\partial x_j}$$

for $j = 2, 3, 4$. The set $\{\xi_2, \xi_3, \xi_4\}$ is a basis of $L|S_1(f) \cap U$ for some neighborhood U of p. We omit the definition of the third differential $d^3 f_p$ in this note. The point is that it is known that p is a cusp if and only if $d^3 f_p$ is surjective, and the surjectivity is equivalent to the inequality

$$\frac{\partial}{\partial x_4}(\xi_4(\xi_4(h)))(p) \neq 0.$$

We then have the following criterion to check if a singularity is a cusp or not.

Lemma 3.3. *In the above setting, $p \in S_1(f)$ is a cusp if and only if* $\operatorname{rank} M = 3$, $\operatorname{rank} H = 2$ *and* $\frac{\partial}{\partial x_4}(\xi_4(\xi_4(h)))(p) \neq 0$.

4. Outline of the proofs

To prove Theorem 1.1, we need to determine if a polynomial map of the form $f(z, w) = z^p + w^q + a\bar{z} + b\bar{w}$ is a generic map or not. Since the assertion in Theorem 1.1 is for generic a and b, we may assume that $ab \neq 0$. Let c_1 and c_2 be non-zero complex numbers satisfying $c_1^p = a\bar{c}_1$ and $c_2^q = b\bar{c}_2$, respectively. By changing the coordinates as $z = c_1 u$ and $w = c_2 v$ and setting $\mu = a\bar{c}_1/(b\bar{c}_2)$, we have

$$\begin{aligned} f(z, w) &= (c_1 u)^p + a\overline{c_1 u} + (c_2 v)^q + b\overline{c_2 v} \\ &= a\bar{c}_1(u^p + \bar{u}) + b\bar{c}_2(v^q + \bar{v}) \\ &= b\bar{c}_2(\mu(u^p + \bar{u}) + v^q + \bar{v}). \end{aligned}$$

Now we set

$$P(u, v; \mu) = \mu(u^p + \bar{u}) + v^q + \bar{v},$$

with $p, q \geq 2$ and $\mu \in \mathbb{C} \setminus \{0\}$. The mixed polynomial f is a generic map in general if and only if P is. Hence hereafter we study the map P instead of f.

Remark 4.1. If $P(u, v; \mu)$ is a generic map then, by changing the radii of a and b with keeping their ratio, we can obtain a linear deformation $f_t(u, v)$ of $f(u, v)$ consisting of generic maps with the same property for $t \in (0, 1]$. Hence to prove Theorem 1.1, it is enough to show that $P(u, v; \mu)$ is a generic map for a generic choice of μ.

Thanks to Lemma 2.1, the set $S(P)$ of singular point of P can be described explicitly as follows:

Lemma 4.2. $z_0 = (u_0, v_0) \in S(P)$ *if and only if*

$$\begin{cases} p|u_0|^{p-1} = q|v_0|^{q-1} = 1, \\ \dfrac{p-1}{2}\arg u_0 + \arg\mu = \dfrac{q-1}{2}\arg v_0 + \kappa\pi, \end{cases}$$

where κ is some integer.

To apply the recipe explained in Section 3, we first need to choose local coordinates of $\mathbb{R}^4$ at $z_0 \in S(P)$ and coordinates of $\mathbb{R}^2$ at $P(z_0)$ such that $P : \mathbb{R}^4 \to \mathbb{R}^2$ satisfies $\text{grad}(\text{Re}P)(z_0) = (1,0,0,0)$ and $\text{grad}(\text{Im}P)(z_0) = (0,0,0,0)$. Set $Q(u,v;\mu)$ and $R(u,v;\mu)$ to be the real and imaginary part of $P(u,v;\mu)$ respectively, i.e., $P(u,v;\mu) = Q(u,v;\mu)+iR(u,v;\mu)$. Set $r_1 = |u|$, $\theta_1 = \arg u$, $r_2 = |v|$ and $\theta_2 = \arg v$, so that $(r_1, \theta_1, r_2, \theta_2)$ are regarded as the polar coordinates of $\mathbb{C}^2$. Since

$$\begin{aligned} P = Q + iR &= \mu(u^p + \bar{u}) + (v^q + \bar{v}) \\ &= |\mu| r_1^p e^{i(p\theta_1 + \theta_\mu)} + |\mu| r_1 e^{i(-\theta_1+\theta_\mu)} + r_2^q e^{iq\theta_2} + r_2 e^{-i\theta_2}, \end{aligned}$$

we have

$$\begin{cases} Q = |\mu| r_1^p \cos(p\theta_1 + \arg\mu) + |\mu| r_1 \cos(-\theta_1 + \arg\mu) + r_2^q \cos(q\theta_2) + r_2 \cos(-\theta_2), \\ R = |\mu| r_1^p \sin(p\theta_1 + \arg\mu) + |\mu| r_1 \sin(-\theta_1 + \arg\mu) + r_2^q \sin(q\theta_2) + r_2 \sin(-\theta_2). \end{cases}$$

Then

$$\begin{aligned} &\text{grad}\, Q(z_0) = (k_1, k_2, k_3, k_4) \\ &= (2|\mu| \cos\Theta_1 \cos\Theta_2, -2|\mu||u_0| \sin\Theta_1 \cos\Theta_2, 2\cos\Theta_3 \cos\Theta_4, -2|v_0| \sin\Theta_3 \cos\Theta_4), \end{aligned}$$

where

$$\Theta_1 = \frac{p+1}{2} \arg u_0, \qquad \Theta_2 = \frac{p-1}{2} \arg u_0 + \arg\mu,$$
$$\Theta_3 = \frac{q+1}{2} \arg v_0, \qquad \Theta_4 = \frac{q-1}{2} \arg v_0.$$

Now we change the coordinates as

$$(r_1', \theta_1', r_2', \theta_2') = (k_1 r_1 + k_2\theta_1 + k_3 r_2 + k_4\theta_2, \theta_1, r_2, \theta_2),$$

so that we have $\text{grad}\, Q(z_0) = (1,0,0,0)$. Set $\hat{R} = R - sQ$ with $s = \frac{\partial R}{\partial r_1'}(z_0)$. Then $(\hat{R}, Q)$ is regarded as new coordinates of $\mathbb{R}^2$ at $P(z_0)$, and it satisfies $\text{grad}\, \hat{R}(z_0) = (0,0,0,0)$. We need to use the condition $k_1 \neq 0$ in these changes of coordinates. The case $k_1 = 0$ can be discussed by choosing other suitable coordinates.

Suppose $k_1 \neq 0$. Then the matrix H in Section 3 is calculated as follows:

Lemma 4.3. *The Hessian H of $\hat{R}$ with variables $(\theta_1', r_2', \theta_2')$ is*

$$H = \begin{pmatrix} k_2^2 A - 2k_2 B + C & k_3(k_2 A - B) & k_4(k_2 A - B) \\ k_3(k_2 A - B) & k_3^2 A + D & k_3 k_4 A + E \\ k_4(k_2 A - B) & k_3 k_4 A + E & k_4^2 A + F \end{pmatrix},$$

where

$$A = \frac{1}{k_1^2} \frac{\partial^2 \hat{R}}{\partial r_1^2}, \quad B = \frac{1}{k_1} \frac{\partial^2 \hat{R}}{\partial r_1 \partial \theta_1}, \quad C = \frac{\partial^2 \hat{R}}{\partial \theta_1^2}$$
$$D = \frac{\partial^2 \hat{R}}{\partial r_2^2}, \quad E = \frac{\partial^2 \hat{R}}{\partial r_2 \partial \theta_2}, \quad F = \frac{\partial^2 \hat{R}}{\partial \theta_2^2}.$$

Its determinant is

$$\det H = (k_4^2 D - 2k_3 k_4 E + k_3^2 F)(AC - B^2) + (k_2^2 A - 2k_2 B + C)(DF - E^2).$$

By Lemma 3.2, we can conclude that $z_0 \in S(P)$ is a fold if and only if $\det H(z_0) = 0$. A point z_0' with $\det H(z_0') = 0$ is possibly a cusp. To know if it is actually a cusp, we need to check the inequality $\frac{\partial}{\partial x_4}(\xi_4(\xi_4(h)))(p) \neq 0$ mentioned in Lemma 3.3 after applying further change of coordinates. See [7] in detail.

We shortly explain about the proof of Theorem 1.2. By Lemma 4.2, we see that the set of singular points of P consists of r parallel curves C_k, $k = 0, \cdots, r-1$, on the torus $\{(u,v) \in \mathbb{C}^2 \mid |u| = A, |v| = B\}$, each of which is parametrized, with parameter $e^{i\theta} \in S^1$, as

$$(u,v) = \left(Ae^{(\frac{q-1}{r}\theta + c_k)i}, Be^{\frac{p-1}{r}\theta i}\right),$$

where $r = \gcd(p-1, q-1)$, $A = 1/p^{1/(p-1)}$, $B = 1/q^{1/(q-1)}$ and $c_k = \frac{1}{p-1}(-2\arg\mu + 2\pi k)$. Set the map $P_k : C_k \to \mathbb{C}$ as

$$\begin{aligned} P_k(\theta) &= P\left(Ae^{(\frac{q-1}{r}\theta + c_k)i}, Be^{\frac{p-1}{r}\theta i}\right) \\ &= \mu\left(A^p e^{(\frac{p(q-1)}{r}\theta + pc_k)i} + Ae^{-(\frac{q-1}{r}\theta + c_k)i}\right) + B^q e^{\frac{q(p-1)}{r}\theta i} + Be^{-\frac{p-1}{r}\theta i}. \end{aligned}$$

Since P is assumed to be a generic map, the set of cusps of P on C_k corresponds to the roots of $dP_k/d\theta = 0$. The left hand side is calculated as

$$\frac{dP_k}{d\theta} = -2e^{\frac{(p-1)(q-1)}{2r}\theta i}\Phi(\theta)$$

with

$$\begin{aligned} \Phi(\theta) = (-1)^k|\mu|\frac{q-1}{r}A\sin\left(\frac{(p+1)(q-1)}{2r}\theta + \frac{p+1}{2}c_k\right) \\ + \frac{p-1}{r}B\sin\left(\frac{(p-1)(q+1)}{2r}\theta\right). \end{aligned}$$

Hence, to determine the number of cusps, it is enough to count the number of roots of this equation. Theorem 1.2 is proved by observing this number explicitly, see [7] in detail.

5. Questions

It is interesting to consider how we can generalize the results in Theorem 1.1 and 1.2 to more general settings. We close this note with proposing a few questions.

Question 5.1. *Let $f(z,w) = z^p + w^q$ be a Brieskorn polynomial with $p \geq q \geq 2$ and f_t be a linear deformation of f into generic maps. Does the number $c(f_t)$ of cusps of f_t, $t \in (0,1]$, appearing in a previously fixed small neighborhood of the origin satisfy the inequalities $(p+1)(q-1) \leq c(f_t) \leq (p-1)(q+1)$?*

Question 5.2. *Estimate the number of cusps appearing in a linear deformation of a Brieskorn type singularity in higher dimension.*

The third author studied the second question in the case where $f(z_1, \ldots, z_n) = z_1^q + \cdots + z_n^q$ with $q \geq 2$ and the linear terms for the deformation have only complex conjugate variables. In that case, a generic map obtained by a linear deformation has $(q+1)(q-1)^n$ cusps. See [8].

Question 5.3. *Is a linear deformation obtained in Theorem 1.1 a stable map?*

In Theorem 1.1, we proved that the map is a generic map by using Levine's criterion. However, since the souce manifold is open, it seems to be difficult to determine if the map is stable or not.

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