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Geometry of timelike Bertrand curves in Anti de Sitter 3-Space

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Abstract

We investigate the properties of the timelike Bertrand curves in Anti de Sitter 3-space and give a sufficient and necessary condition for a timelike curve is a Bertrand curve, i.e., a timelike curve α in Anti de Sitter 3-space is a Bertrand curve if and only if either (1) torsion $\tau = 0$, curvature $\kappa \neq -\coth a, a \in \mathbb{R}$ or (2)there exist two constants $\lambda \neq 0$ and μ such that $\mu\tau - \lambda\kappa_g = 1$. Moreover, we also characterize the relationship between timelike Bertrand curves in Anti de Sitter 3-space and spacelike Bertrand curves in semi-Euclidean 4-space with index two.

Keywords: Anti de Sitter 3-space; Semi-Euclidean space; Timelike Bertrand curves; Spacelike Bertrand curves.

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1 Introduction

It is well known that there are two kinds of space form with constant sectional curvature which are Riemannian space from and Lorentizian space from. The Lorentzian space form with the negative constant curvature is called Anti de Sitter space which is one of the vacuum solutions of the Einstein equation in the theory of relativity. The third author and his collaborators had investigated the surfaces in Anti de Sitter 3-space from the viewpoint of singularity theory[1, 2]. As results, they studied the contact of surfaces with some models (invariant under the action of a suitable transformation group).

In this paper we consider the geometric properties of a special classes of curves in Anti de Sitter 3-space, so called *timelike Bertrand curves*. Since the Bertrand curves have many applications in nature science, such as in CAGD (computer-aided geometric design, see [3, 4]), the notion of Bertrand curves play important roles in the classical differential geometry for curves in Euclidean space. The history of the study of Bertrand curves is from the beginning of the study of helix. B. de Saint-Venant proved in 1845 that a curve is a general helix if and only if the ratio of its curvature κ to its torsion τ is a constant, i.e., $\tau/\kappa = c$ for c is a constant in [5]. In 1850, J. Bertrand investigated another geometric property of helices. If a curve satisfies this property it is called Bertrand curve named after his study in [6]. As we know, a curve in $\mathbb{R}^3$

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with the curvature κ and the torsion τ is a Bertrand curve if and only if it is a plane curve or it satisfies $\kappa + a\tau = b$ for constant a and $b \neq 0$. Moreover, there are several articles concerning the Bertrand curves immersed in different ambient space [7, 8, 9, 10, 11, 12, 13]. Especially, Y.H. Kim and P.Lucas had studied the Bertrand curves in three dimensional sphere respectively in [14, 15]. Motivated by their studies, we define the timelike Bertrand curves in Anti de Sitter 3-space and investigate their properties. As results, we prove that a timelike curve α in Anti de Sitter 3-space is a Bertrand curve if and only if either (1) $\tau = 0, \kappa \neq -\coth a, a \in \mathbb{R}$ or (2)there exist two constants $\lambda \neq 0$ and μ such that $\mu\tau - \lambda\kappa = 1$. Moreover, we also characterize the relationship between timelike Bertrand curves in Anti de Sitter 3-space and spacelike Bertrand curves in semi-Euclidean 4-space with index two.

We shall assume throughout the whole paper that all the maps are C^∞ and all the curves immersed in Anti de Sitter 3-space are timelike curves unless the contrary is explicitly stated.

2 The local differential geometry of timelike curves in Anti de Sitter 3-space

In this section we prepare some basic notions on semi-Euclidean 4-space with index 2 and introduce the local differential geometry of timelike curves in Anti de Sitter 3-space.

Let $\mathbb{R}^4 = \{(x_1, \cdots, x_4) | x_i \in \mathbb{R}\ (i = 1, \cdots, 4)\ \}$ be a 4-dimensional vector space. For any vectors $\boldsymbol{x} = (x_1, \cdots, x_4)$ and $\boldsymbol{y} = (y_1, \cdots, y_4)$ in $\mathbb{R}^4$, the *pseudo scalar product* of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined to be $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -x_1y_1 - x_2y_2 + x_3y_3 + x_4y_4$. We call $(\mathbb{R}^4, \langle , \rangle)$ a *smei-Euclidean 4-space with index 2* and write $\mathbb{R}^4_2$ instead of $(\mathbb{R}^4, \langle , \rangle)$.

We say that a non-zero vector $\boldsymbol{x}$ in $\mathbb{R}^4_2$ is *spacelike*, *null* or *timelike* if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0, \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0$ or $\langle \boldsymbol{x}, \boldsymbol{x} \rangle < 0$ respectively. The norm of the vector $\boldsymbol{x} \in \mathbb{R}^4_2$ is defined by $\|\boldsymbol{x}\| = \sqrt{|\langle \boldsymbol{x}, \boldsymbol{x} \rangle|}$.

For any $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3 \in \mathbb{R}^4_2$. We define a vector $\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \boldsymbol{x}_3$ by

$$\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \boldsymbol{x}_3 = \begin{vmatrix} -\boldsymbol{e}_1 & -\boldsymbol{e}_2 & \boldsymbol{e}_3 & \boldsymbol{e}_4 \\ x_1^1 & x_2^1 & x_3^1 & x_4^1 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 \\ x_1^3 & x_2^3 & x_3^3 & x_4^3 \end{vmatrix},$$

where $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4\}$ is the canonical basis of $\mathbb{R}^4_2$ and $\boldsymbol{x}_i = (x_1^i, x_2^i, x_3^i, x_4^i)$. We can easily check that

$$\langle \boldsymbol{x}, \boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \boldsymbol{x}_3 \rangle = \det(\boldsymbol{x}, \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3),$$

so that $\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \boldsymbol{x}_3$ is pseudo-orthogonal to any $\boldsymbol{x}_i$ (for $i = 1, 2, 3$).

We now define *Anti de Sitter 3-space* (briefly, *AdS 3-space*) by

$$H_1^3 = \{\boldsymbol{x} \in \mathbb{R}^4_2 \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1\},$$

a *unit pseudo 3-sphere with index 2* by

$$S_2^3 = \{\boldsymbol{x} \in \mathbb{R}^4_2 \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 1\}.$$

We now introduce the local differential geometry of timelike curves in H_1^3. Let $\boldsymbol{\gamma} : I \longrightarrow H_1^3$ be a regular curve (i.e., an embedding). The regular curve $\boldsymbol{\gamma}$ is said to be timelike if $\dot{\boldsymbol{\gamma}}$ is a timelike vector at any $t \in I$, where $\dot{\boldsymbol{\gamma}} = d\boldsymbol{\gamma}/dt$. Since $\boldsymbol{\gamma}$ is a timelike regular curve, it may admit an arc length parametrization $s = s(t)$. Therefore, we can assume that $\boldsymbol{\gamma}(s)$ is a unit speed curve. Now we have the unit tangent vector $\boldsymbol{t}(s) = \boldsymbol{\gamma}'(s)$. Since $\langle \boldsymbol{\gamma}(s), \boldsymbol{\gamma}(s) \rangle \equiv -1$, we

have $\langle \boldsymbol{\gamma}(s), \boldsymbol{t}(s) \rangle \equiv 0$. From a direct calculation we have $\langle \boldsymbol{\gamma}(s), \boldsymbol{t}'(s) \rangle = 1$. In the case when $\langle \boldsymbol{t}'(s), \boldsymbol{t}'(s) \rangle \neq -1$, we can define a unit spacelike vector $\boldsymbol{n}(s)$ by

$$\boldsymbol{n}(s) = \frac{\boldsymbol{t}'(s) + \boldsymbol{\gamma}(s)}{\|\boldsymbol{t}'(s) + \boldsymbol{\gamma}(s)\|}$$

and call it *principle normal vector* of $\boldsymbol{\gamma}$. We denote $\|\boldsymbol{t}'(s) + \boldsymbol{\gamma}(s)\|$ by $k(s)$. Moreover, we define a vector $\boldsymbol{e}(s) = \boldsymbol{\gamma}(s) \wedge \boldsymbol{t}(s) \wedge \boldsymbol{n}(s)$ and call it *binormal vector* of $\boldsymbol{\gamma}$, then we have a pseudo orthonormal frame $\{\boldsymbol{\gamma}(s), \boldsymbol{t}(s), \boldsymbol{n}(s), \boldsymbol{e}(s)\}$ of $\mathbb{R}^4_2$ along $\boldsymbol{\gamma}$. By the standard arguments, under the assumption that $\langle \boldsymbol{t}'(s), \boldsymbol{t}'(s) \rangle \neq -1$, we can give the following *Frenet-Serret type formula*:

$$\begin{cases} \boldsymbol{\gamma}'(s) = \boldsymbol{t}(s) \\ \boldsymbol{t}'(s) = -\boldsymbol{\gamma}(s) + k(s)\boldsymbol{n}(s) \\ \boldsymbol{n}'(s) = k(s)\boldsymbol{t}(s) + \tau(s)\boldsymbol{e}(s) \\ \boldsymbol{e}'(s) = -\tau(s)\boldsymbol{n}(s) \end{cases}$$

where $\tau(s) = -\dfrac{1}{k^2(s)} \det(\boldsymbol{\gamma}(s), \boldsymbol{\gamma}'(s), \boldsymbol{\gamma}''(s), \boldsymbol{\gamma}'''(s))$.

Since $\langle \boldsymbol{t}'(s) + \boldsymbol{\gamma}(s), \boldsymbol{t}'(s) + \boldsymbol{\gamma}(s) \rangle = \langle \boldsymbol{t}'(s), \boldsymbol{t}'(s) \rangle + 1$, the condition $\langle \boldsymbol{t}'(s), \boldsymbol{t}'(s) \rangle \neq -1$ is equivalent to the condition $k(s) \neq 0$. We can show that timelike curve $\boldsymbol{\gamma}$ is a geodesic in H^3_1 if $k(s) = 0$ and $\boldsymbol{t}'(s) + \boldsymbol{\gamma}(s) = \boldsymbol{0}$.

3 Bertrand curves in Anti de Sitter 3-space

In this section, we will introduce the notion of timelike Bertrand curves in H^3_1 and study their properties.

Definition 3.1 *A timelike curve $\boldsymbol{\gamma}$ in H^3_1 with non-zero curvature is said to be a timelike Bertrand curve if there exists another immersed timelike curve $\tilde{\boldsymbol{\gamma}}$ in H^3_1 and one-to-one correspondence $\phi : I \longrightarrow J$, $s \mapsto \phi(s)$ between $\boldsymbol{\gamma}$ and $\tilde{\boldsymbol{\gamma}}$ such that both curves have common principal normal geodesics at corresponding point. The curves $\boldsymbol{\gamma}$ and $\tilde{\boldsymbol{\gamma}}$ are called a pair of timelike Bertrand curves.*

Let $\boldsymbol{\gamma}$ and $\tilde{\boldsymbol{\gamma}}$ be a pair of timelike Bertrand curves then there exists a differentiable function $a(s)$ such that

$$\tilde{\boldsymbol{\gamma}}(\phi(s)) = \cosh a(s)\boldsymbol{\gamma}(s) + \sinh a(s)\boldsymbol{n}_{\boldsymbol{\gamma}}(s),$$

where $\{\boldsymbol{\gamma}(s), \boldsymbol{t}_{\boldsymbol{\gamma}}(s), \boldsymbol{n}_{\boldsymbol{\gamma}}(s), \boldsymbol{e}_{\boldsymbol{\gamma}}(s)\}$ is the Frenet frame along $\boldsymbol{\gamma}$ and $\tilde{\boldsymbol{\gamma}}(\phi(s))$ is the corresponding point to $\boldsymbol{\gamma}(s)$. For any point $\boldsymbol{\gamma}(s_0)$ on $\boldsymbol{\gamma}$, we define the geodesics from $\boldsymbol{\gamma}(s_0)$ by

$$\Gamma_s^{\boldsymbol{\gamma}(s_0)}(u) = \cosh u\boldsymbol{\gamma}(s) + \sinh u\boldsymbol{n}_{\boldsymbol{\gamma}}(s),$$

then we have the following proposition.

Proposition 3.2 *Let $\boldsymbol{\gamma}$ and $\tilde{\boldsymbol{\gamma}}$ be a pair of timelike Bertrand curves in H^3_1, we have the following*:

(1) *The differentiable function $a(s)$ is constant*;

(2) *The angle between the tangent vectors at corresponding points is constant* ;

(3) *The angle between the binormal vectors at corresponding points (considered as vectors in $\mathbb{R}^4_2$) is constant.*

Proof. (1) Since $\boldsymbol{\gamma}$ and $\tilde{\boldsymbol{\gamma}}$ have common principal normal geodesics at corresponding points, we have

$$\frac{d}{du}\,|_{u=0}\, \Gamma_s^{\boldsymbol{\gamma}(s_0)} = \boldsymbol{n}_{\boldsymbol{\gamma}}(s), \quad \frac{d}{du}\,|_{u=a(s)}\, \Gamma_s^{\boldsymbol{\gamma}(s_0)} = \boldsymbol{n}_{\tilde{\boldsymbol{\gamma}}}(\phi(s)),$$

so that

$$\boldsymbol{n}_{\tilde{\gamma}}(\phi(s)) = \sinh a(s)\boldsymbol{\gamma}(s) + \cosh a(s)\boldsymbol{n}_{\gamma}(s),$$

where $\{\tilde{\boldsymbol{\gamma}}(s), \boldsymbol{t}_{\tilde{\gamma}}(s), \boldsymbol{n}_{\tilde{\gamma}}(s), \boldsymbol{e}_{\tilde{\gamma}}(s)\}$ denotes the Frenet frame along $\tilde{\boldsymbol{\gamma}}$.

On the other hand,

$$\frac{d}{ds}\tilde{\boldsymbol{\gamma}}(\phi(s)) = a'(s)\sinh a(s)\boldsymbol{\gamma}(s) + [\cosh a(s) + \kappa(s)\sinh a(s)]\boldsymbol{t}_{\gamma}(s)$$

$$+a'(s)\cosh a(s)\boldsymbol{n}_{\gamma}(s) + \tau_g(s)\sinh a(s)\boldsymbol{e}_{\gamma}(s).$$

Moreover, $\frac{d}{ds}\tilde{\boldsymbol{\gamma}}(\phi(s)) = \phi'(s)\boldsymbol{t}_{\tilde{\gamma}}(\phi)$, so that

$$0 = \langle \frac{d}{ds}\tilde{\boldsymbol{\gamma}}(\phi), \boldsymbol{n}_{\tilde{\gamma}}(\phi)\rangle = a'(s).$$

This means that $a(s) = Constant$.

(2) Since $\tilde{\boldsymbol{\gamma}}(\phi(s)) = \cosh a_0\boldsymbol{\gamma}(s) + \sinh a_0\boldsymbol{n}_{\gamma}(s)$, we have

$$\boldsymbol{t}_{\tilde{\gamma}}(\phi) = \frac{1}{\phi'(s)}(\cosh\cosh a_0 + \kappa(s)\sinh a_0)\boldsymbol{t}_{\gamma}(s) + \tau_g(s)\sinh a_0\boldsymbol{e}_{\gamma}(s).$$

Therefore,

$$\frac{d}{ds}\langle \boldsymbol{t}_{\gamma}(s), \boldsymbol{t}_{\tilde{\gamma}}(\phi)\rangle = \langle -\boldsymbol{\gamma}(s) + \kappa(s)\boldsymbol{n}_{\gamma}(s), \boldsymbol{t}_{\tilde{\gamma}}(\phi)\rangle + \phi'(s)\langle \boldsymbol{t}_{\gamma}(s), -\tilde{\boldsymbol{\gamma}}(\phi) + \tilde{\kappa}(\phi)\boldsymbol{n}_{\tilde{\gamma}}(\phi)\rangle = 0.$$

(3) Let θ_0 denote the constant angle between $\boldsymbol{t}_{\gamma}(s)$ and $\boldsymbol{t}_{\tilde{\gamma}}(\phi(s))$, then we can get

$$\boldsymbol{t}_{\tilde{\gamma}}(\phi) = \cosh\theta\boldsymbol{t}_{\gamma}(s) + \sinh\theta\boldsymbol{e}_{\gamma}(s).$$

By using the wedge product, we can compute the binormal vector $\boldsymbol{e}_{\tilde{\gamma}} = \tilde{\gamma} \times \boldsymbol{t}_{\tilde{\gamma}} \times \boldsymbol{n}_{\tilde{\gamma}}$. Then we have $\boldsymbol{e}_{\tilde{\gamma}}(s)(\sigma) = \sinh\theta\boldsymbol{t}_{\gamma}(s)\alpha + \cosh\theta\boldsymbol{e}_{\gamma}(s)\alpha$. Thus

$$\frac{d}{ds}\langle \boldsymbol{e}_{\gamma}(s), \boldsymbol{e}_{\tilde{\gamma}}(\phi)\rangle = \langle -\tau(s)\boldsymbol{n}_{\gamma}(s), \boldsymbol{e}_{\tilde{\gamma}}(\phi)\rangle + \phi'(s)\langle \boldsymbol{e}_{\gamma}(s), -\tilde{\tau}(\phi)\boldsymbol{n}_{\tilde{\gamma}}(\phi)\rangle = 0.$$

This completes the proof. □

Theorem 3.3 *Let $\boldsymbol{\gamma}(s)$ and $\tilde{\boldsymbol{\gamma}}$ be a pair of timelike Bertrand curves in H_1^3. Then there exist two constants a and θ such that the following relations hold*

(1) $(\cosh a + \sinh a\kappa(s))\sinh\theta = \sinh a\cosh\theta\tau(s)$;

(2) $(\cosh a - \sinh a\tilde{\kappa}(\phi))\sinh\theta = \sinh a\cosh\theta\tilde{\tau}(\phi)$;

(3) $(\cosh a + \sinh a\kappa(s))(\cosh a - \sinh a\tilde{\kappa}(\phi)) = \cosh^2\theta$;

(4) $\sinh^2 a\tau(s)\tilde{\tau}(\phi) = \sinh^2\theta$;

where $\kappa(s), \tilde{\kappa}(\phi), \tau(s)$ and $\tilde{\tau}(\phi)$ denote the curvature and torsion of $\boldsymbol{\gamma}$ and $\tilde{\boldsymbol{\gamma}}$, respectively.

Proof. (1) Since $t_{\tilde{\gamma}}(\phi) = \cosh\theta\boldsymbol{t}_{\gamma}(s) + \sinh\theta\boldsymbol{e}_{\gamma}(s)$, we have

$$\frac{d\tilde{\boldsymbol{\gamma}}}{ds} = \boldsymbol{t}_{\tilde{\gamma}}(\phi)\phi'(s) = \phi'(s)\cosh\theta\boldsymbol{t}_{\gamma}(s) + \phi'(s)\sinh\theta\boldsymbol{e}_{\gamma}(s).$$

Moreover, $\tilde{\boldsymbol{\gamma}}(\phi(s)) = \cosh a\boldsymbol{\gamma}(s) + \sinh a\boldsymbol{n}_{\gamma}(s)$, then

$$\frac{d\tilde{\boldsymbol{\gamma}}}{ds} = (\cosh a + \kappa(s)\sinh a)\boldsymbol{t}_{\gamma}(s) + \tau(s)\sinh a\boldsymbol{e}_{\gamma}(s).$$

Therefore, we get

$$\begin{cases} \phi'(s)\cosh\theta = \cosh a + \kappa(s)\sinh a \\ \phi'(s)\sinh\theta = \tau(s)\sinh a. \end{cases}$$

So that the first assertion holds.

(2) Since

$$\begin{cases} \tilde{\boldsymbol{\gamma}}(\phi(s)) = \cosh a\boldsymbol{\gamma}(s) + \sinh a\boldsymbol{n}_{\gamma}(s) \\ \boldsymbol{n}_{\tilde{\gamma}}(\phi(s)) = \sinh a\boldsymbol{\gamma}(s) + \cosh a\boldsymbol{n}_{\gamma}(s), \end{cases}$$

we have

$$\begin{cases} \boldsymbol{\gamma}(s) = \cosh a\tilde{\boldsymbol{\gamma}}(\phi) - \sinh a\boldsymbol{n}_{\tilde{\gamma}}(\phi) \\ \boldsymbol{n}_{\gamma}(\phi) = -\sinh a\tilde{\boldsymbol{\gamma}}(\phi) + \cosh a\boldsymbol{n}_{\tilde{\gamma}}(\phi). \end{cases}$$

On the other hand,

$$\begin{cases} \boldsymbol{t}_{\tilde{\gamma}}(\phi) = \cosh\theta\boldsymbol{t}_{\gamma}(s) + \sinh\theta\boldsymbol{e}_{\gamma}(s) \\ \boldsymbol{e}_{\tilde{\gamma}}(\phi) = \sinh\theta\boldsymbol{t}_{\gamma}(s) + \cosh\theta\boldsymbol{e}_{\gamma}(s), \end{cases}$$

we have

$$\begin{cases} \boldsymbol{t}_{\gamma}(s) = \cosh\theta\boldsymbol{t}_{\tilde{\gamma}}(\phi) - \sinh\theta\boldsymbol{e}_{\tilde{\gamma}}(\phi) \\ \boldsymbol{e}_{\gamma}(s) = -\sinh\theta\boldsymbol{t}_{\tilde{\gamma}}(\phi) + \cosh\theta\boldsymbol{e}_{\tilde{\gamma}}(\phi). \end{cases}$$

Then we can get

$$\begin{cases} s'(\phi)\cosh\theta = \cosh a - \tilde{\kappa}(\phi)\sinh a \\ -s'(\phi)\sinh\theta = -\tilde{\tau}(\phi)\sinh a. \end{cases}$$

Therefore we complete the proof of (2).

(3)By using

$$\begin{cases} \phi'(s)\cosh\theta = \cosh a + \kappa(s)\sinh a \\ s'(\phi)\cosh\theta = \cosh a - \tilde{\kappa}(\phi)\sinh a, \end{cases}$$

we have $\cosh^2\theta = (\cosh a + \kappa(s)\sinh a)(\cosh a - \tilde{\kappa}(\phi)\sinh a)$.

(4) By using

$$\begin{cases} \phi'(s)\sinh\theta = \tau(s)\sinh a \\ -s'(\phi)\sinh\theta = -\tilde{\tau}(\phi)\sinh a, \end{cases}$$

we have $\sinh^2\theta = \sinh^2 a\tau(s)\tilde{\tau}(\phi)$. □

We remark that this theorem is similar to the theorems given by H. F. Lai [16] and P. Lucas *et al* [14] for Bertrand curves in Euclidean space and in three-dimensional sphere, respectively. Moreover, if $\boldsymbol{\gamma}$ and $\tilde{\boldsymbol{\gamma}}$ are timelike Bertrand curves in H_1^3, part (4) of the above theorem implies that the product of their torsions at corresponding points is constant and non-negative. This is often known as Schell's theorem.

A timelike curve $\boldsymbol{\gamma}$ in H_1^3 is said to be a *timelike plane curve* if its torsion is zero at all points.

Proposition 3.4 (1) *Every timelike plane curve in H_1^3 with $\kappa(s) \neq -\coth a$, $\forall a \in \mathbb{R}$, is a Bertrand curve and it has infinite timelike Bertrand conjugate plane curve.*

(2) *If a timelike Bertrand curve $\boldsymbol{\gamma}$ has a timelike Bertrand conjugate plane curve, then $\boldsymbol{\gamma}$ is a timelike plane curve located on the same totally geodesic two-dimensional Anti de Sitter*

space.

Proof. (1) Let $\boldsymbol{\gamma}$ be a timelike plane curve in H_1^3. For any $a \in \mathbb{R}$, suppose $\tilde{\boldsymbol{\gamma}}_a(s)$ be a timelike curve in H_1^3 defined by

$$\tilde{\boldsymbol{\gamma}}_a(s) = \cosh a\boldsymbol{\gamma}(s) + \sinh a\boldsymbol{n}_\gamma(s).$$

Then we obtain $\frac{d\tilde{\boldsymbol{\gamma}}_a(s)}{ds} = (\cosh a + \sinh a\kappa(s))\boldsymbol{t}_\gamma(s)$.

We assume that ϕ is the arc-length parameter, then $\phi'(s) = \| \frac{d\tilde{\boldsymbol{\gamma}}_a(s)}{ds} \| = \cosh a + \sinh a\kappa(s)$. By using $\frac{d\tilde{\boldsymbol{\gamma}}_a}{d\phi}\frac{d\phi}{ds} = \boldsymbol{t}_{\tilde{\gamma}_a}(\phi)\phi' = \phi'\boldsymbol{t}_\gamma(s)$, so we have $\boldsymbol{t}_{\tilde{\gamma}_a}(\phi) = \boldsymbol{t}_\gamma(s)$. This means that

$$[-\tilde{\boldsymbol{\gamma}}_a(s) + \tilde{\kappa}(\phi)\boldsymbol{n}_{\tilde{\gamma}_a}(\phi)]\phi'(s) = -\boldsymbol{\gamma}(s) + \kappa(s)\boldsymbol{n}_\gamma(s).$$

Therefore, we can deduce

$$\tilde{\kappa}(\phi)\boldsymbol{n}_{\tilde{\gamma}_a}(\phi)\phi'(s) = [\sinh a + \cosh a\kappa(s)][\sinh a\boldsymbol{\gamma}(s) + \cosh a\boldsymbol{n}_\gamma(s)],$$

then

$$\boldsymbol{n}_{\tilde{\gamma}_a}(\phi) = \sinh a\boldsymbol{\gamma}(s) + \cosh a\boldsymbol{n}_\gamma(s),\ \tilde{\kappa}(\phi) = \frac{\sinh a + \cosh a\kappa(s)}{\cosh a + \sinh a\kappa(s)}.$$

So that the principal normal geodesic starting at a point $\tilde{\boldsymbol{\gamma}}_a(\phi_0)$, $\phi_0 = \phi(s_0)$, is given by

$$\Gamma(u) = \cosh(u + a)\boldsymbol{\gamma}(s_0) + \sinh(u + a)\boldsymbol{n}_\gamma(s_0).$$

This means that $\tilde{\boldsymbol{\gamma}}_a(s)$ is the Bertrand conjugate of $\boldsymbol{\gamma}(s)$.

Furthermore, since $\boldsymbol{n}_{\tilde{\gamma}_a}(\phi) = \sinh a\boldsymbol{\gamma}(s) + \cosh a\boldsymbol{n}_\gamma(s)$, by using Frenet equation, we can get

$$\phi'(s)\frac{d\boldsymbol{n}_{\tilde{\gamma}_a}(\phi)}{d\phi} = (\sinh a + \cosh a\kappa(s))\boldsymbol{t}_\gamma(s) + \cosh a\tau(s)\boldsymbol{e}_\gamma(s).$$

For the reason of $\tau(s) = 0$, we have

$$\phi'(s)(\tilde{\kappa}(\phi)\boldsymbol{t}_{\tilde{\gamma}_a}(\phi) + \tilde{\tau}(\phi)\boldsymbol{e}_{\tilde{\gamma}_a}(\phi)) = (\sinh a + \cosh a\kappa(s))\boldsymbol{t}_\gamma(s).$$

From above we have $\tilde{\tau}(\phi) = 0$. So the timelike curve $\tilde{\boldsymbol{\gamma}}_a$ is a timelike plane curve in H_1^3.

(2) Since $\tilde{\tau}(\phi) = 0$, by Theorem 3.3(4), we have $\sinh\theta = 0$, so $\cosh\theta = 1$. Moreover, by Theorem 3.3(1), we get $\sinh a\tau(s) = 0$. Either $\sinh a = 0$, then $\tilde{\boldsymbol{\gamma}}_a(s) = \boldsymbol{\gamma}(s)$, or $\tau(s) = 0$, we can obtain the same result. □

Theorem 3.5 *A timelike curve $\boldsymbol{\gamma}$ in H_1^3 is a Bertrand curve if and only if either* (1) $\tau(s) = 0, \kappa(s) \neq -\coth a$, $\forall a \in \mathbb{R}$ *or* (2) *there exist two constants $\lambda \neq 0$ and μ such that $\mu\tau(s) - \lambda\kappa(s) = 1$.*

Proof. Let $\boldsymbol{\gamma}$ be a timelike Bertrand curve. If $\boldsymbol{\gamma}$ is not a plane curve, then from Theorem 3.3(1) we have that $\sinh a\cosh\theta\tau(s) - \sinh a\sinh\theta\kappa(s) = \cosh a\sinh\theta$. We can deduce $\tanh a\coth\theta\tau(s) - \tanh a\kappa(s) = 1$. Let $\tanh a = \lambda$, $\tanh a\coth\theta = \mu$, we have $\mu\tau(s) - \lambda\kappa(s) = 1$.

On the other hand, we assume that $\mu\tau(s) - \lambda\kappa(s) = 1$, where $\lambda = \tanh a \neq 0$. Let $\tilde{\boldsymbol{\gamma}} = \cosh a\boldsymbol{\gamma}(s) + \sinh a\boldsymbol{n}_\gamma(s)$, then by using the Frenet equations we obtain

$$\frac{d\tilde{\boldsymbol{\gamma}}}{ds} = (\cosh a + \sinh a\kappa(s))\boldsymbol{t}_\gamma(s) + \sinh a\tau(s)\boldsymbol{e}_\gamma(s).$$

Let ϕ be the arc-length parameter of $\tilde{\gamma}$. We deduce

$$\phi'(s) = \| \frac{d\tilde{\gamma}}{ds} \| = \sqrt{| -(\cosh a + \sinh a\kappa(s))^2 + (\sinh a\tau(s))^2 |}.$$

Since $\mu\tau(s) - \lambda\kappa(s) = 1$, $\tanh a = \lambda$, we have $\mu \cosh a\tau(s) - \sinh a\kappa(s) = \cosh a$. Therefore, we get

$$\phi'(s) = \sqrt{| -(\mu \cosh a\tau(s))^2 + (\sinh a\tau(s))^2 |} = \tau(s)\sqrt{(\mu \cosh a)^2 - \sinh^2 a}.$$

So that

$$t_{\tilde{\gamma}}(\phi)\phi'(s) = \mu \cosh a\tau(s)\boldsymbol{t}_{\gamma}(s) + \sinh a\tau(s)\boldsymbol{e}_{\gamma}(s).$$

Taking the derivative in above we have

$$\begin{aligned}(-\tilde{\gamma} + \tilde{\kappa}(\phi)\boldsymbol{n}_{\tilde{\gamma}}(\phi))\phi'(s) = &(\frac{\mu \cosh a}{\sqrt{(\mu \cosh a)^2 - \sinh a^2}})(-\boldsymbol{\gamma}(s) + \kappa(s)\boldsymbol{n}_{\gamma}(s)) \\ &+(\frac{\sinh a}{\sqrt{(\mu \cosh a)^2 - \sinh^2 a}})(-\tau(s)\boldsymbol{n}_{\gamma}(s)).\end{aligned}$$

So

$$\tilde{\kappa}(\phi)\boldsymbol{n}_{\tilde{\gamma}}(\phi)\phi'(s) = (\frac{\cosh a(\mu \cosh a\kappa(s) + \mu \sinh a - \tau(s) \sinh a)}{\sqrt{(\mu \cosh a)^2 - \sinh^2 a}})(\sinh a\boldsymbol{\gamma}(s) + \cosh a\boldsymbol{n}_{\gamma}(s)).$$

This means that

$$\boldsymbol{n}_{\tilde{\gamma}}(\phi) = \sinh a\boldsymbol{\gamma}(s) + \cosh a\boldsymbol{n}_{\gamma}(s),\ \tilde{\kappa}(\phi)\phi'(s) = \frac{\cosh a(\mu \cosh a\kappa(s) + \mu \sinh a - \tau(s) \sinh a)}{\sqrt{(\mu \cosh a)^2 - \sinh^2 a}}.$$

Then the principal normal geodesic starting at a point $\tilde{\gamma}(\phi_0), \phi_0 = \phi(s_0)$, is given by

$$\gamma(u) = \cosh(u + a)\gamma(s_0) + \sinh(u + a)\boldsymbol{n}_{\gamma}(s_0).$$

This complete the proof. □

A curve γ in H_1^3 is called a *helix* if $\tau(s)$, $\kappa(s)$ are both non-zero constant. For details of helix immersed in Anti de Sitter 3-space, please see [17]

Theorem 3.6 *Let γ be a timelike curve in H_1^3. The following conditions are equivalent:*

(1) *γ is a helix.*

(2) *γ has infinite number Bertrand conjugate curves.*

(3) *γ has at least two Bertrand conjugate curves.*

Proof. (1)$\Longrightarrow$(2) we assume that $\kappa(s)$ and $\tau(s)$ are non-zero constants. Since there are infinite number of μ and λ such that $\mu\tau(s) - \lambda\kappa(s) = 1$, we can construct infinite number different Bertrand conjugate curves.

(2)$\Longrightarrow$(3) It is obviously.

(3)$\Longrightarrow$(1) If γ has two Bertrand conjugate curves $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$, then we can find four constants $a_1 \neq 0, a_2 \neq 0, \theta_1$ and θ_2 such that

$$\begin{cases} \tanh a_1 \coth \theta_1\tau(s) - \tanh a_1\kappa(s) = 1 \\ \tanh a_2 \coth \theta_2\tau(s) - \tanh a_2\kappa(s) = 1, \end{cases}$$

where $a_1 \neq a_2, \theta_1 \neq \theta_2$. Since $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ are two different Bertrand conjugate curves. By taking the derivative in these equations we obtain

$$\begin{cases} \tanh a_1 \coth \theta_1\tau'(s) - \tanh a_1\kappa'(s) = 0 \\ \tanh a_2 \coth \theta_2\tau'(s) - \tanh a_2\kappa'(s) = 0. \end{cases}$$

Therefore $\kappa'(s) = \tau'(s) = 0$, this means that $\kappa(s)$ and τ are both constant. That concludes the proof. □

Example 3.7 We define $\gamma : I \longrightarrow H_1^3$ by

$$\gamma(s) = (2\cosh 2s, \sqrt{2}\cosh\sqrt{\frac{17}{3}}s + \sqrt{5}\sinh\sqrt{\frac{17}{3}}s, 2\sinh 2s, \sqrt{2}\sinh\sqrt{\frac{17}{3}}s + \sqrt{5}\cosh\sqrt{\frac{17}{3}}s).$$

By straightforward calculation, we get

$$\boldsymbol{t}(s) = (4\sinh 2s, \sqrt{2}\sqrt{\frac{17}{3}}\sinh\sqrt{\frac{17}{3}}s + \sqrt{5}\sqrt{\frac{17}{3}}\cosh\sqrt{\frac{17}{3}}s,$$
$$4\cosh 2s, \sqrt{2}\sqrt{\frac{17}{3}}\cosh\sqrt{\frac{17}{3}}s + \sqrt{5}\sqrt{\frac{17}{3}}\sinh\sqrt{\frac{17}{3}}s),$$

and $\langle \boldsymbol{t}(s), \boldsymbol{t}(s)\rangle = -1$. Therefore, γ is timelike curve in H_1^3 and s is the arc-length parameter of γ. Furthermore, by calculations we can obtain that

$$\kappa = \frac{10}{\sqrt{3}}, \ \tau = -2\sqrt{\frac{17}{3}}.$$

So that, γ is a helix in H_1^3. By Theorem 3.6, it has infinite number Bertrand conjugate curves in H_1^3.

4 The relationship between timelike Bertrand curves in H_1^3 and spacelike Bertrand curves in $\mathbb{R}_2^4$

In this section, we will investigate the relationship between timelike Bertrand curves in H_1^3 and spacelike Bertrand curves in $\mathbb{R}_2^4$. Firstly, we will review the basic definitions and notations about spacelike Bertrand curves in $\mathbb{R}_2^4$.

We first introduce the local differential geometry of spacelike curves in $\mathbb{R}_2^4$. Let $\alpha : L \longrightarrow \mathbb{R}_2^4$ be a regular curve (i.e., an embedding). The regular curve α is said to be spacelike if $\dot{\alpha}$ is a spacelike vector at any $t \in L$, where $\dot{\alpha} = d\alpha/dt$. Since α is a spacelike regular curve, it may admit an arc length parametrization $s = s(t)$. Therefore, we can assume that $\alpha(s)$ is a unit speed curve. Now we have the unit tangent vector $\boldsymbol{t}^\alpha(s) = \alpha'(s)$. We can also choose the unit normal vectors $\boldsymbol{n}_1^\alpha, \boldsymbol{n}_2^\alpha, \boldsymbol{n}_3^\alpha$, where $\langle \boldsymbol{n}_1^\alpha, \boldsymbol{n}_1^\alpha\rangle = 1$, $\langle \boldsymbol{n}_i^\alpha, \boldsymbol{n}_i^\alpha\rangle = -1$, $i = 2, 3$. Then we have the Frenet frame $\{\boldsymbol{t}^\alpha, \boldsymbol{n}_1^\alpha, \boldsymbol{n}_2^\alpha, \boldsymbol{n}_3^\alpha\}$ and the following Frenet-Serret formula:

$$\begin{cases} \boldsymbol{t}^{\alpha\prime}(s) = k_1(s)\boldsymbol{n}_1^\alpha(s) \\ \boldsymbol{n}_1^{\alpha\prime}(s) = -k_1(s)\boldsymbol{t}^\alpha(s) + k_2(s)\boldsymbol{n}_2^\alpha(s) \\ \boldsymbol{n}_2^{\alpha\prime}(s) = k_2(s)\boldsymbol{n}_1^\alpha(s) + k_3(s)\boldsymbol{n}_3^\alpha(s) \\ \boldsymbol{n}_3^{\alpha\prime}(s) = -k_3(s)\boldsymbol{n}_2^\alpha(s). \end{cases}$$

Let $\alpha : L \longrightarrow \mathbb{R}_2^4, s \mapsto \alpha(s)$ be a Frenet curve in $\mathbb{R}_2^n$ with Frenet frame $\{\boldsymbol{t}^\alpha, \boldsymbol{n}_1^\alpha, \boldsymbol{n}_2^\alpha, \boldsymbol{n}_3^\alpha\}$ and curvatures $\kappa_1, \kappa_2, \kappa_3$, where s is the arc-length parameter. We call α a *special Frenet curve*, if curvatures $\kappa_i > 0$, $i = 1, 2$ and $\kappa_3 \neq 0$ for any point $p = \alpha(s)$. Moreover, a plane generated by normal vectors $\boldsymbol{n}_j(s)$ and $\boldsymbol{n}_k(s)$ is called *Frenet* $(j, k)-$*normal plane* of curve at the point p.

Definition 4.1 *A special Frenet spacelike curve α in $\mathbb{R}_2^4$ is said to be a spacelike Frenet $(1, 3)-$Bertrand curve if there exists another immersion speical Frenet spacelike curve $\tilde{\alpha}$ in $\mathbb{R}_2^4$*

such that both curves have common Frenet $(1,3)-$*normal plane at corresponding point. The curves* α *and* $\tilde{\alpha}$ *are called a pair of spacelike* $(1,3)-$*Bertrand curves.*

We have the following characterisation about the spacelike $(1,3)-$Bertrand curves.

Theorem 4.2 *A spacelike curve* α *in* $\mathbb{R}^4_2$ *with arc-length parameter* s *is a spacelike* $(1,3)-$*Bertrand curve if and only if there exist four constants* a, b, c, d *such that the following conditions are held.*

(i) $a\kappa_2(s) - b\kappa_3(s) \neq 0$,

(ii) $c(a\kappa_2(s) - b\kappa_3(s)) + a\kappa_1(s) = 1$,

(iii) $d\kappa_3(s) = c\kappa_1(s) + \kappa_2(s)$,

(iv) $(c^2+1)\kappa_1(s)\kappa_2(s) + c(\kappa_1^2(s) + \kappa_2^2(s) - \kappa_3^2(s)) \neq 0$.

Since the proof is analogue to the proof of Theorem 4.1 in [12], so we omit it.

We now use spacelike $(1,3)-$Bertrand curves in $\mathbb{R}^4_2$ to construct timelike Bertrand curves in H^3_1 as follows. We assume that $\boldsymbol{\gamma}(s) = \boldsymbol{n}^{\alpha}_3(s)$ and σ is the arc-length parameter of $\boldsymbol{\gamma}$. By taking derivative on both sides of the above equation we get $\boldsymbol{t}_{\gamma}(\sigma)\sigma'(s) = -\kappa_3(s)\boldsymbol{n}^{\alpha}_2(s)$. Therefore

$$\sigma'(s) = \varepsilon_3\kappa_3(s),\ \boldsymbol{t}_{\gamma}(\sigma) = -\varepsilon_3\boldsymbol{n}^{\alpha}_2, \text{ where } \varepsilon_3 = \pm 1.$$

This means that $\langle \boldsymbol{t}_{\gamma}, \boldsymbol{t}_{\gamma}\rangle = -1$. According to the similar calculation we get

$$\sigma'(s)\kappa(\sigma) = \varepsilon_2\kappa_2(s),\ \boldsymbol{n}_{\gamma}(\sigma) = -\varepsilon_2\varepsilon_3\boldsymbol{n}^{\alpha}_1, \text{ where } \varepsilon_2 = \pm 1,$$

$$\sigma'(s)\tau(\sigma) = \varepsilon_1\kappa_1(s),\ \boldsymbol{e}_{\gamma}(\sigma) = \varepsilon_1\varepsilon_2\varepsilon_3\boldsymbol{t}^{\alpha}(s), \text{ where } \varepsilon_1 = \pm 1.$$

So that we first construct the timelike curve $\boldsymbol{\gamma}$ in H^3_1 by using spacelike curve α in $\mathbb{R}^4_2$. Furthermore, we can prove the following theorem.

Theorem 4.3 *Let* α *be a spacelike* $(1,3)-$*Bertrand curves in* $\mathbb{R}^4_2$ *with Frenet frame* $\{\boldsymbol{t}^{\alpha}, \boldsymbol{n}^{\alpha}_1, \boldsymbol{n}^{\alpha}_2, \boldsymbol{n}^{\alpha}_3\}$. *Then every timelike curve in* H^3_1 *defined by* $\boldsymbol{\gamma}(s) = \boldsymbol{n}^{\alpha}_3(s)$ *is a timelike Bertrand curve, where* s *is the arc-length parameter of the curve* α.

Proof. Since α is a spacelike $(1,3)-$Bertrand curves in $\mathbb{R}^4_2$, by Theorem 4.2, there exist four constants a, b, c, d which satify the conditions (i), (ii), (iii), (iv) of Theorem 4.2. According to condition (iii), we have $d \neq 0$.

Let $\boldsymbol{\gamma}(s) = \boldsymbol{n}^{\alpha}_3(s)$. Since $c\kappa_1(s) + \kappa_2(s) = d\kappa_3(s)$, we have

$$c\frac{\sigma'(s)\tau_g(\sigma)}{\varepsilon_1} + \frac{\sigma'(s)\kappa_g(\sigma)}{\varepsilon_2} = d\frac{\sigma'(s)}{\varepsilon_3}.$$

Therefore,

$$(c\varepsilon_2\varepsilon_3)\tau_g(\sigma) + \varepsilon_1\varepsilon_3\kappa_g(\sigma) = d\varepsilon_1\varepsilon_2,\ \tau_g(\sigma)\frac{c\varepsilon_2}{d\varepsilon_1} - \kappa_g(\sigma)\frac{-\varepsilon_3}{d\varepsilon_2} = 1.$$

We assume that $\lambda = -\frac{-\varepsilon_3}{d\varepsilon_2}, \mu = \frac{c\varepsilon_2}{d\varepsilon_1}$, then we have

$$\mu\tau_g(\sigma) - \lambda\kappa_g(\sigma) = 1.$$

This finished the proof. □

On the other hand, we can also use timelike curves in H^3_1 to construct spacelike $(1,3)-$Bertrand curves in $\mathbb{R}^4_2$. Let $\boldsymbol{\gamma} = \boldsymbol{\gamma}(t)$ be a timelike curve in H^3_1 with Frenet frame $\{\boldsymbol{\gamma}, \boldsymbol{t}_{\gamma}, \boldsymbol{n}_{\gamma}, \boldsymbol{e}_{\gamma}\}$. Suppose $\alpha(t) = \int_{t_0}^{t} \boldsymbol{e}_{\gamma}(s(u))du$, where $s = s(t)$ is the arc-length parameter of the curve $\boldsymbol{\gamma}$.

Without loss of generality, we assume that $s' > 0$. Then we have $\alpha'(t) = \boldsymbol{e}_\gamma(s(t))$. Since $\| \alpha'(t) \| = \| \boldsymbol{e}_\gamma(s(t)) \| = 1$, t is the arc-length parameter of the curve α and α is the spacelike curve in $\mathbb{R}^4_2$. By using the Frenet-Serret type formulas of γ and α we get

$$\kappa_1(t) = s'(t)\tau(s) > 0,\ \boldsymbol{n}_1^\alpha(t) = -\varepsilon \boldsymbol{n}_\gamma(s), \text{ where } \varepsilon = \pm 1$$
$$\kappa_2(t) = s'(t)\kappa(s) > 0,\ \boldsymbol{n}_2^\alpha(t) = -\varepsilon \boldsymbol{t}_\gamma(s)$$
$$\kappa_3(t) = -\varepsilon s'(t) \neq 0,\ \boldsymbol{n}_3^\alpha(t) = -\boldsymbol{\gamma}(s).$$

Therefore, we had used the timelike curve γ in H_1^3 to construct the spacelike curve α in $\mathbb{R}^4_2$. Moreover, we have the following result.

Theorem 4.4 *Let γ be a non-planar timelike Bertrand curves in H_1^3 with non-constant curvature. Then there exists a regular differential mapping $s = s(t)$, such that the curve defined by $\alpha(t) = \int_{t_0}^t \boldsymbol{e}_\gamma(s(u))du$ is a spacelike $(1,3)-$Bertrand curves in $\mathbb{R}^4_2$ with arc-length parameter s.*

Proof. Since γ is a non-planar timelike Bertrand curves in H_1^3 with non-constant curvature, according to Theorem 3.5, there exist two constants $\lambda \neq 0$ and μ such that $\mu\tau(s) - \lambda\kappa(s) = 1$. Taking two constants a and b, such that

$$\lambda[\varepsilon a(\lambda\tau_g(s) - \mu\kappa_g(s)) - b\mu] > 0,$$

$$s'(t) = \frac{\lambda}{\varepsilon a(\lambda\tau_g(s) - \mu\kappa_g(s)) - b\mu}, s'(t) > 0.$$

We can also take another two constants $c = -\frac{\varepsilon\mu}{\lambda}$ and $d = \frac{\varepsilon}{\lambda}$, then we have the following.

$$(i)\ a\kappa_2 - b\kappa_3 = \frac{\lambda(a\kappa(s) + b\varepsilon)}{\varepsilon a(\lambda\tau(s) - \mu\kappa(s)) - b\mu} \neq 0,$$
$$(ii)\ a\kappa_1 - c(a\kappa_2 - b\kappa_3) = 1,$$
$$(iii)\ c\kappa_1 + \kappa_2 = d\kappa_3,$$
$$(iv)\ (c^2+1)\kappa_1\kappa_2 + c[\kappa_1^2 + \kappa_2^2 - \kappa_3^2] = \varepsilon(s')^2\frac{\mu\kappa_g - \lambda\tau_g + \lambda\mu}{\lambda^2} \neq 0.$$

Therefore, by Theorem 4.2, we complete the proof. □

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