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THE EXISTENCE OF A NON SPECIAL ARONSZAJN TREE AND TODORČEVIĆ ORDERINGS

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ABSTRACT. It is proved that it is consistent that every forcing notions with $R_{1,\aleph_{1}}$ has precaliber $\aleph_{1}$, every Todorcevic ordering for any second countable Hausdorff space also has precaliber $\aleph_{1}$, and there exists a non-special Aronszajn tree. This slightly extends the previous work [16, 18].

1. INTRODUCTION

Martin’s Axiom was introduced by Martin and Solovay to solve Suslin’s problem in [5]. In 1980’s, Todorcevic investigated Martin’s Axiom from the view point of Ramsey theory, and introduced the countable chain condition for partitions on the set $[\omega_{1}]^{<\aleph_{0}}$. In [13], Todorcevic and Veličković proved that $MA_{\aleph_{1}}$, which is Martin’s Axiom for $\aleph_{1}$ many dense sets, is equivalent to the statement $\mathcal{K}_{<\omega}$ that every ccc partition $K_{0}\cup K_{1}$ on $[\omega_{1}]^{<\aleph_{0}}$ has an uncountable $K_{0}$-homogeneous set. Todorcevic also introduced many fragments of $MA_{\aleph_{1}}$ in his many papers e.g. [9, 13]. Some of them are as follows(1): $\mathcal{K}_{<\omega}$ is the statement that every ccc forcing notion has precaliber $\aleph_{1}$. For each $n \in \omega$, $\mathcal{K}_{n}$ is the statement that every uncountable subset of a ccc forcing notion has an uncountable $n$-linked subset, and $\mathcal{K}_{n}'$ is the statement that every ccc partition $K_{0}\cup K_{1} = [\omega_{1}]^{n}$ has an uncountable $K_{0}$-homogeneous set. $C^{2}$ is the statement that every product of ccc forcing notions has the countable chain condition. We note that they have many applications. For example, $C^{2}$ implies Suslin’s Hypothesis, every $(\omega_{1},\omega_{1})$-gap is indestructible, and the bounding number $b$ is greater than $\aleph_{1}$, and $\mathcal{K}_{n}'$ implies that every Aronszajn tree is special. (For other applications, see e.g. [3].) We also note the following diagram of implications

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(1) They are defined by Todorcevic in several papers. In [3, Definition 4.9] and [13, §2], $\mathcal{K}_{n}$’s are defined as statements for ccc forcing notions, however in [4, §4] and [9, §7], $\mathcal{K}_{n}$’s are defined as statements for ccc partitions. To separate them, we use notation as above. In [13], $\mathcal{K}_{<\omega}$ above is denoted by $\mathcal{H}$.

A forcing notion $P$ has precaliber $\aleph_{1}$ if every uncountable subset $I$ of $P$ has an uncountable subset $I'$ of $I$ such that every finite subset of $I'$ has a common extension in $P$. A subset $I$ of a forcing notion $P$ is called $n$-linked if every member of the set $[I]^{n}$ has a common extension in $P$. A forcing notion $P$ has property K if every uncountable subset of $P$ has an uncountable 2-linked subset.
between them:

\[
\begin{array}{ccccccc}
\mathcal{K}_{<\omega} & \rightarrow & \cdots & \rightarrow & \mathcal{K}_{n+1} & \rightarrow & \mathcal{K}_n & \rightarrow & \cdots & \rightarrow & \mathcal{K}_2 & \rightarrow & C^2 \\
\text{MA}_{R_1} & \rightarrow & \mathcal{K}'_{<\omega} & \rightarrow & \cdots & \rightarrow & \mathcal{K}'_{n+1} & \rightarrow & \mathcal{K}'_n & \rightarrow & \cdots & \rightarrow & \mathcal{K}'_2 \\
\end{array}
\]

The equivalence of MA$_{R_1}$, $\mathcal{K}_{<\omega}$ and $\mathcal{K}'_{<\omega}$ are the theorem due to Todor\'evi\'c and Veli\'cki\'c [13]. Other implications follows from definitions or trivial arguments. It is unknown whether any other implications hold in ZFC.

The author studied about this problem in [14, 15, 16, 17, 18]. In [16, 18], The author introduced the following property on chain conditions [16, 18, Definition 2.6]: A forcing notion $\mathbb{P}$ has the property $R_{1,N_1}$ if conditions of $\mathbb{P}$ are finite sets of countable ordinals, the order $\leq_\mathbb{P}$ is equal to the superset relation $\supseteq$, and for any large enough regular cardinal $\theta$, any countable elementary submodel $N$ of $H(\theta)$, any uncountable subset $I$ of $\mathbb{P}$ which forms a $\Delta$-system with root $\nu$ and any $\sigma \in \mathbb{P}$ with $\sigma \cap N = \nu$, there exists an uncountable subset $I'$ of $I$ such that every condition in $I'$ is compatible with $\sigma$ in $\mathbb{P}$. It is proved that $\mathcal{K}_2(R_{1,N_1})^{(2)}$ also implies that Suslin's Hypothesis holds, every $(\omega_1, \omega_1)$-gap is indestructible and $b > \aleph_1$. It is also proved that it is consistent that every forcing notion with the property $R_{1,N_1}$ has precaliber $\aleph_1$ and there exists a non-special Aronszajn tree. This says that $\mathcal{K}_{<\omega}(R_{1,N_1})$ doesn't imply MA$_{R_1}(R_{1,N_1})$.

In this paper, we slightly develop this result by dealing with not only forcing notions with $R_{1,N_1}$ but also forcing notions defined due to Todor\'evi\'c and Balcar-Paz\'ak-Th\"ummel [10, 1], so called Todor\'evi\'c orderings. Namely, it is shown that it is consistent that every forcing notion with the property $R_{1,N_1}$ has precaliber $\aleph_1$, Todor\'evi\'c orderings for second countable Hausdorff spaces also have precaliber $\aleph_1$, and there exists a non-special Aronszajn tree.

2. Preliminaries

2.1. Todor\'evi\'c orderings. As said in [1], when a topological space is applied to Todor\'evi\'c ordering, it is natural to require it to be sequential and have the unique limit property. A topological space $X$ is called sequential if for any $Z \subseteq X$, $Z$ is closed in $X$ iff for any $A \subseteq Z$ and $x \in X$ to which $A$ converges, $x$ belongs to $Z$. A topological space $X$ has the unique limit property if any converging subset of $X$ converges to the unique point. For example, Hausdorff spaces have the unique limit property. For a subset $F$ of a topological space, let $F^d$ denote the first Cantor-Bendixson derivative of $F$, that is, the set of all accumulation points of $F$.

**Definition 2.1** (Todor\'evi\'c [10], see also [1, 8]). For a topological space $X$, $T(X)$ is the set of all subsets of $X$ which are unions of finitely many converging sequences.

\(_{(2)}\) $\mathcal{K}_2(R_{1,N_1})$ is the statement that every forcing notion with the property $R_{1,N_1}$ has property K.
including their limit points, and for each \( p \) and \( q \) in \( T(X) \), \( q \leq_{T(X)} p \) iff \( q \supseteq p \) and \( q^d \cap p = p^d \).\(^{(3)}\)

For \( p, q \in T(X) \), the statement \( q \leq_{T(X)} p \) means that \( q \) is an extension of \( p \) (as the subset relation) and the isolated points in \( p \) are still isolated in \( q \). \( T(X) \) is called Todorčević ordering for the space \( X \) in [1, 8] (and [19]).

Todorčević orderings were firstly introduced by Todorčević in [10]. The motivation is to demonstrate a Borel definable ccc forcing which consistently does not have property K. He defined it on a separable metric space. By generalizing it and applying it to other topological spaces, Thümmel discovered a forcing notion which has the \( \sigma \)-finite chain condition but does not have the \( \sigma \)-bounded chain condition, and so he solved the problem of Horn and Tarski [8]. (For Horn-Tarski’s problem, see [2, 11].) Right after Thümmel’s result, Todorčević introduced a Borel definable solution of the problem of Horn and Tarski [12].

In [12], Todorčević introduced the Borel definable version of Todorčević orderings, which consists of all countable compact subsets whose first Cantor-Bendixson derivative is finite. In [1], Balcar-Pazák-Thümmel introduced a separative version of Todorčević orderings, which consists of all functions \( f \) from members \( p \) of \( T(X) \) into \( \{0, 1\} \) such that \( f^{-1}(1) \) is a finite set including \( p^d \) as a subset, ordered by the function-extension. In this paper, as in [19], we adopt the definition of Todorčević orderings in Definition 2.1.

Some of Todorčević orderings may not be ccc [1, Theorem 2.3], but many of them are ccc. From the proof of [10], we note that for a space \( X \), if each of finite powers of \( X \) is hereditarily separable, then Todorčević ordering for \( X \) has the ccc. In [1, Definition 2.1], Balcar-Pazák-Thümmel introduced the property of topological spaces which is a sufficient condition to introduce Todorčević orderings to have the ccc (see also [19]). In this paper, we use the following property of Todorčević orderings.

**Lemma 2.2.** For a second countable Hausdorff space \( X \), \( T(X) \) is powerfully ccc, that is, a finite support product of any number of copies of \( T(X) \) has the countable chain condition.

*Proof.* It suffices to show that for any \( n \in \omega \), the finite support product \( nT(X) \) is ccc. Let \( I \) be an uncountable subset of \( nT(X) \). By shrinking \( I \) if necessary, we may assume that for each \( i < n \), the set \( \{p^d_i; \langle p_j; j < n \rangle \in I \} \) forms a \( \Delta \)-system with root \( d_i \). Take a countable elementary submodel \( N \) of \( H(\theta) \) (for some large enough regular cardinal \( \theta \)) such that \( \{X, I\} \in N \).

Take \( \langle p_i; i < n \rangle \) and \( \langle q_i; i < n \rangle \) in \( I^{(4)} \) such that for each \( i < n \),

\[
\bullet (p^d_i \setminus d_i) \cap N = \emptyset, \text{ and}
\]

\((3)\)This definition is slightly different from the original one, in [10], which consists of all finite sets \( \sigma \) of convergent sequences in \( X \) including their limit points such that for any \( A, B \in \sigma \),

\[\lim(A) \notin (B \setminus \lim(B)),\]

ordered by the reverse inclusion. But essentially, both are same. In fact, both are forcing-equivalent.

\((4)\)Since the set \( \{p^d_i; \langle p_j; j < n \rangle \in I \} \) forms an uncountable \( \Delta \)-system for each \( i < n \) and \( N \) is countable, we can find such a \( \langle p_i; i < n \rangle \in I \). Similarly, since the set \( N \cup \bigcup_{i<n} p_i \) is countable, we can find such a \( \langle q_i; i < n \rangle \in I \).
\begin{itemize}
  \item \((q_i^d \setminus d_i) \cap (N \cup p_i) = \emptyset\).
\end{itemize}

Since \(X\) is second countable Hausdorff and \(N\) is an elementary submodel, there exists a sequence \(\langle U_i, V_i; i < n \rangle \in N\) of open subsets of \(X\) such that for each \(i < n\),

\begin{itemize}
  \item \(U_i \cap V_i = \emptyset\),
  \item \(p_i^d \setminus d_i \subseteq U_i\),
  \item \(q_i^d \setminus d_i \subseteq V_i\), and
  \item \(V_i \cap (p_i \setminus U_i) = \emptyset\).
\end{itemize}

This can be done because the sets \(p_i^d \setminus d_i\), \(q_i^d \setminus d_i\) and \(p_i \setminus U_i\) are finite and \((q_i^d \setminus d_i) \cap p_i = \emptyset\). By the elementarity of \(N\), there exists \(\langle q_i^i; i < n \rangle \in I \cap N\) such that for each \(i < n\), \((q_i^i)^d \setminus d_i \subseteq V_i\). Then for each \(i < n\),

\[
q_i^i \cup p_i \leq_{T(X)} p_i.
\]

Since \(q_i^i \subseteq N^{(5)}\) and \((p_i^d \setminus d_i) \cap N = \emptyset\) for each \(i < n\), we notice that

\[
q_i^i \cup p_i \leq_{T(X)} q_i^i.
\]

Thus the condition \(\langle q_i^i \cup p_i; i < n \rangle\) is a common extension of conditions \(\langle p_i; i < n \rangle\) and \(\langle q_i^i; i < n \rangle\) in \(n^T(X)\). \(\Box\)

### 2.2. The chapter IX of [6]: Souslin Hypothesis Does Not Imply “Every Aronszajn Tree Is Special.”

In this section, we summarize Shelah’s approach to show the consistency that Suslin’s Hypothesis holds and there exists a non-special Aronszajn tree. All of definitions and proofs in this section are in [6, IX. Souslin Hypothesis Does Not Imply “Every Aronszajn Tree Is Special”].

**Definition 2.3** (Shelah, [6, IX 3.3 Definition]). *For an Aronszajn tree \(T\) and a subset \(S\) of \(\omega_1\), \(T\) is called \(S\)-st-special if there exists a function \(f\) from the set \(\{t \in T; rk_T(t) \in S\}\) into \(\omega\) such that for each \(n \in \omega\), the set \(f^{-1}\{\{n\}\}\) forms an antichain in \(T\).*

We note that if \(S\) is uncountable and an Aronszajn tree \(T\) is \(S\)-st-special, then \(T\) is still Aronszajn in the forcing extension where \(S\) is still uncountable. And then \(T\) has an uncountable antichain, hence then \(T\) is not a Suslin tree. For a costationary subset \(S\) of \(\omega_1\), if \(T\) is a special Aronszajn tree, then there exists an antichain \(A\) through \(T\) such that the set \(rk_T[A] \setminus S^{(6)}\) is stationary. Therefore if \(S\) is an uncountable costationary subset of \(\omega_1\) and \(T^*\) satisfies the property

\[
(\ast) \text{ for every antichain } A \text{ through } T^*, \text{ the set } rk_{T^*}[A] \setminus S \text{ is nonstationary,}
\]

then \(T^*\) is a non-special Aronszajn tree.

In [6, IX 4.8 Conclusion], Shelah introduced the iterated proper forcing which forces that Suslin’s Hypothesis holds and there are a stationary and costationary subset \(S\) of \(\omega_1\) and an \(S\)-st-special Aronszajn tree \(T^*\) which satisfies the property \((\ast)\). The \(S\)-st-speciality of \(T^*\) guarantees that \(T^*\) is still Aronszajn in any proper forcing extension. To guarantee the property \((\ast)\) of \(T^*\), we shoot a club on \(\omega_1\) for the complement of \(rk_{T^*}[A]\) which is disjoint from \(S\) in some intermediate stage of the iteration [6, IX 4.7, 4.8]. However, the iteration is required to be a proper forcing. To do this, Shelah introduced the following preservation property.

\((5)\) \(q_i^i\) is a countable subset of \(X\).

\((6)\) \(rk_T[A] := \{rk_T(t); t \in A\}\).
Definition 2.4 (Shelah [6, IX 4.5 Definition]). Let $T$ be an Aronszajn tree and $S$ a subset of $\omega_1$.

A forcing notion $\mathbb{P}$ is $(T, S)$-preserving if for a large enough regular cardinal $\theta$, a countable elementary submodel $N$ of $H(\theta)$ which has the set $\{\mathbb{P}, T, S\}$ and $p \in \mathbb{P} \cap N$, there exists $q \leq p$ $p$ which is $(\mathbb{N}, \mathbb{P})$-generic such that if $\omega_1 \cap N \notin S$, then

for any $x \in T$ of height $\omega_1 \cap N$,
if $\forall A \in \mathbb{P}(T) \cap N (x \in A \rightarrow \exists y \in A (y <_T x))$,
then for every $\mathbb{P}$-name $\dot{A}$, which is in $N$, for a subset of $T$,
\[ q \Vdash \text{"} x \in \dot{A} \rightarrow \exists y \in \dot{A} (y <_T x) \text{"}. \]

If $T^*$ is a Suslin tree, then for every countable elementary submodel $N$ of $H(\theta)$ (for some large enough regular cardinal $\theta$) and $x \in T^*$ of height $\omega_1 \cap N$ and $A \in \mathbb{P}(T^*) \cap N$, if $x \in A$, then there exists $y \in A$ such that $y <_{T^*} x$.(7) It follows that $T^*$ satisfies $(\ast)$. So we start from a Suslin tree $T^*$ and a stationary and costationary subset $S$ of $\omega_1$ and make each Aronszajn tree to be $S$-st-special and $T^*$ to be $S$-st-special which satisfies the property $(\ast)$ by the iterated proper forcing extension such that each iteration is $(T^*, S)$-preserving and the whole iteration is also $(T^*, S)$-preserving. For Aronszajn trees $T$ and $T^*$ and a stationary subset $S$ of $\omega_1$, Shelah introduced the forcing notion $Q(T, S)$ which forces $T$ to be $S$-st-special and is $(T^*, S)$-preserving [6, IX 4.2, 4.3, 4.6]. Moreover, Shelah introduced the new forcing iteration, so called a free limit iteration, which preserves the $(T^*, S)$-preserving property [6, IX §1, §2 and 4.7].

The following is Shelah's iterated forcing in [6, Chapter IX, 4.8 Conclusion] (8). We start in the ground model where $2^{\aleph_0} = \mathfrak{N}_1$, $2^{\aleph_1} = \mathfrak{N}_2$, and there exists a Suslin tree $T^*$. Let $S$ be a stationary and costationary subset of the set $\omega_1$. We define an $\aleph_1$-free iteration $(P_\xi, Q_\eta; \xi \leq \omega_2 \& \eta < \omega_2)$ such that

- $Q_0 = Q(T^*, S),
- each$ $Q_\eta$ satisfies one of the following:
  1. $Q_\eta$ is proper and $(T^*, S)$-preserving of size $\aleph_1$,
  2. for some $\mathcal{P}_\xi$-name of an antichain $\dot{A}$ of $T^*$, $\mathcal{P}_{\eta}
  [\dot{A}] \cap S = \emptyset$ and $Q_\eta = Q_{\text{club}}(\omega_1 \setminus \mathcal{P}_{\eta}[\dot{A}])$, which shoots a club through the set $\omega_1 \setminus \mathcal{P}_{\eta}[\dot{A}]$ by countable approximations.

In this extension (with some bookkeeping argument), $S$ is still stationary and costationary, every Aronszajn tree is $S$-st-special (hence not Suslin), and $T^*$ is an $S$-st-special Aronszajn tree which satisfies $(\ast)$.

Combining Shelah's iteration above, some bookkeeping device, theorems in [16, 18] and the next section, we can conclude the following.

Theorem 2.5. It is consistent that every forcing notions with $R_{1, \aleph_1}$ has precaliber $\aleph_1$, every Todorcević ordering for any second countable Hausdorff space also has precaliber $\aleph_1$ and there exists a non-special Aronszajn tree.

---

(7) Let $D := \{ t \in T^*; t \in A \text{ or for every } s \in T^* \text{ with } t <_{T^*} s, s \notin A \}$. Since $D$ is a dense subset of $T^*$ and $T^*$ is Suslin, there exists $y \in D \cap N$ which is compatible with $x$ in $T^*$. Then it have to be true that $y <_{T^*} x$. Since $x \in A$, it have to be true that $y \in A$.

This statement is equivalent that there are no uncountable antichain through $T^*$.

(8) Shelah's proof uses an $\aleph_1$-free iteration. This is different from a countable support iteration. But Schindlein proved in [7] that the same proof works for a countable support iterations. So our theorem can be shown by a countable support iteration.
3. PROOF

Suppose that $S$ is a stationary subset of $\omega_1$, $X$ is a second countable Hausdorff space and $I$ is an uncountable subset of $T(X)$. By shrinking $I$ if necessary, we may assume that

- the size of $I$ is $\aleph_1$,
- the set $\{p^d; p \in I\}$ forms a $\Delta$-system with root $d$,
- for some $q \in \mathcal{T}(X)$,

$$ q \models_{\mathcal{T}(X)} "I \cap \tilde{G} \text{ is uncountable}". $$

Let $\tilde{M} = \langle M_\alpha; \alpha \in \omega_1 \rangle$ be a sequence of countable elementary submodels of $H(\aleph_2)$ such that $\{S, X, I\} \in M_0$, and for every $\alpha \in \omega_1$, $\langle M_\beta; \beta \in \alpha \rangle \in M_\alpha$. By shrinking $I$ if necessary again, we may assume that

- for each $p \in I$ and $\alpha \in \omega_1$, if $p^d \cap (M_{\alpha+1} \setminus M_\alpha) \neq \emptyset$, then $p^d \subseteq M_{\alpha+1} \setminus M_\alpha$.

We have to notice then that it may happen that $I$ does not belong to $M_0$. From now on, we do not assume that $I \in M_\alpha$ for any $\alpha \in \omega_1$.

We define the forcing notion $\mathcal{Q}(\mathcal{T}(X), I, \tilde{M}, S)$ which consists of pairs $\langle h, f \rangle$ such that

- $h$ is a finite partial function from $\omega_1$ into $\omega_1$,
- for any $\alpha, \beta \in \text{dom}(h)$, $\alpha \leq h(\alpha)$, and if $\alpha < \beta$, then $h(\alpha) < \beta$,
- for any $\alpha \in \text{dom}(h) \cap S$, $h(\alpha) = \alpha$,
- $f$ is a finite partial function from $I$ into $\omega$,
- for any $\alpha \in \text{dom}(h)$ and $p \in \text{dom}(f)$,

$$ p^d \cap (M_{h(\alpha)} \setminus M_\alpha) = \emptyset, $$

- for any $p \in \text{dom}(f)$, the set $\bigcup f^{-1}\{\{f(p)\}\}$ is a common extension of members of the set $f^{-1}\{\{f(p)\}\}$ in $\mathcal{T}(X)$, ordered by extension, that is, for any $\langle h, f \rangle$ and $\langle h', f' \rangle$ in $\mathcal{Q}(\mathcal{T}(X), I, \tilde{M}, S)$,

$$ \langle h, f \rangle \leq_{\mathcal{Q}(\mathcal{T}(X), I, \tilde{M}, S)} \langle h', f' \rangle : \iff h \supseteq h' \text{ and } f \supseteq f'. $$

By a density argument, if $\mathcal{Q}(\mathcal{T}(X), I, \tilde{M}, S)$ is proper, then $\mathcal{Q}(\mathcal{T}(X), I, \tilde{M}, S)$ adds an uncountable subset of $I$ which satisfies the finite compatibility property. Therefore, under the approach due to Shelah in §2, it suffices to show that $\mathcal{Q}(\mathcal{T}(X), I, \tilde{M}, S)$ is proper and $(T^*, S)$-preserving.

Lemma 3.1. $\mathcal{Q}(\mathcal{T}(X), I, \tilde{M}, S)$ is proper.

Proof. Let $\theta$ be a large enough regular cardinal, a countable elementary submodel $N$ of $H(\theta)$ which has the set $\{X, I, \tilde{M}, S\}$, $\langle h, f \rangle \in \mathcal{Q}(\mathcal{T}(X), I, \tilde{M}, S)$, and $\delta$ a countable ordinal not smaller than the ordinal $\omega_1 \cap N$ (if $\omega_1 \cap N \in S$, then we define $\delta := \omega_1 \cap N$). We show that $\langle h \cup \{\langle \omega_1 \cap N, \delta \rangle, f \rangle \rangle (N, \mathcal{Q}(\mathcal{T}(X), I, \tilde{M}, S))$-generic.

Let $\langle h', f' \rangle \leq_{\mathcal{Q}(\mathcal{T}(X), I, \tilde{M}, S)} \langle h \cup \{\langle \omega_1 \cap N, \delta \rangle, f \rangle \rangle$ and $D$ a dense open subset of $\mathcal{Q}(\mathcal{T}(X), I, \tilde{M}, S)$. We will find a condition in $D \cap N$ which is compatible with $\langle h', f' \rangle$ in $\mathcal{Q}(\mathcal{T}(X), I, \tilde{M}, S)$.

By extending the condition $\langle h', f' \rangle$ if necessary, we may assume that $\langle h', f' \rangle \in D$. We note that $\langle h' \upharpoonright N, f' \upharpoonright N \rangle$ is in $\mathcal{Q}(\mathcal{T}(X), I, \tilde{M}, S) \cap N$ because $\omega_1 \cap N \in \text{dom}(h')$. 

Let
\[ D' := \{ (k, g) \in D; \langle k, g \rangle \leq_{Q(T(X), I, \vec{M}, S)} \langle h'|N, f'|N \rangle \& \text{ran}(g) = \text{ran}(f') \}. \]

We note that $D'$ is in $N^{(9)}$, $\langle h', f' \rangle \in D'$ and $D'$ is dense in $Q(Q, I, \vec{M})$ below $\langle h'|N, f'|N \rangle$. Since the product forcing $\text{ran}(f')\mathbb{T}(X)$ of $\mathbb{T}(X)$ is ccc in the model $N$, by the elementarity of $N$, there exists a countable subset $J$ of $\text{ran}(f')\mathbb{T}(X)$ in $N$ such that

- $J$ is a subset of the set
  \[ \left\{ \left( \bigcup g^{-1}\{n\}; n \in \text{ran}(f') \right); (k, g) \in D' \right\}, \]
  
- for every $(k, g) \in D'$, there exists $\langle \mu_n; n \in \text{ran}(f') \rangle \in J$ such that for each $n \in \text{ran}(f')$, $\mu_n$ and $\bigcup g^{-1}\{n\}$ are compatible in $\mathbb{T}(X)$.

Since $\langle h', f' \rangle \in D'$, there exists $\langle \mu_n; n \in \text{ran}(f') \rangle \in J$ such that for each $n \in \text{ran}(f')$, $\mu_n$ and $\bigcup (f')^{-1}\{n\}$ are compatible in $\mathbb{T}(X)$. Since $\langle \mu_n; n \in \text{ran}(f') \rangle \in J$ holds in $N$, there exists $(k, g) \in D' \cap N$ such that

\[ \left( \bigcup g^{-1}\{n\}; n \in \text{ran}(f') \right) = \langle \mu_n; n \in \text{ran}(f') \rangle. \]

Then $\langle h' \cup k, f' \cup g \rangle$ is a common extension of $\langle h', f' \rangle$ and $(k, g)$ in $Q(T(X), I, \vec{M}, S)$.

\[ \square \]

Lemma 3.2. For any Aronszajn tree $T$, $Q(T(X), I, \vec{M}, S)$ is $(T, S)$-preserving.

Proof. Let $T$, $\theta$, $N$ be as in the statement of the definition of the $(T, S)$-preservation, (moreover we suppose $\vec{M} \in N$) and $\langle h, f \rangle \in Q(T(X), I, \vec{M}, S) \cap N$. Suppose that $\omega_1 \cap N \notin S$, because if $\omega_1 \cap N \in S$, then the condition $\langle h \cup \{\omega_1 \cap N, \delta\}, f \rangle$ is as desired.

Let
\[ \delta := \sup \{ F(\omega_1 \cap N) + 1; F \in (\omega_1) \cap N \}. \]

Since $N$ is countable, $\delta$ is a countable ordinal. We will show that the condition $\langle h \cup \{\omega_1 \cap N, \delta\}, f \rangle$ of $Q(Q, I, \vec{M}, S)$ is our desired one.

As seen in the proof of the previous lemma, the condition $\langle h \cup \{\omega_1 \cap N, \delta\}, f \rangle$ is $(N, Q(T(X), I, \vec{M}, S))$-generic. Suppose that $x \in T$ of height $\omega_1 \cap N$ such that for any subset $A \in N$ of $T$, if $x \in A$, then there is $y \in A$ such that $y \prec_T x$. Let $\dot{A} \in N$ be a $Q(T(X), I, \vec{M}, S)$-name for a subset of $T$. We will show that
\[ \langle h \cup \{\omega_1 \cap N, \delta\}, f \rangle \models_{Q(T(X), I, \vec{M}, S)} "x \notin \dot{A}\text{ or } \exists y \in \dot{A}(y \prec_T x)". \]

Let $\langle h', f' \rangle \leq_{Q(Q, I, \vec{M}, S)} \langle h \cup \{\omega_1 \cap N, \delta\}, f \rangle$, and assume that
\[ \langle h', f' \rangle \not\models_{Q(T(X), I, \vec{M}, S)} "x \notin \dot{A}". \]

By strengthening $\langle h', f' \rangle$ if necessary, we may assume that
\[ \langle h', f' \rangle \models_{Q(T(X), I, \vec{M}, S)} "x \in \dot{A}". \]

\[ (9)\text{ran}(f') \text{ is a finite subset of } \omega. \]
We note that $\langle h'|N, f'|N \rangle$ is in $N$ (because $\omega_1 \cap N \in \text{dom}(h')$), and by the definition of $Q(T(X), I, \vec{M}, S)$, for every $p \in \text{dom}(f')$, if $\text{ran}(p) \not\subseteq N$, then

$$(p^d \setminus d) \cap M_\delta = \emptyset.$$ 

Let $\gamma \in \omega_1 \cap N$ be such that for every $p \in \text{dom}(f')$, if the set $p^d \setminus d$ intersects $N$, then $p^d \subseteq M_\gamma$\(^{(10)}\). Since $X$ is second countable Hausdorff and $N$ is an elementary submodel, there exists a finite set $\mathcal{U}$ of pairwise disjoint open subsets of $X$ in $N$ such that for each $n \in \text{ran}(f')$, the finite set $(\bigcup f'^{-1}(\{n\}))^d$ is separated by $\mathcal{U}$. We define a function $F$ with the domain

$$\{ t \in T; \text{ht}_T(t) > \max(\text{dom}(h'|N)) \}$$

such that for each $t \in T$ of height larger than $\max(\text{dom}(h'|N))$,

$$F(t) := \sup \left\{ \beta \in \omega_1; \text{there exists } \langle k, g \rangle \in Q(T(X), I, \vec{M}, S) \text{ such that} \right\}$$

- $\min(\text{dom}(k)) = \text{rk}_T(t),$
- $k(\text{rk}_T(t)) = \beta,$
- $\langle (h'|N) \cup k, (f'|N) \cup g \rangle$ is a condition of $Q(T(X), I, \vec{M}, S),$
- for each $p \in \text{dom}(g)$, $(p^d \setminus d) \cap M_\gamma = \emptyset,$
- $\text{ran}(g) = \text{ran}(f' \setminus N),$
- for each $n \in \text{ran}(f' \setminus N)$, the set $(\bigcup g^{-1}(\{n\}))^d \setminus d$ is separated by $\mathcal{U}$, and
- $\langle (h'|N) \cup k, (f'|N) \cup g \rangle \models_{Q(T(X), I, \vec{M}, S)} \{ t \in \dot{A} \}$.

Then $F$ belongs to $N$. Let

$$B := \{ t \in T; \text{rk}_T(t) > \max(\text{dom}(h'|N)) \land F(t) = \omega_1 \},$$

which is also in $N$. We define a function $F'$ with the domain

$$\{ \max(\text{dom}(h'|N)) + 1, \omega_1 \}$$

such that for a countable ordinal $\beta$ larger than $\max(\text{dom}(h'|N))$,

$$F'(\beta) := \sup \left\{ F(t) + 1; t \in T \setminus B \land \text{rk}_T(t) \in \{ \max(\text{dom}(h'|N)), \beta \} \right\}.$$ 

This $F'$ is a function from $\omega_1$ into $\omega_1$ and also in $N$. Hence $F'(\omega_1 \cap N) < \delta$ by the definition of $\delta$. Since $\langle h', f' \rangle \models_{Q(T(X), I, \vec{M}, S)} \{ x \in \dot{A} \}$ and $h'(\text{rk}_T(x)) = h'(\omega_1 \cap N) = \delta$, $F(x) \geq \delta$ holds. Therefore $x$ have to belong to $B$. Thus by our assumption, there exists $y \in B$ such that $y <_T x$.

Take $\varepsilon \in \omega_1$ such that $f' \subseteq M_\varepsilon$. Let

$$E := \left\{ \langle k, g \rangle \in Q(Q, I, \vec{M}, S); \right\}$$

- $\min(\text{dom}(k)) = \text{rk}_T(y),$
- $\langle (h'|N) \cup k, (f'|N) \cup g \rangle$ is a condition of $Q(T(X), I, \vec{M}, S),$
- for each $p \in \text{dom}(g)$, $(p^d \setminus d) \cap M_\gamma = \emptyset,$
- $\text{ran}(g) = \text{ran}(f'),$
- for each $n \in \text{ran}(f')$, the set $(\bigcup g^{-1}(\{n\}))^d \setminus d$ is separated by $\mathcal{U}$, and
- $\langle (h'|N) \cup k, (f'|N) \cup g \rangle \models_{Q(T(X), I, \vec{M}, S)} \{ y \in \dot{A} \}.$

\(^{(10)}\) Then for every $p \in \text{dom}(f')$, $(p^d \setminus d) \cap M_\gamma = \emptyset$ if $(p^d \setminus d) \cap M_\delta = \emptyset.$
We note that $E$ is in $N$, and the set
$$\{k(rk_T(y)); \langle k, g \rangle \in E\}$$
is uncountable because $F(y) = \omega_1$. So there exists $\langle k, g \rangle \in E$ such that for each $p \in \text{dom}(g)$, $(p^d \setminus d) \cap M_d = \emptyset$. Then for each $n \in \text{ran}(f' \setminus N)$,
$$\left(\left(\bigcup g^{-1}[[n]]\right)^d \setminus d\right) \cap \left(\bigcup (f' \setminus N)^{-1}[[n]]\right) = \emptyset.$$Since $X$ is second countable Hausdorff and $N$ is an elementary submodel, there exists disjoint open subsets $U$ and $V$ of $X$ in $N^{(11)}$ such that for each $n \in \text{ran}(f' \setminus N)$,
$$\left(\left(\bigcup (f' \setminus N)^{-1}[[n]]\right)^d \setminus d\right) \subseteq U,$$$$
\left(\left(\bigcup g^{-1}[[n]]\right)^d \setminus d\right) \subseteq V$$and
$$V \cap \left(\left(\bigcup (f' \setminus N)^{-1}[[n]]\right) \setminus U\right) = \emptyset.$$By the elementarity of $N$, we can find $\langle k', g' \rangle \in E$ such that
$$\left(\bigcup (g')^{-1}[[n]]\right)^d \setminus d \subseteq V.$$Then for each $n \in \text{ran}(f' \setminus N)$, the set
$$\bigcup (f')^{-1}[[n]] \cup \bigcup (g')^{-1}[[n]] \leq_{T(X)} \bigcup (f')^{-1}[[n]].$$Since $g' \subseteq N$ and $(\bigcup (f' \setminus N)^{-1}[[n]])^d \cap N = \emptyset$, we note that for each $n \in \text{ran}(f' \setminus N)$, the set
$$\bigcup (f')^{-1}[[n]] \cup \bigcup (g')^{-1}[[n]] \leq_{T(X)} \bigcup (g')^{-1}[[n]].$$Therefore $\langle k' \cup h', g' \cup f' \rangle$ is an extension of $\langle h', f' \rangle$ in $Q(T(X), I, \vec{M}, S)$ and $\langle k' \cup h', g' \cup f' \rangle \models_{Q(T(X), I, \vec{M}, S)} "y \in \dot{A}".$

\[\square\]

REFERENCES


(11) This can be done because the set $(\bigcup (f' \setminus N)^{-1}[[n]]) \setminus U$ is finite if $U$ satisfies that $(\bigcup (f' \setminus N)^{-1}[[n]])^d \setminus d \subseteq U.$


[15] T. Yorioka. The inequality \( b > \aleph_1 \) can be considered as an analogue of Suslin’s Hypothesis. Axiomatic Set Theory and Set-theoretic Topology (Kyoto 2007), Sūrikaisekikenkyūsho Kökyūroku No. 1595 (2008), 84–88.


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