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INTUITIONISTIC TREE SEQUENT CALCULUS AND INTUITIONISTIC LAMBDA-RHO-CALCULUS

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ABSTRACT. In [8], the author gave a subsystem of the $\lambda\rho$-calculus [7], and showed that the subsystem corresponds to intuitionistic logic. The proof was given with the intuitionistic tree sequent calculus [4, 6]. In this paper, we give a observation to these two proof systems, and show there is a close connection between them.

1. INTRODUCTION

The *$\lambda\rho$-calculus* [1] [7] is a natural deduction style proof system for implicational fragment of classical logic. In [8], the author gave a natural subsystem, called *intuitionistic $\lambda\rho$-calculus*, and showed that the subsystem corresponds to intuitionistic logic. In [8], the proof was given with the intuitionistic tree sequent calculus **TLJ** [4, 6]. On the other hand, in [1], the same fact was showed by purely proof theoretic way. In this paper, we try to give another proof to the fact that **TLJ** corresponds to intuitionistic logic by applying the technique in [1]. With observing this proof, we can invent a close connection between the intuitionistic $\lambda\rho$-calculus and **TLJ**.

We introduce the basic notions which we are going to use, before getting to the main point.

1.1. TLJ.

The set Fml of *formulas* (*types*) is defined, with a given countable set P of *propositional variables* (*atomic types*), as follows.

$$\alpha, \beta \in \mathrm{Fml} ::= p \mid (\alpha \to \beta), \quad p \in \mathrm{P}.$$

We use metavariables $p, q, \ldots$ to stand for arbitrary propositional formulas, $\alpha, \beta, \ldots$ for arbitrary formulas, $\Gamma, \Delta, \ldots$ for arbitrary finite sets of formulas. Parentheses are omitted as follows: $\alpha \to \beta \to \gamma \equiv (\alpha \to (\beta \to \gamma))$.

A *finite tree* is a structure $\langle T, \leq, r\rangle$ such that:

- T is a nonempty finite set. Each element of T is called a *node* of $\mathcal{T}$.
- $\leq$ is a reflexive partial order on T such that, for every $t \in T$, the relation $\leq$ is a total order on $\{s \in T \mid s \leq t\}$.
- r is a least element of T with respect to the relation $\leq$. We call this element the *root node* of $\mathcal{T}$.

[1]In [8], the author treat not only the $\lambda\rho$-calculus but also the $\lambda\mu$-calculus [10]. Although we treat only the $\lambda\rho$-calculus in this paper, we can treat the $\lambda\mu$-calculus in the same way through the following translation m from $\lambda\rho$-terms into $\lambda\mu$-terms: $x^m \equiv x$, $(MN)^m \equiv M^m N^m$, $(\lambda x.M)^m \equiv \lambda x.M^m$, $(aM)^m \equiv \mu b.aM^m$ $(b \notin \mathrm{FV}_\mu(M))$, $(\rho a.M)^m \equiv \mu a.aM^m$.

Given a reflexive partial order $\leq$, we write $t < s$ if both $t \leq s$ and $t \neq s$ hold, and write $t <_1 s$ if $t < s$ and there are no elements $u \in T$ such that $t < u < s$. We say t_2 is a *descendant* of t_1 if $t_1 < t_2$, and say t_2 is a *son* of t_1 if $t_1 <_1 t_2$. A *leaf node* of $\mathcal{T}$ is an element which has no descendants.

We write the tree $\langle \{r\}, \{(r,r)\}, r\rangle$ as $\mathcal{R}(r)$ (or write $\mathcal{R}$ simply).

Let $\mathcal{T} = \langle T, \leq_t, r_t\rangle, \mathcal{S} = \langle S, \leq_s, r_s\rangle$ be trees. We write $\mathcal{T} \sqsubseteq \mathcal{S}$ if all of the following conditions hold.

- $T \subseteq S$, $\leq_t \subseteq \leq_s$ and $r_t = r_s$.
- If $t \in T$ then $\{s \in S \mid s \leq_s t\} \subseteq T$.

We say $\mathcal{T}$ is a *subtree* of $\mathcal{S}$ if $\mathcal{T} \sqsubseteq \mathcal{S}$ holds.

Let t be a leaf node of $\mathcal{T} = \langle T, \leq, r\rangle$. We define the subtree $\mathcal{T}_{-t} = \langle T', \leq', r'\rangle$ as follows.

- $T' = T \setminus \{t\}$.
- $\leq' = \leq \setminus \{(t', t) \in \leq\}$.
- $r' = r$.

A *tree sequent* is an expression of the form $\Gamma \vdash^{\mathcal{T}} \Delta$ where:

- $\mathcal{T} = \langle T, \leq, r\rangle$ is a finite tree.
- Γ, Δ are finite subsets of $\{t : \alpha \mid t \in \mathcal{T}, \alpha \in \mathrm{Fml}\}$.

Then the tree sequent calculus **TLJ**, which derives tree sequents, is defined by the following rules.

Definition 1.1. The system **TLJ** consists of the following rules.
[axiom] $\qquad \{t : \alpha\} \vdash^{\mathcal{T}} \{t : \alpha\}.$
[inference rule]

$$\frac{\Gamma \vdash^{\mathcal{T}} \Delta}{\Gamma' \cup \Gamma \vdash^{\mathcal{T}} \Delta \cup \Delta'}\ (\text{Weakening})$$

$$\frac{\Gamma \vdash^{\mathcal{T}} \Delta}{\Gamma \vdash^{\mathcal{S}} \Delta}\ (\text{Grow})\quad (\mathcal{T} \sqsubseteq \mathcal{S})$$

$$\frac{\{t : \alpha\} \cup \Gamma \vdash^{\mathcal{T}} \Delta}{\{s : \alpha\} \cup \Gamma \vdash^{\mathcal{T}} \Delta}\ (\text{Heredity})_l\quad (s < t)$$

$$\frac{\Gamma \vdash^{\mathcal{T}} \Delta \cup \{t : \alpha\}}{\Gamma \vdash^{\mathcal{T}} \Delta \cup \{s : \alpha\}}\ (\text{Heredity})_r\quad (t < s)$$

$$\frac{\Gamma_1 \vdash^{\mathcal{T}} \Delta_1 \cup \{t : \alpha\} \quad \{t : \beta\} \cup \Gamma_2 \vdash^{\mathcal{T}} \Delta_2}{\{t : \alpha \to \beta\} \cup \Gamma_1 \cup \Gamma_2 \vdash^{\mathcal{T}} \Delta_1 \cup \Delta_2}\ (\to\vdash)$$

$$\frac{\{t : \alpha\} \cup \Gamma \vdash^{\mathcal{T}} \Delta \cup \{t : \beta\}}{\Gamma \vdash^{\mathcal{T}_{-t}} \Delta \cup \{s : \alpha \to \beta\}}\ (\vdash\to)\quad (s <_1 t \text{ and } t \text{ is a leaf})$$

In the $(\vdash\to)$-scheme, because the lower sequent is also a tree sequent, we implicitly impose the following condition: if u is a descendant of t then u does not occur in the lower sequent.

Lemma 1.2. The cut rule

$$\frac{\Gamma_1 \vdash^{\mathcal{T}} \Delta_1 \cup \{t : \alpha\} \quad \{t : \alpha\} \cup \Gamma_2 \vdash^{\mathcal{T}} \Delta_2}{\Gamma_1 \cup \Gamma_2 \vdash^{\mathcal{T}} \Delta_1 \cup \Delta_2}\ (\text{Cut})$$

is admissible in **TLJ**.

Theorem 1.3 ([4, 6, 9]). φ is intuitionistically valid if and only if $\emptyset \vdash^{\mathcal{R}} \{r : \varphi\}$ is derivable in **TLJ**.

1.2. **$\lambda\rho$-calculus.** Suppose that there exist a countable set V_λ of *λ-variables* and a set V_ρ of *ρ-variables*. The set Λ_ρ of all *$\lambda\rho$-terms* is defined by the following grammar.

$$M, N \in \Lambda_\rho ::= x \mid (MN) \mid (\lambda x.M) \mid (aM) \mid \rho a.M,$$
$$x \in V_\lambda,\ a \in V_\rho.$$

We use metavariables $x, y, z, \ldots$ to stand for arbitrary λ-variables, $a, b, c, \ldots$ for arbitrary ρ-variables, $M, N, P, \ldots$ for arbitrary $\lambda\rho$-terms. We use the abbreviations such as:

$$MNP \equiv ((MN)P)$$
$$\lambda x.MN \equiv (\lambda x.(MN))$$
$$\lambda xy.M \equiv (\lambda x.(\lambda y.M))$$

We also use notations such as $[N/x]M$ (the substitution of N for free occurrences of x in M), $\mathrm{Sub}(M)$ (the set of all subterms of M), $\mathrm{FV}_{\lambda(\rho)}(M)$ (the set of all free $\lambda(\rho)$-variables in M), $\mathrm{BV}_{\lambda(\rho)}(M)$ (the set of all bound $\lambda(\rho)$-variables in M). In the following argument, we assume that each $\lambda\rho$-term follows the Barendregt's convention [2].

A typing judgement is an expression of the form $\Gamma \vdash M : \alpha, \Delta$ where:

- Γ is a finite subset of the set $\{x : \alpha \mid x \in V_\lambda, \alpha \in \mathrm{Fml}\}$.
- M is a $\lambda\rho$-term.
- α is a formula.
- Δ is a finite subset of the set $\{a : \alpha \mid a \in V_\rho, \alpha \in \mathrm{Fml}\}$.

Definition 1.4. The typing system $\mathbf{TA}_{\lambda\rho}$ consists of the following rules.
[axiom] $\qquad \{x : \alpha\} \vdash x : \alpha,\ \emptyset$
[inference rule]

$$\frac{\Gamma_1 \vdash M : \alpha \to \beta,\ \Delta_1 \quad \Gamma_2 \vdash N : \alpha,\ \Delta_2}{\Gamma_1 \cup \Gamma_2 \vdash MN : \beta,\ \Delta_1 \cup \Delta_2}\ (\mathrm{App})$$

$$\frac{\Gamma \vdash M : \beta,\ \Delta}{\Gamma \setminus \{x : \alpha\} \vdash \lambda x.M : \alpha \to \beta,\ \Delta}\ (\mathrm{Abs})$$

$$\frac{\Gamma \vdash M : \alpha,\ \Delta}{\Gamma \vdash aM : \beta,\ \Delta \cup \{a : \alpha\}}\ (\text{App-}\rho) \qquad \frac{\Gamma \vdash M : \alpha,\ \Delta}{\Gamma \vdash \rho a.M : \alpha,\ \Delta \setminus \{a : \alpha\}}\ (\text{Abs-}\rho)$$

Note 1.5. The rule (App-ρ) works like the right-side weakening rule of **LJ**. Similarly, we can see that the rule (Abs-ρ) works like the right-side contraction rule.

Note 1.6. If $\{x_1 : \alpha_1, \ldots, x_n : \alpha_n\} \vdash M : \beta,\ \{a_1 : \gamma_1, \ldots, a_m : \gamma_m\}$ is derivable in $\mathbf{TA}_{\lambda\rho}$, then $\mathrm{FV}_\lambda(M) = \{x_1, \ldots, x_n\}$ and $\mathrm{FV}_\rho(M) = \{a_1, \ldots, a_m\}$.

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[2] M follows the Barendregt's convention if the following two conditions hold: (1) All bound variables in M are all different from each other. (2) $\mathrm{FV}_\lambda(M) \cap \mathrm{BV}_\lambda(M) = \emptyset$, $\mathrm{FV}_\rho(M) \cap \mathrm{BV}_\rho(M) = \emptyset$.

For every M, there is a $\lambda\rho$-term N which is α equivalent to M and follows the Barendregt convention.

Note 1.7. If $\Gamma \vdash M : \alpha, \emptyset$ is derivable in $\mathbf{TA}_{\lambda\rho}$ and M is a λ-term, then this typing judgement can be derived in $\mathbf{TA}_{\lambda}$ [3] .

Komori [7] showed that $\lambda\rho$-terms correspond to classical logic in the following sense: φ is classically valid if and only if there exists a closed $\lambda\rho$-term M such that $\emptyset \vdash M : \varphi,\ \emptyset$ is derivable in $\mathbf{TA}_{\lambda\rho}$.

1.3. **Intuitionistic $\lambda\rho$-calculus.**

Definition 1.8. Define the subset $\Lambda_\rho^{\mathrm{Int}}$ of Λ_ρ as follows.

- $\mathrm{V}_\lambda \subseteq \Lambda_\rho^{\mathrm{Int}}$.
- $M, N \in \Lambda_\rho^{\mathrm{Int}} \implies MN, aM, \rho a.M \in \Lambda_\rho^{\mathrm{Int}}$.
- $M \in \Lambda_\rho^{\mathrm{Int}},\ x \in \bigcup_{a \in \mathrm{FV}_\rho(M)} \mathrm{FV}_\lambda^a(M) \implies \lambda x.M \in \Lambda_\rho^{\mathrm{Int}}$.

Here $\mathrm{FV}^a(M) = \{x \in \mathrm{FV}_\lambda(M) \mid$ there is a subterm of M of the form $a(\ldots x \ldots)\}$.

Theorem 1.9 ([8])**.** φ is intuitionistically valid if and only if there exists a closed term $M \in \Lambda_\rho^{\mathrm{Int}}$ such that $\emptyset \vdash M : \varphi,\ \emptyset$ is derivable in $\mathbf{TA}_{\lambda\rho}$.

2. Proof of theorem 1.9 with tree sequent

In this subsection, we give a sketch of the proof of theorem 1.9 given in [8]. The key of our proof is the following notion.

Definition 2.1. For each $M \in \Lambda_\rho$ following the Barendregt's convention, we define the tree structure $\mathcal{T}^M = \langle T^M, <^M, r^M \rangle$ as follows.

- $T^M = \{r^M\} \cup \{\bar{x} \mid x \in \mathrm{BV}_\lambda(M)\}$.
- $r^M <^M x$ for every $x \in \mathrm{BV}_\lambda(M)$.
- $\bar{x} <^M \bar{y}$ $(x, y \in \mathrm{BV}_\lambda(M))$ if λy is in the scope of λx in M.

In addition, for each subterm N of M, we define the subtree $\mathcal{T}^M(N) = \langle T^N, <^M, r^N \rangle$ of $\mathcal{T}^M$ as follows.

- $T^N = \{r^M\} \cup \{\bar{x} \in T^M \mid \bar{x} \leq^M \bar{y}$ for some $y \in \mathrm{FV}_\lambda(N)\}$.
- $<^N = \{(\bar{x}, \bar{y}) \mid \bar{x} <^M \bar{y}$ and $\bar{x}, \bar{y} \in T^N\}$.
- $r^M = r^N$.

Example 2.2. Let $M \equiv \lambda x.(\lambda y.\rho a.ay)(\lambda z.xz)$ then the tree $\mathcal{T}(M)$ is the structure $\langle T^M, <^M, r^M \rangle$ (see also figure 1) where:

- $T^M = \{r^M, \bar{x}, \bar{y}, \bar{z}\}$.
- $<_1^M = \{(r^M, \bar{x}), (\bar{x}, \bar{y}), (\bar{x}, \bar{z})\}$.

Proof of theorem 1.9 with tree sequent. The "only if" part is obvious, because $\Lambda_\rho^{\mathrm{Int}}$ includes all λ-terms. Hence we show "if" part.

Suppose M is a closed $\lambda\rho$-term following the Barendregt's convention, and there is a $\mathbf{TA}_{\lambda\rho}$-derivation Σ of $\emptyset \vdash M : \varphi,\ \emptyset$. We first define $[N] \in \mathcal{T}(M)$ and $[a]_N \in \mathcal{T}(M)$, for each $N \in \mathrm{Sub}(M)$ and $a \in \mathrm{FV}_\rho(N)$, as follows.

- $[N]$ is the greatest element, with respect to $<^M$, of the set $\{r^M\} \cup \{\bar{x} \mid x \in \mathrm{FV}_\lambda(N)\}$.
- $[a]_N$ is the greatest element, with respect to $<^M$, of the set $\{r^M\} \cup \{\bar{x} \mid x \in \mathrm{FV}_\lambda^a(N)\}$.

[3] $\mathbf{TA}_\lambda$ is a typing system for λ-terms, see [5] for TA_λ

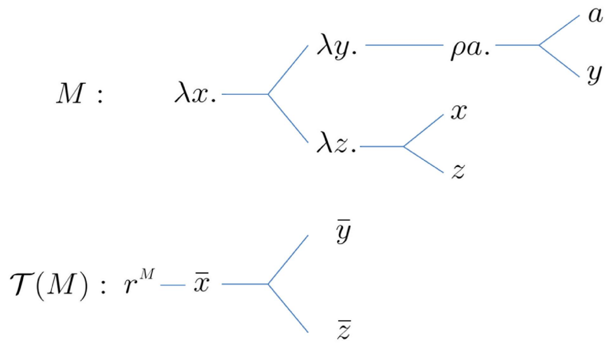


FIGURE 1. $M \equiv \lambda x.(\lambda y.\rho a.ay)(\lambda z.xz)$

Then we can show, by induction on the size of the $\mathbf{TA}_{\lambda\rho}$-derivation, that if

$$\{x_1 : \alpha_1, \ldots, x_n : \alpha_n\} \vdash N : \beta,\ \{a_1 : \gamma_1, \ldots, a_m : \gamma_m\}$$

occurs in Σ then the tree sequent

$$\{\bar{x_1} : \alpha_1, \ldots, \bar{x_n} : \alpha_n\} \vdash^{\mathcal{T}^M(N)} \{[N] : \beta\} \cup \{[a_1]_N : \gamma_1, \ldots, [a_m]_N : \gamma_m\}$$

is derivable in **TLJ**. In particular, we can see that $\emptyset \vdash^{\mathcal{R}(r^M)} \{r^M : \varphi\}$ is derivable in **TLJ**. Hence φ is intuitionistically valid. $\square$

3. PROOF OF THEOREM 1.9 WITH REDUCTION

In section 2, we give a proof to theorem 1.9, by use of the sequent calculus **TLJ**. On the other hand, in [1], the theorem was proved by purely proof theoretic method. In this subsection, we give a sketch of the proof.

First, we give a derivation reduction for the $\lambda\rho$-calculus as follows.

Definition 3.1. We define a binary relation $\triangleright_1$ on Λ_ρ by the following rules.

(1) $aMN \triangleright_1 aM$.
(2) $N(aM) \triangleright_1 aM$.
(3) $\lambda x.aM \triangleright_1 aM$ if $x \notin \mathrm{FV}_\lambda(M)$.
(4) $b(aM) \triangleright_1 aM$.
(5) $\rho a.M \triangleright_1 M$ if $a \notin \mathrm{FV}_\rho(M)$.
(6) $\rho a.aM \triangleright_1 M$ if $a \notin \mathrm{FV}_\rho(M)$.
(7) $M \triangleright_1 N \implies PM \triangleright_1 PN$.
(8) $M \triangleright_1 N \implies MP \triangleright_1 NP$.
(9) $M \triangleright_1 N \implies \lambda x.M \triangleright_1 \lambda x.N$.
(10) $M \triangleright_1 N \implies aM \triangleright_1 aN$.
(11) $M \triangleright_1 N \implies \rho a.M \triangleright_1 \rho a.N$.

Furthermore, we define the relation $\triangleright$ as the reflexive transitive closure of $\triangleright_1$.

We can easily check the following properties.

Lemma 3.2.

(1) If $M \triangleright_1 N$, then $\mathrm{FV}_\lambda(M) \supseteq \mathrm{FV}_\lambda(N)$ and $\mathrm{FV}_\rho(M) \supseteq \mathrm{FV}_\rho(N)$.

(2) If $M \rhd_1 N$ and $\Gamma \vdash M : \varphi, \Delta$ is derivable in $\mathbf{TA}_{\lambda\rho}$, then

$\{x : \alpha \mid x : \alpha \in \Gamma, x \in \mathrm{FV}_\lambda(N)\} \vdash N : \varphi, \{a : \beta \mid a : \beta, b \in \mathrm{FV}_\rho(N)\}$

is derivable in $\mathbf{TA}_{\lambda\rho}$.

(3) If $M \rhd_1 N$ and $M \in \Lambda_\rho^{\mathrm{Int}}$, then $N \in \Lambda_\rho^{\mathrm{Int}}$.

By use of these properties, we can give another proof to theorem 1.9 as follows.

Proof of theorem 1.9 with reduction. Suppose that there is a closed $\lambda\rho^{\mathrm{Int}}$-term M such that $\emptyset \vdash M : \varphi,\ \emptyset$ is derivable in $\mathbf{TA}_{\lambda\rho}$. First we can easily check that M can be reduced to a closed λ-term N with the reduction $\rhd$. From lemma 3.2, we have:

- N is closed.
- $\emptyset \vdash N : \varphi, \emptyset$ is derivable in $\mathbf{TA}_{\lambda\rho}$.

With note 1.7, it can be checked that $\emptyset \vdash N : \varphi, \emptyset$ is derivable in $\mathbf{TA}_\lambda$. Therefore φ is intuitionistically valid. □

4. Proof of theorem 1.3

In section 2, theorem 1.9 is proved with a close connection between the systems **TLJ** and the intuitionistic $\lambda\rho$-calculus. On the other hand, in section 3, theorem 1.9 is proved by purely proof theoretic method. Then a question arises:

($\sharp$) Can we prove the "if" part of theorem 1.3 with technique in section 3?

In this section, we give an affirmative answer to this question.

4.1. **On the reduction $\rhd_1$.** First, we explain what role the reduction $\rhd_1$ play in the proof. The rules (1)-(6) in definition 3.1. gives the following derivation reduction.

(1)

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Gamma_1 \vdash M : \alpha, \Delta_1\end{array}}{\Gamma_1 \vdash aM : \beta \to \gamma, \Delta_1 \cup \{a : \alpha\}} \quad \begin{array}{c}\vdots\\ \Gamma_2 \vdash N : \beta\end{array}}{\begin{array}{c}\Gamma_1 \cup \Gamma_2 \vdash aMN : \gamma, \Delta_1 \cup \{a : \alpha\} \cup \Delta_2\\ \vdots\end{array}} \quad \rhd_1 \quad \dfrac{\begin{array}{c}\vdots\\ \Gamma_1 \vdash M : \alpha, \Delta_1\end{array}}{\begin{array}{c}\Gamma_1 \vdash aM : \gamma, \Delta_1 \cup \{a : \alpha\}\\ \vdots\end{array}}$$

(2)

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma_1 \vdash M : \beta \to \gamma, \Delta_1\end{array} \quad \dfrac{\begin{array}{c}\vdots\\ \Gamma_2 \vdash N : \alpha, \Delta_2\end{array}}{\Gamma_2 \vdash aN : \beta, \Delta_2 \cup \{a : \alpha\}}}{\begin{array}{c}\Gamma_1 \cup \Gamma_2 \vdash M(aN) : \gamma, \Delta_1 \cup \Delta_2 \cup \{a : \alpha\}\\ \vdots\end{array}} \quad \rhd_1 \quad \dfrac{\begin{array}{c}\vdots\\ \Gamma_2 \vdash N : \alpha, \Delta_2\end{array}}{\begin{array}{c}\Gamma_2 \vdash aN : \gamma, \Delta_2 \cup \{a : \alpha\}\\ \vdots\end{array}}$$

(3)

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Gamma \vdash M : \alpha, \Delta\end{array}}{\Gamma \vdash aM : \beta, \Delta \cup \{a : \alpha\}}}{\begin{array}{c}\Gamma \vdash \lambda x.aM : \gamma \to \beta, \Delta \cup \{a : \alpha\}\\ \vdots\end{array}} \quad \rhd_1 \quad \dfrac{\begin{array}{c}\vdots\\ \Gamma \vdash M : \alpha, \Delta\end{array}}{\begin{array}{c}\Gamma \vdash aM : \gamma \to \beta, \Delta \cup \{a : \alpha\}\\ \vdots\end{array}}$$

(4)

$$\begin{array}{c}\vdots\\ \dfrac{\dfrac{\Gamma \vdash M : \alpha, \Delta}{\Gamma \vdash aM : \beta, \Delta \cup \{a : \alpha\}}}{\Gamma \vdash b(aM) : \gamma, \Delta \cup \{a : \alpha, b : \beta\}}\\ \vdots\end{array} \quad \rhd_1 \quad \begin{array}{c}\vdots\\ \dfrac{\Gamma \vdash M : \alpha, \Delta}{\Gamma \vdash aM : \gamma, \Delta \cup \{a : \alpha\}}\\ \vdots\end{array}$$

(5)

$$\begin{array}{c}\vdots\\ \dfrac{\Gamma \vdash M : \alpha, \Delta}{\Gamma \vdash \rho a.M : \alpha, \Delta}\\ \vdots\end{array} \quad \rhd_1 \quad \begin{array}{c}\vdots\\ \Gamma \vdash M : \alpha, \Delta\\ \vdots\end{array}$$

(6)

$$\begin{array}{c}\vdots\\ \dfrac{\dfrac{\Gamma \vdash M : \alpha, \Delta}{\Gamma \vdash aM : \alpha, \Delta \cup \{a : \alpha\}}}{\Gamma \vdash \rho a.aM : \alpha, \Delta}\\ \vdots\end{array} \quad \rhd_1 \quad \begin{array}{c}\vdots\\ \Gamma \vdash M : \alpha, \Delta\\ \vdots\end{array}$$

By observing the above figures, we can see that the reduction $\rhd_1$ gives the operation to remove a redundant weakening from proof terms. Then the proof in section 3 tells us that closed intuitionistic $\lambda\rho$-calculus can be transformed into a closed λ-term (i.e. an **NJ**-derivation) by removing all redundant weakening.

Then, in the following argument, we show that **TLJ**-derivation can be transformed into an **LJ**-derivation [4] by removing all redundant weakening rules from the **TLJ**-derivation.

4.2. **Extracting an LJ-derivation from a TLJ-derivation.** First, to simplify the argument, we introduce a new proof system.

Definition 4.1. The tree sequent calculus **TLJ**$'$ is obtained from **TLJ** by:

- removing the rules (Heredity)$_l$, (Heredity)$_r$ and (Grow).
- modifying the axiom scheme as: $\{t : \alpha\} \vdash^{\mathcal{T}} \{s : \alpha\}$ $(t \leq s)$.
- modifying the $(\vdash\rightarrow)$-rule as follows.

$$\frac{\{t : \alpha\} \cup \Gamma \vdash^{\mathcal{T}} \Delta \cup \{t : \beta\}}{\Gamma \vdash^{\mathcal{T}} \Delta \cup \{s : \alpha \to \beta\}} \ (\vdash\rightarrow)' \quad (s <_1 t)$$

Here we impose the following condition (called label condition): no descendants of t occur in the lower sequent.

Then, we can easily show the following lemma.

Lemma 4.2. If $\Gamma \vdash^{\mathcal{T}} \Delta$ is derivable in **TLJ**, then $\Gamma \vdash^{\mathcal{S}} \Delta$ is derivable in **TLJ**$'$ for some $\mathcal{S} \sqsupseteq \mathcal{T}$.

From this lemma, for our purpose, it suffices to show the following theorem.

Theorem 4.3. If $\emptyset \vdash^{\mathcal{T}} \{r : \varphi\}$ is derivable in **TLJ**$'$, then φ is provable in **LJ**.

[4] **LJ** and **NJ** are proof systems for intuitionistic logic originally introduced by Gentzen [2, 3].

We then specify the notion "redundant weakening".

Definition 4.4. Let Σ be a $\mathbf{TLJ}'$-derivation and $\mathcal{W}$ be an occurrence of weakening rule in Σ. We say $\mathcal{W}$ is *necessary* if $\mathcal{W}$ occurs as the lowest inference rule in Σ (i.e. $\mathcal{W}$ derives the endsequent of Σ) or $\mathcal{W}$ satisfies the following conditions.

- A $(\vdash\rightarrow)$-rule $\mathcal{I}$ occurs immediately after $\mathcal{W}$.
- $\mathcal{W}$ adds exactly one side formula of $\mathcal{I}$.

We say $\mathcal{W}$ is *redundant* if it is not necessary. An *essential* $\boldsymbol{TLJ'}$*-derivation* is a $\mathbf{TLJ}'$-derivation in which no redundant weakening rules occur.

Then we start proving the theorem.

Lemma 4.5. A **TLJ**-derivation Σ can be transformed into an essential derivation.

Proof. The theorem is proved by double induction on:

- the number n of the redundant weakening rules occurring in Σ.
- the height h of the lowest occurrence of redundant weakening in Σ.

It is easy to see that Σ can be transformed into another derivation, without changing the endsequent, which we can apply the inductive hypothesis (see [9] in detail). As an example, we treat the following two cases.

(1) Suppose a lowest occurrence $\mathcal{W}$ of redundant weakening has the following form.

$$\cfrac{\cfrac{\begin{array}{c}\vdots\ (A)\\ \Gamma \vdash^{\mathcal{T}} \Delta\end{array}}{\{t:\alpha\}\cup\Gamma'\cup\Gamma \vdash^{\mathcal{T}} \Delta\cup\Delta'\cup\{t:\beta\}}\ \mathcal{W}}{\begin{array}{c}\Gamma'\cup\Gamma \vdash^{\mathcal{T}} \Delta\cup\Delta'\cup\{s:\alpha\rightarrow\beta\}\\ \vdots\ (B)\end{array}}\ (\vdash\rightarrow)'$$

Then we transform this derivation into the following derivation.

$$\cfrac{\begin{array}{c}\vdots\ (A)\\ \Gamma \vdash^{\mathcal{T}} \Delta\end{array}}{\begin{array}{c}\Gamma'\cup\Gamma \vdash^{\mathcal{T}} \Delta\cup\Delta'\cup\{s:\alpha\rightarrow\beta\}\\ \vdots\ (B)\end{array}}\ (\text{Weakening})$$

Obviously, we can apply the inductive hypothesis to the lower derivation.

(2) Suppose a lowest occurrence $\mathcal{W}$ of redundant weakening has the following form.

$$\cfrac{\cfrac{\begin{array}{c}\vdots\ (C)\\ \Gamma_1 \vdash^{\mathcal{T}} \Delta_1\end{array}}{\Gamma_1\cup\Gamma_2 \vdash^{\mathcal{T}} \Delta_1\cup\Delta_2\cup\{t:\alpha\}}\ \mathcal{W} \qquad \begin{array}{c}\vdots\ (D)\\ \{t:\beta\}\cup\Gamma_3 \vdash^{\mathcal{T}} \Delta_3\end{array}}{\begin{array}{c}\{t:\alpha\rightarrow\beta\}\cup\Gamma_1\cup\Gamma_2\cup\Gamma_3 \vdash^{\mathcal{T}} \Delta_1\cup\Delta_2\cup\Delta_3\\ \vdots\ (E)\end{array}}\ (\rightarrow\vdash)$$

Then we transform this derivation into the following derivation.

$$\cfrac{\begin{array}{c}\vdots\ (C)\\ \Gamma_1 \vdash^{\mathcal{T}} \Delta_1\end{array}}{\begin{array}{c}\{t:\alpha\to\beta\}\cup\Gamma_1\cup\Gamma_2\cup\Gamma_3 \vdash^{\mathcal{T}} \Delta_1\cup\Delta_2\cup\Delta_3\\ \vdots\ (E)\end{array}}\ \text{(Weakening)}$$

Obviously, we can apply the inductive hypothesis to the lower derivation. □

Lemma 4.6. Suppose that Σ is an essential **TLJ**$'$-derivation of $\emptyset \vdash^{\mathcal{T}} \{r:\varphi\}$. If $\Gamma \vdash^{\mathcal{T}} \Delta$ occurs in Σ, then Δ is a singleton set.

Proof. Suppose there is a sequent whose right side is not singleton and take an uppermost occurrence S of such sequent. From the form of inference rule, S is derived by a weakening rule as follows.

$$\cfrac{\cfrac{\begin{array}{c}\vdots\ (A)\\ \{t:\alpha\}\cup\Gamma \vdash^{\mathcal{T}} \{u:\gamma\}\end{array}}{\{t:\alpha\}\cup\Gamma \vdash^{\mathcal{T}} \{u:\gamma\}\cup\{t:\beta\}}\ \text{(Weakening)}}{\begin{array}{c}\Gamma \vdash^{\mathcal{T}} \{u:\gamma\}\cup\{s:\alpha\to\beta\}\\ \vdots\end{array}}\ (\vdash\to)$$

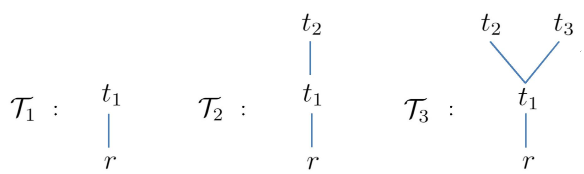
Here, $S = \{t:\alpha\}\cup\Gamma \vdash^{\mathcal{T}} \{u:\gamma\}\cup\{t:\beta\}$. From the condition of S, there are no weakening rules which add a formula in right side of a sequent in the subderivation (A). Then we can show, by checking the form of inference rules, the following fact:

⊙ If the sequent $\{t_1:\delta\}\cup\Pi \vdash^{\mathcal{T}} \{t_2:\varepsilon\}$ occurs in the subderivation (A), then t_2 is a descendant of t_1.

From this observation, we especially obtain the fact that u is a descendant of t. However this violates the label condition of the $(\vdash\to)$-rule. □

Proof of theorem 4.3 (proof of theorem 1.3). Suppose there is a **TLJ**$'$-derivation of $\emptyset \vdash^{\mathcal{T}} \{r:\varphi\}$, then, from lemma 4.5, we can transform this derivation into another derivation Σ which is essential. From lemma 4.6, if $\Gamma \vdash^{\mathcal{T}} \Delta$ occurs in Σ, then Δ is singleton. Hence we obtain an **LJ**-derivation of $\emptyset \vdash \{\varphi\}$ from Σ by replacing each tree sequent $\Gamma \vdash^{\mathcal{T}} \{t:\alpha\}$ by the sequent $\{\beta \mid t:\beta\in\Gamma\} \vdash \{\alpha\}$. □

Example 4.7. Let $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$ be the following tree.



Suppose that the following **TLJ**-derivation Σ is given.

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\{t_2 : p\} \vdash^{\mathcal{T}_3} \{t_2 : p\}}{\{t_1 : p\} \vdash^{\mathcal{T}_3} \{t_2 : p\}}\,(\text{Heredity})_l}{\{t_1 : p, t_3 : q\} \vdash^{\mathcal{T}_3} \{t_2 : p, t_3 : p\}}\,(\text{Weakening})}{\{t_1 : p\} \vdash^{\mathcal{T}_2} \{t_2 : p, t_1 : q \to p\}}\,(\vdash\to)}{\{t_1 : p, t_2 : q\} \vdash^{\mathcal{T}_2} \{t_2 : p, t_1 : q \to p\}}\,(\text{Weakening})}{\{t_1 : p\} \vdash^{\mathcal{T}_1} \{t_1 : q \to p\}}\,(\vdash\to)}{\emptyset \vdash^{\mathcal{R}(r)} \{r : p \to q \to p\}}\,(\vdash\to)$$

Then we extract an **LJ**-derivation from Σ as follows.

$$\Sigma$$

$$\triangledown \text{ Lemma 4.2}$$

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\{t_1 : p\} \vdash^{\mathcal{T}_3} \{t_2 : p\}}{\{t_1 : p, t_3 : q\} \vdash^{\mathcal{T}_3} \{t_2 : p, t_3 : p\}}\,(\text{Weakening})}{\{t_1 : p\} \vdash^{\mathcal{T}_3} \{t_2 : p, t_1 : q \to p\}}\,(\vdash\to)}{\{t_1 : p, t_2 : q\} \vdash^{\mathcal{T}_3} \{t_2 : p, t_1 : q \to p\}}\,(\text{Weakening})}{\{t_1 : p\} \vdash^{\mathcal{T}_3} \{t_1 : q \to p\}}\,(\vdash\to)}{\emptyset \vdash^{\mathcal{T}_3} \{r : p \to q \to p\}}\,(\vdash\to)$$

$$\triangledown \text{ Lemma 4.5}$$

$$\cfrac{\cfrac{\cfrac{\{t_1 : p\} \vdash^{\mathcal{T}_3} \{t_2 : p\}}{\{t_1 : p, t_2 : q\} \vdash^{\mathcal{T}_3} \{t_2 : p\}}\,(\text{Weakening})}{\{t_1 : p\} \vdash^{\mathcal{T}_3} \{t_1 : q \to p\}}\,(\vdash\to)}{\emptyset \vdash^{\mathcal{T}_3} \{r : p \to q \to p\}}\,(\vdash\to)$$

$$\triangledown$$

$$\cfrac{\cfrac{\cfrac{\{p\} \vdash \{p\}}{\{p, q\} \vdash \{p\}}\,(\text{Weakening})}{\{p\} \vdash \{q \to p\}}\,(\vdash\to)}{\emptyset \vdash \{p \to q \to p\}}\,(\vdash\to)$$

4.3. **Advantages of our method.** In [4, 6], the proof of the "if" part of theorem 1.3 is given with a translation, called formulaic translation, from tree sequents into formulas. In comparison with their method, our method has three advantages in comparison with the standard method.

One is that our method can be applied to many other logics. As we showed in the above argument, our method can work well in implicational fragment. On the other hand, the standard method does not work well in disjunction-less fragments or negation-less fragments.

The second advantage is that our method provides a natural transformation from **TLJ**-derivation into **LJ**-derivation in comparison with the standard method. For example, using the formulaic translation, we transform the **TLJ**-derivation Σ given in example 4.7 into the derivation in figure 2. As example 4.7 showed, using our method, we extract the essential part from a given derivation and can give a natural **LJ**-derivation.

The other advantage is that our method give a proof to Gentzen's Hauptsatz [2, 3]. With our method, we can obtain a cut-free **LJ**-derivation. On the other hand, as figure 2 shows, the algorithm with formulaic translation constructs a derivation including a lot of cut-rules.

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\dfrac{\{p\} \vdash \{p\}}{\{\top, p\} \vdash \{p\}}\;(\text{Weakening})
}{\{p\} \vdash \{\top \to p\}}\;(\vdash\to)
}{\emptyset \vdash \{p \to \top \to p\}}\;(\vdash\to)
\qquad
\dfrac{\{p\} \vdash \{p\} \quad \{\top \to p\} \vdash \{\top \to p\}}{\{p \to \top \to p, p\} \vdash \{\top \to p\}}\;(\to\vdash)
}{\{p\} \vdash \{\top \to p\}}\;(\text{Cut})
}{\{p\} \vdash \{(\top \to p) \vee (q \to p)\}}\;(\vdash \vee)
\qquad
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\dfrac{\{\bot\} \vdash \{\bot\}}{\emptyset \vdash \{\top\}}\;(\vdash\to)
\quad \{p\} \vdash \{p\}
}{\{\top \to p\} \vdash \{p\}}\;(\to\vdash)
}{\{q, \top \to p\} \vdash \{p\}}\;(\text{Weakening})
}{\{\top \to p\} \vdash \{p \to p\}}\;(\vdash\to)
\quad \{q \to p\} \vdash \{q \to p\}
}{\{(\top \to p) \vee (q \to p)\} \vdash \{q \to p\}}\;(\vee \vdash)
}{\{(\top \to p) \vee (q \to p)\} \vdash \{(q \to p) \vee (q \to p)\}}\;(\vdash \vee)
}{\{p\} \vdash \{(q \to p) \vee (q \to p)\}}\;(\text{Cut})
\qquad
\dfrac{\{q \to p\} \vdash \{q \to p\} \quad \{q \to p\} \vdash \{q \to p\}}{\{(q \to p) \vee (q \to p)\} \vdash \{q \to p\}}\;(\vee \vdash)
}{
\dfrac{\{p\} \vdash \{q \to p\}}{\emptyset \vdash \{p \to q \to p\}}\;(\vdash\to)
}\;(\text{Cut})
$$

FIGURE 2

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