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Phase transitions in unprovability

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Abstract

Why are some theorems not provable in certain theories of mathematics? Why are most theorems from existing mathematics provable in very weak systems? The area of Concrete Mathematical Independence seeks answers for those questions. One recent tool, developed by Weiermann, for providing more insights into independence is the phase transition. In this note we provide an overview of some interesting results and heuristics.

1 Introduction

Gödel's incompleteness theorems tell us that, as soon as a consistent theory T encompasses some basic arithmetic, there exist statements φ in the language of T which can be neither proven nor disproven in T. Gödel's results brought great interest in finding *natural* examples of such φ in areas of mathematics other than logic for theories T of arithmetic. Since 1977, starting with results by Paris and Harrington [10], a growing number of such φ have been found for Peano Arithmetic, the first order theory based on Peano's axioms from [11]. Peano arithmetic is of particular interest because, before Gödels result, it was one of the candidate theories to answer Hilbert's second problem: to find axioms for mathematics and, most importantly:

> To prove that they are not contradictory, that is, that a finite number of logical steps based upon them can never lead to contradictory results.[1]

[1]English translation by M. W. Newson in [6], page 414

During the study of natural independent statements different variants emerged for, among others, the Paris–Harrington [10] and the Kanamori–McAloon [7] results. One can modify them, inserting functions $f\colon \mathbb{N} \to \mathbb{N}$ as parameter values. The obvious question to ask is to classify the parameter values f according to the independence of the resulting parametrised statement. Andreas Weiermann has started a general programme examining this phenomenon for different statements and theories. The answer to this question pinpoints where the strength of independent statements is located, showing that small modifications can result in huge differences in provability.

In this note we give some examples, with references for the interested reader, of Weiermann-style phase transition results and explore some heuristics for determining the thresholds. This overview is based largely on lectures given by Andreas Weiermann at Ghent University and on results in [12] and [14]. The list of examples is not exhaustive and we have left out the sharpened results.

2 Preliminaries

T is a first order theory which includes $\mathrm{I}\Sigma_1$, for our examples this is PA or a fragment of PA. Parameter values f are functions $\mathbb{N} \to \mathbb{N}$, statements φ_f have the form $\forall x \exists y M_f(x) = y$ where functions M_f are recursive for all parameter values f under consideration.

Transition results have the following shape:

- $T \nvdash \varphi_{f_c}$ for every c, but
- $T \vdash \varphi_f$.

where the f_c's approach f in growth rate.

Definition 1 *For $f\colon \mathbb{N} \to \mathbb{N}$ the inverse is:*

$$f^{-1}(i) = \max\{0\} \cup \{j : f(j) \leq i\}.$$

Definition 2 $2_0(i) = i$, $2_{n+1}(i) = 2^{2_n(i)}$, $\log$ *is the inverse of* $i \mapsto 2^i$, $\log^n$ *is the inverse of* $i \mapsto 2_n(i)$, $\log^*$ *is the inverse of* $i \mapsto 2_i(i)$, $\sqrt[c]{\log^n}$ *is the inverse of* $i \mapsto 2_n(i^c)$ *and* $i \mapsto \frac{i}{c}$ *is the inverse of* $i \mapsto i \cdot c$, *where* $\frac{x}{0} = 1$.

3 Examples

In each case we introduce the parametrised version of the statements, the respective original version are obtained by using $f = \mathrm{id}$.

3.1 Paris–Harrington

This is one of the earliest statements shown to be independent from Peano Arithmetic [10]. The transition results was first proven by Weiermann (see e.g. :[15]).

Definition 3 *$[X]^d$ is the set of d-element subsets of X, $[m, R]^d = [\{m, \ldots, R\}]^d$ and $[R]^d = [0, R]^d$.*

Definition 4 *Given a colouring $C\colon [m, R]^d \to r$, we call a set H homogeneous for C or C-homogeneous if C is constant on $[H]^d$.*

Definition 5 (PH_f) *For every d, r, m there exists an R such that for every colouring $C\colon [m, R]^d \to r$ there exists a C-homogeneous $H \subseteq [m, R]$ of size $f(\min H)$.*

Theorem 1 (Weiermann)

1. $\mathrm{PA} \nvdash \mathrm{PH}_{\log^n}$ *for every n, but*
2. $\mathrm{PA} \vdash \mathrm{PH}_{\log^*}$.

3.2 Kanamori–McAloon

This statement was introduced in [7] as a variant which is more easily demonstrated to be independent of Peano Arithmetic. The transition result by Carlucci, Lee and Weiermann can be found in [1].

Definition 6 (KM_f) *For every d, m, a there exists R such that for every colouring $C\colon [a, R]^d \to \mathbb{N}$ with $C(x) \leq f(\min x)$ there exists $H \subseteq R$ of size m for which for all $x, y \in [H]^d$ with $\min x = \min y$ we have $C(x) = C(y)$.*

Theorem 2 (Carlucci, Lee, Weiermann)

1. $\mathrm{PA} \nvdash \mathrm{KM}_{\log^n}$ *for every n, but*

2. $\mathrm{PA} \vdash \mathrm{KM}_{\log^*}$.

3.3 Adjacent Ramsey

Introduced first by Friedman in a draft on his web-page, with independence of Peano Arithmetic fully shown in [4] . The proof of the transition result can be found in [12].

Definition 7 *For r-tuples a, b:*

$$a \leq b \Leftrightarrow (a)_1 \leq (b)_1 \wedge \cdots \wedge (a)_r \leq (b)_r$$

Definition 8 *A colouring $C\colon \{0, \ldots, R\}^d \to \mathbb{N}^r$ is f-limited if*

$$\max C(x) \leq f(\max x) \text{ for all } x \in \{0, \ldots, R\}^d.$$

Definition 9 (AR_f) *For every d, r there exists R such that for every f-limited colouring $C\colon \{0, \ldots, R\}^d \to \mathbb{N}^r$ there exist $x_1 < \cdots < x_{d+1} \leq R$ with*

$$C(x_1, \ldots, x_d) \leq C(x_2, \ldots, x_{d+1}).$$

Theorem 3 (P.)

1. $\mathrm{PA} \nvdash \mathrm{AR}_{\log^n}$ *for every n, but*
2. $\mathrm{PA} \vdash \mathrm{AR}_{\log^*}$.

3.4 Hydra Battles

Tracing back to Gentzen showing in 1936 that transfinite induction up to ε_0 is not provable in PA formulated with a free predicate variable [5], Kirby and Paris [8] introduced a combinatorial game based on the following statement on ordinals below ε_0:

Definition 10 *A Hydra battle for f is a sequence $h_0, h_1, h_2, \ldots$ of ordinals starting with $h_0 = \omega_k(l)$ such that $h_{i+1} = h_i[f(i) + 1]$ and $h_{i+1} < h_i$ for all $i < R$. Here $\alpha[x]$ denotes the xth element of the canonical fundamental sequence of α as in Definition 2 of* [2] *if α is a limit ordinal. If $\alpha = \beta + 1$ then $\alpha[x] = \beta$. Furthermore* $0[x] = 0$.

Definition 11 (EHD_f) *Every Hydra Battle for f is finite.*

Theorem 4 (Weiermann)

1. $\mathrm{PA} \nvdash \mathrm{EHD}_{\log^n}$ *for every n, but*
2. $\mathrm{PA} \vdash \mathrm{EHD}_{\log^*}$.

3.5 Dickson's Lemma

Dickson's lemma is one of the most rediscovered lemmas, and attributed to L.E. Dickson [3]. Proving its independence from $\mathrm{I}\Sigma_1$ is an exercise for a course in proof theory.

Definition 12 *For d-tuples a take:*

$$\deg(a) = (a)_1 + \cdots + (a)_d$$

Definition 13 (MDL_f) *For every d, l there exists D such that for every sequence $m_0, \ldots, m_D$ of d-tuples such that* $\deg(m_i) \leq l + f(i)$ *there are $i < j \leq R$ with $m_i \leq m_j$.*

Theorem 5 (Weiermann)

1. $\mathrm{I}\Sigma_1 \nvdash \mathrm{MDL}_{\sqrt[c]{}}$ *for every c, but*
2. $\mathrm{I}\Sigma_1 \vdash \mathrm{MDL}_{\log}$.

3.6 Higman's Lemma

Definition 14 *X^* denotes the set of finite strings of elements from X. Given two strings $a, b \in \{0, \ldots, d\}^*$ we define:*

$$a \leq b \Leftrightarrow \exists i_1 < \cdots < i_{|a|} \leq |b|.\forall j \leq |a|.(a)_j = (b)_{i_j}.$$

Definition 15 (MHL_f) *For every l, d there exists H such that for every sequence $s_0, \ldots, s_H$ of strings from $\{0, \ldots, d\}^*$ with $|s_i| \leq l + f(i)$ there exist $i < j \leq H$ with $s_i \leq s_j$.*

Theorem 6 (Weiermann)

1. $\mathrm{I}\Sigma_2 \nvdash \mathrm{MHL}_{\sqrt[c]{\log}}$ *for every c, but*
2. $\mathrm{I}\Sigma_2 \vdash \mathrm{MHL}_{\log^2}$.

3.7 Maclagan's theorem on monomial ideals

This theorem by Maclagan [9] is used in computational algebra. The transition result is shown in [13].

Definition 16 *A polynomial of the form* $m = X_d^{m_d} \dots X_1^{m_1}$ *is a monomial. Given field* F *we call an ideal in the polynomial ring* $F[X_1, \dots X_d]$ *a monomial ideal if it is generated by monomials.*

1. *The degree of a monomial is:*
$$\deg(m) = m_d + \cdots + m_1.$$
2. *The degree of a finite set* G *of monomials is the maximum degree of its elements:*
$$\deg(G) = \max\{\deg(m) : m \in G\}.$$
3. *The degree of a monomial ideal* I *is the smallest degree required to be able to generate it:*
$$\deg(I) = \min\{\deg(G) : I = \langle G \rangle\}.$$

Definition 17 (MM_f) *For every* l, d *there exists* M *such that for every sequence* $I_0, \dots, I_M$ *of monomial ideals in* d *variables with* $\deg(I_i) \leq l + f(i)$ *there exist* $i < j \leq M$ *with* $I_i \supseteq I_j$.

Theorem 7 (P.)

1. $\mathrm{I}\Sigma_2 \nvdash \mathrm{MM}_{\sqrt[c]{\log}}$ *for every c, but*
2. $\mathrm{I}\Sigma_2 \vdash \mathrm{MM}_{\log^2}$.

3.8 Growing trees

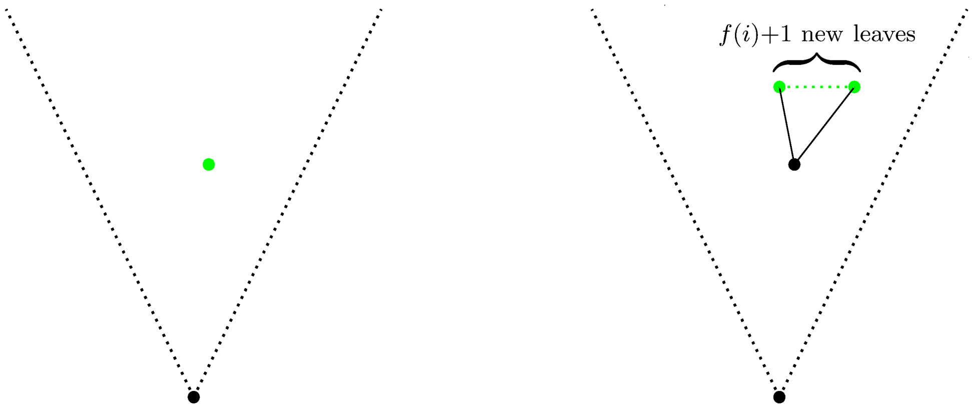
Introduced in [14] as a simple example of a phase transition for independence of $\mathrm{I}\Sigma_1$.

Definition 18 *We call sequence $T_0, \ldots$ of finite trees an f-GT (Growing Tree) if*

1. *$T_0 =$ root and*
2. *For every T_{i+1} from the sequence there exists a leaf $\sigma \in T_i$ such that*

$$T_{i+1} = T_i \cup \{\sigma j : j \leq f(i)\}.$$

We say an f-GT has reached height h if it contains a tree of height at least h.



Tree T_i: Select a leaf   Tree T_{i+1}: Add new leaves

Definition 19 (MKL_f) *For every h there exists a K such that every f-GT of length K has reached height h.*

Theorem 8 (P.)

1. $\mathrm{I}\Sigma_1 \nvdash \mathrm{MKL}_{\sqrt[c]{}}$ *for every c, but*
2. $\mathrm{I}\Sigma_1 \vdash \mathrm{MKL}_{\log}$.

4 Heuristics

Given a parametrised statement of the form $\forall x \exists y M_f(x) = y$ for which M_{id} is not provably total, but $M_{i \mapsto k}$ is. The transition results have the following shape:

- $T \nvdash \varphi_{f_c}$ for every c, but
- $T \vdash \varphi_f$.

To determine a transition threshold we examine the following function:

$$k \mapsto M_{i \mapsto k}(x).$$

Very roughly speaking, functions f_c will be the inverses of lower bounds l and the function f the inverse of an upper bound u. For a proof of the upper bounds lemma, a more detailed examination of the cases for Ramsey-like statements and a method for sharpening these results we refer the reader to [12].

4.1 Provability

For the provability part of the phase transition results, we have the upper bounds lemma, which states the following:

Assuming M_f and $u\colon \mathbb{N} \to \mathbb{N}$ satisfy some technical properties and ordering functions by eventual domination, if:

$$k \mapsto M_{i \mapsto k}(x) \leq u$$

for all x, then:

$$T \vdash \forall x \exists y M_{u^{-1}}(x) = y.$$

Lemma 1 (Upper bounds) *Let* $\mathrm{I}\Sigma_1 \subseteq T$ *and* $M_f(x)$ *be computable for every computable* f. *Suppose* M *has the following properties:*

1. *if* $\forall i \leq M_g(x)$ *we have* $f(i) \leq g(i)$ *then* $M_f(x) \leq M_g(x)$,
2. *there exist* T*-provably recursive, unbounded functions* u, h *such that* $\forall k \geq h(x)$ *we have* $M_k(x) \leq u(k)$,

then:

$$T \vdash \forall x \exists y M_{u^{-1}}(x) = y.$$

The most important technical conditions in the upper bounds lemma are that the upper bound u is provably total in T and that M_f be monotone in the parameter value.

Note that in many cases the u from the lemma can be found in the existing mathematics literature. For example, in the case of the Paris–Harrington theorem, $M_{i \mapsto k}(x)$ can be computed using Ramsey numbers, which are known to be bounded by the tower function when one fixes the dimension.

4.2 Independence

For the unprovability parts there, unfortunately, does not exist a general method. However, in each transition result so far the following has been observed:

Assuming M_f and $l\colon \mathbb{N} \to \mathbb{N}$ satisfy some technical properties, if there exists x such that:

$$l \leq k \mapsto M_{i \mapsto k}(x)$$

then:

$$T \nvdash \forall x \exists y M_{l^{-1}}(x) = y.$$

Conjecture 9 (Lower bounds) *Suppose T is a theory which contains $\mathrm{I}\Sigma_1$, l is unbounded, nondecreasing and M_f is a nondecreasing computable function for every computable f with the following properties:*

1. $T \nvdash \forall x \exists y M_{\mathrm{id}}(x) = y$,
2. $f(i) \leq g(i)$ *for all* $i \leq M_g(x)$ *implies* $M_f(x) \leq M_g(x)$,
3. *there exists* x *such that* l *is eventually strictly dominated by* $k \mapsto M_{i \mapsto k}(x)$,

then:

$$T \nvdash \forall x \exists y M_{l^{-1}}(x) = y.$$

Note again that functions l can often be found in the literature. For example, for the Paris–Harrington theorem, $M_{i \mapsto k}(x)$ can be computed using Ramsey numbers, which are known to have a lower bound $2_n(k)$ for sufficiently high dimension.

We emphasise that this part of the phase transitions often requires the most work as in each case different techniques, sometimes not involving the function $k \mapsto M_{i \mapsto k}(x)$ at all, may be used to prove it. In some cases one may adapt existing independence proofs and in others one uses combinatorics to show $\varphi_{l^{-1}} \to \varphi_{\mathrm{id}}$.

□

References

[1] L. Carlucci, G. Lee, and A. Weiermann, *Sharp thresholds for hypergraph regressive Ramsey numbers*, J. Combin. Theory Ser. A **118** (2011): 558–585.

[2] E. A. Cichon, *A short proof of two recently discovered independence results using recursion theoretic methods*, Proc. Amer. Math. Soc. **87** (1983): 704–706.

[3] L. E. Dickson, *Finiteness of the Odd Perfect and Primitive Abundant Numbers with n Distinct Prime Factors*, Amer. J. Math. **35** (1913): 413–422.

[4] H. M. Friedman and F. Pelupessy, *Independence of Ramsey theorem variants using ε_0*, Accepted for publication in PAMS.

[5] G. Gentzen, *Die Widerspruchsfreiheit der reinen Zahlentheorie*, Math. Ann. **112** (1936): 493–565.

[6] D. Hilbert, *Mathematical problems*, Bull. Amer. Math. Soc. (N.S.) **37**, Reprinted from Bull. Amer. Math. Soc. **8** (1902), 437–479 (2000): 407–436 (electronic).

[7] A. Kanamori and K. McAloon, *On Gödel incompleteness and finite combinatorics*, Ann. Pure Appl. Logic **33** (1987): 23–41.

[8] L. Kirby and J. Paris, *Accessible independence results for Peano arithmetic*, Bull. London Math. Soc. **14** (1982): 285–293.

[9] D. Maclagan, *Antichains of monomial ideals are finite*, Proc. Amer. Math. Soc. **129** (2001): 1609–1615 (electronic).

[10] J. Paris and L. Harrington, *A Mathematical Incompleteness in Peano Arithmetic*, in Handbook for Mathematical Logic (Ed. J. Barwise) Amsterdam, Netherlands: North-Holland, 1977.

[11] G. Peano, *The principles of arithmetic presented by a new method (1889)*, in From Frege to Gödel, edited and translation by van Heijenoort, J. Harvard University Press, Cambridge, Massachusetts (1967).

[12] F. Pelupessy, *Phase transition results for three Ramsey-like theorems*, Accepted for publication in NDJFL.

[13] ———, *Monomial ideals and independence of* $I\Sigma_2$, draft.

[14] ———, *Connecting the Provable with the Unprovable: Phase Transitions in Unprovability*, Thesis.

[15] A. Weiermann and W. Van Hoof, *Sharp phase transition thresholds for the Paris Harrington Ramsey numbers for a fixed dimension*, Proc. Amer. Math. Soc. **140** (2012): 2913–2927.

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