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Many point reflections at infinity of a time changed reflecting diffusion

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1 Introduction

The boundary problem of a Markov process $X$ concerns all possible Markovian prolongations of $X$ beyond its life time $\zeta$ whenever $\zeta$ is finite. Let $Z = (Z_t, Q_z)$ be a conservative right process on a locally compact separable metric space $E$ and $\Delta$ be the point at infinity of $E$. Suppose $Z$ is transient relative to an excessive measure $m$: for the 0-order resolvent $R$ of $Z$, $Rf(z) < \infty$, $m$-a.e. for some strictly positive function (or equivalently, for any non-negative function) $f \in L^1(E;m)$. Then

$$Q_z(\lim_{t\to\infty} Z_t = \Delta) = 1 \quad \text{for q.e. } x \in E,$$

if $Rf$ is lower semicontinuous for any non-negative Borel function $f$ ([FTa]). The last condition is not needed when $X$ is $m$-symmetric ([CF2]).

Take any strictly positive bounded function $f \in L^1(E;m)$. Then $A_t = \int_0^t f(Z_s)ds, \ t \geq 0$ is a strictly increasing PCAF of $Z$ with $E_z[A_x] = Rf(x) < \infty$ for q.e. $x \in E$. The time changed process $X = (X_t, \zeta, P_x)$ of $Z$ by means of $A$ is defined by

$$X_t = Z_{\tau t}, \ t \geq 0, \ \tau = A^{-1}, \ \zeta = A_{\infty}, \ P_x = Q_z, \ x \in E. \quad (1.1)$$

Since $P_z(\zeta < \infty, \lim_{t\to\infty} X_t = \Delta) = P_z(\zeta < \infty) = 1$, the boundary problem for $X$ at $\Delta$ makes perfect sense. For different choices of $f$, the corresponding processes $X$ have the same geometric shapes related each other only by time changes. Thus a study of the boundary problem for $X$ is a good way to make a close look at a geometric picture of a conservative transient process $Z$ around $\Delta$.

When a right process $Z$ is $m$-symmetric, we can work with the associated Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(E;m)$. Let $\mathcal{F}$ and $\mathcal{F}^{ref}$ be its extended Dirichlet space and its reflected Dirichlet space ([CF2]). Then $\mathcal{F} \subset \mathcal{F}_c \subset \mathcal{F}^{ref}$ and the inner product $\mathcal{E}$ is extended from $\mathcal{F}$ to both spaces. Define the subspace $\mathcal{H}^*$ of $\mathcal{F}^{ref}$ by

$$\mathcal{H}^* = \{u \in \mathcal{F}^{ref} : \mathcal{E}(u, v) = 0 \ \text{for any} \ v \in \mathcal{F}_c\}. \quad (1.2)$$

The stated boundary problem for $Z$ is closely related to dim($\mathcal{H}^*$). The process $Z$ or the associated Dirichlet form $(\mathcal{E}, \mathcal{F})$ is said to satisfy a Liouville property if dim($\mathcal{H}^*$) = 1. We will be concerned with the cases where $Z$ are the reflecting Brownian motion on an unbounded domain of $\mathbb{R}^n$ and the distorted Brownian motion on the whole space $\mathbb{R}^n$.

We first consider the reflecting Brownian motion (RBM) $Z$ on the closure $\overline{D}$ of a Lipschitz domain $D \subset \mathbb{R}^n$ that is a special case of the reflecting diffusion process constructed in [FTo]. $Z$ is always conservative. $Z$ is symmetric with respect to the Lebesgue measure on $D$ and the Dirichlet form $\mathcal{E}$ of $Z$ on $L^2(D)$ is given by

$$\mathcal{E} = \frac{1}{2} D, \quad D(\mathcal{E}) = H^1(D) = BL(D) \cap L^2(D),$$
where
\[ D(u, v) = \int_D \nabla u(x) \cdot \nabla v(x) dx, \quad \text{BL}(D) = \{ u \in L^2_{\text{loc}}(D) : |\nabla u| \in L^2(D) \}. \]

BL(D) is the reflected Dirichlet space of Z.

We requires that
(A.1) Z is transient,
and accordingly it must be that \( n \geq 3 \) and \( D \) is unbounded. When \( d \geq 3 \), an infinite cone \( D \) satisfies (A.1) but an infinite cylinder does not. Under (A.1), the extended Sobolev space \( H^1_e(D) \) is a Hilbert space with inner product \( \frac{1}{2} D(u, v) \) so that it does not contain any non-zero constant, while \( \text{BL}(D) \) does. Hence \( H^1_e(D) \) is a proper subspace of \( \text{BL}(D) \) and the space \( \mathcal{H}^*(D) \) defined by
\[
\mathcal{H}^*(D) = \{ u \in \text{BL}(D) : D(u, v) = 0 \text{ for every } v \in H^1_e(D) \},
\]
is a non-trivial family of harmonic functions on \( D \).

In what follows, we assume that \( n \geq 3 \). A domain \( D \subset \mathbb{R}^d \) is called a uniform domain if there exists \( C > 0 \) such that, for every \( x, y \in D \), there is a rectifiable curve \( \gamma \) in \( D \) connecting \( x \) and \( y \) with length(\( \gamma \)) \( \leq C|x - y| \), and moreover
\[
\min \{|x - z|, |z - y|\} \leq C \text{dist}(z, D^c) \quad \text{for every } z \in \gamma.
\]

A typical example of a unbounded uniform domain is an infinite cone.

According to [CF1],
- a domain \( D \) containing a unbounded uniform domain satisfies (A.1).
- \( Z \) satisfies the Liouville property \( \dim(\mathcal{H}^*(D)) = 1 \) whenever \( D \setminus \overline{B_r(0)} \) is a unbounded uniform domain, for some \( r > 0 \).

The proof used the two facts that
- for an unbounded uniform domain \( D \), any \( u \in \text{BL}(D) \) admits a bounded linear extension to \( \text{BL}(\mathbb{R}^d) \) ([HK]).
- any harmonic function on \( \mathbb{R}^d \) with finite Dirichlet integral is constant, namely, the RBM on \( \mathbb{R}^n \) satisfies the Liouville property \( \dim(\mathcal{H}^*(\mathbb{R}^n)) = 1 \) ([B]).

On the other hand, \( \dim(\mathcal{H}^*(D)) = 2 \) for a domain with two symmetric cone branches ([CF2]):
\[
D = B_1(0) \cup \left\{ x \in \mathbb{R}^n : x_n^2 > \left( \sum_{k=1}^{n-1} x_k^2 \right)^{1/2} \right\}, \quad n \geq 3.
\]

This domain is not uniform because of the presence of a bottleneck.

2 RBM on a domain with \( N \) unbounded uniform branches

In this section, we consider a Lipschitz domain \( D \) of \( \mathbb{R}^n \), \( n \geq 3 \), such that
(A.2) \( D \setminus \overline{B_r(0)} = \bigcup_{j=1}^N C_j \)
for some \( r > 0 \) and an integer \( N \), where \( C_1, \ldots, C_N \) are unbounded uniform domains whose closures are mutually disjoint.

Obviously \( D \) has the property (A.1).

Let \( \partial_j \) be the point at infinity of the unbounded closed set \( \overline{C}_j \) for each \( 1 \leq j \leq N \). Denote the \( N \)-points set \( \{\partial_1, \ldots, \partial_N\} \) by \( F \) and put \( \overline{D}^* = \overline{D} \cup F \). \( \overline{D}^* \) can be made to be a compact Hausdorff space if we employ as a local base of neighborhoods of each point \( \partial_j \in F \) the neighborhoods of \( \partial_j \in \overline{C}_j \cup \{\partial_j\} \). \( \overline{D}^* \) may be called the \( N \)-points compactification of \( \overline{D} \).

For the RBM \( Z = (Z_t, Q_x) \) on \( \overline{D} \), define the approaching probabilities \( \varphi_j(x) \) by

\[
\varphi_j(x) = Q_x \left( \lim_{t \to \infty} Z_t = \partial_j \right), \quad x \in \overline{D}, \quad 1 \leq j \leq N.
\]

**Theorem 2.1.** It holds that

\[
\begin{cases}
\sum_{j=1}^{N} \varphi_j(x) = 1, & \varphi_j(x) > 0, \quad 1 \leq j \leq N, \quad \text{for every } x \in \overline{D}, \\
\dim(H^*(D)) = N, & H^*(D) = \{\sum_{j=1}^{N} c_j \varphi_j : c_j \in \mathbb{R}\}.
\end{cases}
\]

We fix a strictly positive \( f \in L^1(D) \) and let \( X = (X_t, \zeta, P_x) \) be the time changed process of \( Z \) by the PCAF \( A_t = \int_0^t f(Z_s)ds \). \( X \) is then symmetric with respect to \( m(dx) = f(x)dx \) and its Dirichlet form \( (\mathcal{E}^X, \mathcal{F}^X) \) on \( L^2(D; m) \) is given by \( \mathcal{E}^X = \frac{1}{2}D, \quad \mathcal{F}^X = H^2_0(D) \cap L^2(D; m) \).

The reflected Dirichlet space of \( X \) is still BL(D). \( \varphi_j(x) \) can be rewritten as

\[
\varphi_j(x) = P_x(\zeta < \infty, X_{\zeta-} = \partial_j), \quad x \in \overline{D}, \quad 1 \leq j \leq N.
\]

A map \( \Pi \) from the boundary set \( F = \{\partial_1, \ldots, \partial_N\} \) onto a finite set \( \hat{F} = \{\hat{\partial}_1, \ldots, \hat{\partial}_\ell\} \) with \( \ell \leq N \) is called a partition of \( F \). We let \( \overline{D}^{\Pi,*} = \overline{D} \cup \hat{F} \). We extend the map \( \Pi \) from \( F \) to \( \overline{D}^* \) by setting \( \Pi x = \hat{\partial}_i, \quad x \in \overline{D} \), and introduce the quotient topology on \( \overline{D}^{\Pi,*} \) by \( \Pi \), in other words,

\[
U_{\Pi} = \{U \subset \overline{D}^{\Pi,*} : \Pi^{-1}(U) \text{ is an open subset of } \overline{D}^* \}
\]

is taken to be the family of open subsets of \( \overline{D}^{\Pi,*} \).

\( \overline{D}^{\Pi,*} \) is a compact Hausdorff space and may be called an \( \ell \)-points compactification of \( \overline{D} \) obtained from \( \overline{D}^* \) by identifying the points in the set \( \Pi^{-1}\hat{\partial}_i \subset F \) as a single point \( \hat{\partial}_i \) for each \( 1 \leq i \leq \ell \).

Given a partition \( \Pi \) of \( F \), the approaching probabilities \( \hat{\varphi}_i \) of the time changed RBM \( X = (X_t, \zeta, P_x) \) to \( \hat{\partial}_i \in \hat{F} \) are defined by

\[
\hat{\varphi}_i(x) = \sum_{j \in \Pi^{-1}\hat{\partial}_i} \varphi_j(x), \quad x \in \overline{D}, \quad 1 \leq i \leq \ell.
\]

The measure \( m(dx) = f(x)dx \) is extended from \( \overline{D} \) to \( \overline{D}^{\Pi,*} \) by setting \( m(\hat{F}) = 0 \).

- \( \hat{\varphi}_i \) is strictly positive on \( \overline{D} \) for every \( 1 \leq i \leq N \),
- \( m \) is a finite measure on \( \overline{D} \),
- \( G^X g = G^Z(fg) \) is lower semicontinuous for the \( 0 \)-order resolvent \( G^X \) (resp. \( G^Z \)) of \( X \) (resp. \( Z \)) and any non-negative Borel function \( g \) on \( \overline{D} \).

Thus all requirements for the unique existence of \( \ell \)-point extension of \( X \) from \( \overline{D} \) to \( \overline{D}^{\Pi,*} \) in Section 7.7 of [CF2] are fulfilled.
Theorem 2.2. There exists a unique $m$-symmetric recurrent diffusion extension $X^{\Pi,*}$ of $X$ from $\overline{D}$ to $\overline{D}^{\Pi,*}$. The Dirichlet form $(\mathcal{E}^{\Pi,*}, \mathcal{F}^{\Pi,*})$ of $X^{\Pi,*}$ on $L^2(\overline{D}^{\Pi,*};m)$ ($= L^2(D;m)$) admits the extended Dirichlet space expressed as
\[
\begin{aligned}
&\mathcal{F}^{\Pi,*} = H_e^1(D) \oplus \{ \sum_{i=1}^{\ell} c_i \varphi_i : c_i \in \mathbb{R} \} \subset \text{BL}(D), \\
&\mathcal{E}^{\Pi,*}(u,v) = \frac{1}{2} D(u,v), \quad u,v \in \mathcal{F}^{\Pi,*}.
\end{aligned}
\]

Actually the family $\{X^{\Pi,*} : \Pi \text{ is a partition of } F\}$ exhausts all possible $m$-symmetric conservative diffusion extensions of the time changed RBM $X$ on $\overline{D}$ as will be formulated below. Let $E$ be a Lusin space into which $\overline{D}$ is homeomorphically embedded as an open subset. The measure $m(dx) = f(x)dx$ on $\overline{D}$ is extended to $E$ by setting $m(E \setminus \overline{D}) = 0$. Let $Y = (Y_t, \mathcal{F}_t^Y)$ be an $m$-symmetric conservative diffusion process on $E$ whose state process on $\overline{D}$ is identical in law with $X$. The following theorem extends Theorem 3.4 in [CF1] (the case that $N = 1$).

Theorem 2.3. There exists a partition $\Pi$ of $F$ such that $E$ is quasi-homeomorphic with $\overline{D}^{\Pi,*}$ and $Y$ is a quasi-homeomorphic image of $X^{\Pi,*}$.

Outline of a proof of Theorem 2.3

Let $\mathcal{E}^Y$ be the Dirichlet form of $Y$ on $L^2(E;m)$. Since $\mathcal{E}^Y$ is quasi-regular, we can use a quasi homeomorphism to assume

- $E$ is a locally compact separable metric space,
- $\mathcal{E}^Y$ is a regular Dirichlet form on $L^2(E;m),$
- $Y$ is an associated Hunt process on $E$,
- $\tilde{F} := E \setminus \overline{D}$ is quasi-closed.

As $Y$ is a conservative extension of the non-conservative process $X$, $\tilde{F}$ is not $\mathcal{E}^Y$-polar. Every function in $\mathcal{F}^Y_e$ will be taken to be $\mathcal{E}^Y$-quasi continuous. By Theorem 7.1.6 of [CF2], one can conclude that
\[
\begin{aligned}
&\mathcal{F}^Y_e \subset \text{BL}(D), \quad \mathcal{H}^Y := \{ Hu : u \in \mathcal{F}^Y_e \} \subset \mathcal{H}^*, \\
&\mathcal{E}^Y(u,v) = \frac{1}{2} D(u,u) + \frac{1}{2} \mu^c_{\langle Hu \rangle}((\tilde{F})) , \quad u,v \in \mathcal{F}^Y_e,
\end{aligned}
\]
where $Hu(x) = E^Y_x[u(Y_{\tau_{\tilde{F}}})], \quad x \in E$. We show that
\[
\mu^c_{\langle u \rangle}(\tilde{F}) = 0 \quad u \in \mathcal{H}^Y .
\]

Take any $u \in \mathcal{H}^Y$. Theorem 2.1 and the above inclusion imply that $u = \sum_{j=1}^{N} c_j \varphi_j$ for some constants $c_j$. As $u$ is continuous along the sample path of $Y$, $u$ takes only the values $\{c_1, \cdots, c_N\}$ on the boundary $\tilde{F}$ $\nu$-almost everywhere where
\[
\nu(B) = \int_{B} \mathcal{P}^Y_x (Y_{\sigma_{\tilde{F}}} \in B, \sigma_{\tilde{F}} < \infty) m(dx), \quad B \in \mathcal{B}(E).
\]

Since $\tilde{F}$ is a quasi-support of $\nu$, $u$ takes only the values $\{c_1, \cdots, c_N\}$ quasi-everywhere on $\tilde{F}$. (2.1) then follows from the image measure density property of $\mu^c_{\langle u \rangle}$ due to Bouleau-Hirsch.

Define a partition $\Pi$ of $F$ by means of the values taken by functions in $\mathcal{H}^Y$ along the path of $X$ to obtain
\[
(\mathcal{F}^Y_e, \mathcal{E}^Y) = (\mathcal{F}^{\Pi,*}_e, \mathcal{E}^{\Pi,*}).
\]

Both being quasi-regular, they are related by a quasi-homeomorphism of their underlying spaces.
Remark 2.4. Given measurable functions $a_{ij}(x)$, $1 \leq i, j \leq n$, on $D$ such that
\[ a_{ij}(x) = a_{ji}(x), \quad A^{-1} |\xi|^2 \leq \sum_{1 \leq i,j \leq n} a_{ij}(x) \xi_i \xi_j \leq A |\xi|^2, \quad x \in D, \xi \in \mathbb{R}^n, \]
for some constant $A \geq 1$, we define a Dirichlet form $(\mathcal{A}, H^1(D))$ on $L^2(D)$ by
\[ \mathcal{A}(u, v) = \int_D \sum_{i,j=1}^n a_{ij}(x) \frac{\partial u}{\partial x_i}(x) \frac{\partial v}{\partial x_j}(x) dx, \quad u, v \in H^1(D). \]
If we replace the Dirichlet form $(\frac{1}{2}D, H^1(D))$ on $L^2(D)$ and the associated RBM $Z$ on $\overline{D}$, respectively, by $(\mathcal{A}, H^1(D))$ and the associated reflecting diffusion process on $\overline{D}$ constructed in [FTo], all assertions stated above remain valid with no essential change.

By this replacement, the extended Dirichlet space and the reflected Dirichlet space are still $H^1_0(D)$ and $\text{BL}(D)$, respectively, although the inner product $\frac{1}{2}D$ is replaced by $\mathcal{A}$. It suffices to notice that any function in $\text{BL}(\mathbb{R}^n)$ is a sum of a function in $H^1_0(\mathbb{R}^n)$ and a constant $c$ and $\mathcal{A}(c, c) = 0$.

3 Liouville property of energy forms on $\mathbb{R}^n$

In this section, we consider a positive Borel function $\rho$ on $\mathbb{R}^n$ that is locally bounded above and locally uniformly bounded away from 0, and an associated form
\[ \mathcal{E}^\rho(u, v) = \int_{\mathbb{R}^n} \nabla u(x) \cdot \nabla v(x) \rho(x) dx. \quad (3.1) \]
$(\mathcal{E}^\rho, C^1_b(\mathbb{R}^n))$ is closable on $L^2(\mathbb{R}^n) = L^2(\mathbb{R}^n, dx)$ and the closure $(\mathcal{E}^\rho, \mathcal{F}^\rho)$ (called an energy form) is a strongly local regular Dirichlet form on $L^2(\mathbb{R}^n)$. It is irreducible ([FOT, Theorem 4.6.4]). In general, an irreducible recurrent Dirichlet form enjoys the Liouville property in view of [CF2, Lemma 6.7.3]. It therefore suffices to consider only the transient case in order to study the Liouville property of $\mathcal{E}^\rho$. We shall examine this property when $\rho(x)$ is a positive smooth function depending only on the radial part of the variable $x \in \mathbb{R}^n$.

Theorem 3.1. For any positive smooth function $\eta$ on $[0, \infty)$, let $\rho(x) = \eta(|x|)$, $x \in \mathbb{R}^n$. Then $\mathcal{E}^\rho$ satisfies the Liouville property when $n \geq 2$.
When $n = 1$, $\mathcal{E}^\rho$ satisfies the Liouville property in recurrent case but $\text{dim}(\mathcal{H}^*) = 2$ in transient case.

Proof. According to Theorem 1.6.7 in the first edition of [FOT], $\mathcal{E}^\rho$ is transient if and only if
\[ (T) \quad \int_1^\infty \frac{1}{\eta(r)r^{n-1}} dr < \infty. \]
In what follows, we assume that $\eta$ satisfies condition (T).

It then follows from $1/r = (r^{n-3} \eta(r))^{1/2}(\eta(r)r^{n-1})^{-1/2}$ and the Schwarz inequality that
\[ \int_1^\infty r^{n-3} \eta(r) dr = \infty. \quad (3.2) \]
We use the polar coordinate

\[
\begin{align*}
x_1 &= r \cos \theta_1 \\
x_2 &= r \sin \theta_1 \cos \theta_2 \\
x_3 &= r \sin \theta_1 \sin \theta_2 \cos \theta_3 \\
&\quad \quad \quad \quad \quad \vdots \\
x_{n-1} &= r \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{n-2} \cos \theta_{n-1} \\
x_n &= r \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{n-2} \sin \theta_{n-1}.
\end{align*}
\]

Then, for \(u, v \in C_0^1(\mathbb{R}^n)\),

\[
\mathcal{E}^\rho(u, v) = \int_{[0, \infty) \times [0, \pi]^{n-2} \times [0, 2\pi]} \left[ u_r v_r + \frac{u_{\theta_1} v_{\theta_1}}{r^2} + \frac{u_{\theta_2} v_{\theta_2}}{r^2 \sin^2 \theta_1} + \cdots + \frac{u_{\theta_{n-1}} v_{\theta_{n-1}}}{r^2 \sin^2 \theta_1 \cdots \sin^2 \theta_{n-2}} \right] \times \eta(r) r^{n-1} \sin^{n-2} \theta_1 \cdots \sin \theta_{n-2} dr d\theta_1 \cdots d\theta_{n-1}.
\]

For a smooth function \(u\) on \(\mathbb{R}^n\), we denote by \(\mathcal{E}^\eta(u, u)\) the value of the integral of the right hand side of (3.3) for \(v = u\).

As in the case that \(\rho = 1\), the reflected Dirichlet space of \(\mathcal{E}^\rho\) is given by

\[
\mathcal{F}^{\rho, \text{ref}} = \{u \in L^2_{\text{loc}}(\mathbb{R}^n) : \int_{\mathbb{R}^n} |\nabla u(x)|^2 \eta(|x|) dx < \infty\}.
\]

Since \(\mathcal{H}^* = \{u \in \mathcal{F}^{\rho, \text{ref}} : \mathcal{E}^\rho(u, v) = 0 \text{ for every } v \in C_0^\infty(\mathbb{R}^n)\}\), it follows from (3.3) that \(u \in \mathcal{H}^*\) if and only if

\[
\begin{align*}
u \text{ is smooth, } & \mathcal{E}^\eta(u, u) < \infty \quad \text{and} \quad Lu(x) = 0, \ x \in \mathbb{R}^n, \quad (3.4)
\end{align*}
\]

where

\[
Lu(r, \theta_1, \cdots, \theta_{n-1}) = \frac{1}{r^{n-1}} (u_r \cdot \eta(r) r^{n-1})_r + \frac{\eta(r)}{r^2 \sin^{n-2} \theta_1} (u_{\theta_1} \sin^{n-2} \theta_2)_{\theta_1} + \frac{\eta(r)}{r^2 \sin^2 \theta_1 \sin^{n-3} \theta_2} (u_{\theta_2} \sin^{n-3} \theta_2)_{\theta_2} + \cdots + \frac{\eta(r)}{r^2 \sin^2 \theta_1 \cdots \sin^2 \theta_{n-3} \sin \theta_{n-2}} (u_{\theta_{n-2}} \sin \theta_{n-2})_{\theta_{n-2}} + \frac{\eta(r)}{r^2 \sin^2 \theta_1 \cdots \sin^2 \theta_{n-2}} (u_{\theta_{n-1}})_{\theta_{n-1}} \quad (3.5)
\]

Now take any function \(u \in \mathcal{H}^*\). We claim that

\[
u_{\theta_{n-1}} = 0. \quad (3.6)
\]

Put \(w = u_{\theta_{n-1}}\). Due to the expression (3.5) of \(L\), \(Lw = (Lu)_{\theta_{n-1}} = 0\), namely, \(w\) is \(L\)-harmonic. For \(B_r = \{x \in \mathbb{R}^n ; |x| < r\}\) and the uniform probability measure \(\Pi(d\xi)\) on \(\partial B_1\), \(w\) therefore admits the Poisson integral formula

\[
w(x) = \int_{\partial B_1} K_r(x, r\xi) w(r\xi) \Pi(d\xi). \quad x \in B_r, \quad (3.7)
\]

where \(K_r(x, r\xi)\) is the Poisson kernel for \(B_r\) with respect to \(L\), which is known to be continuous in \((x, \xi) \in B_r \times \partial B_1\). We also note that \(K_r(0, r\xi) = 1\) for any \(\xi \in \partial B_1\) by the rotation invariance of \(L\) around the origin 0.
Fix $a > 0$. It then holds for any $r > a$ that

$$K_r(x, r\xi_2) = \int_{\partial B_a} K_a(x, a\xi_1)K_r(a\xi_1, r\xi_2)\Pi(d\xi_1), \quad x \in B_a, \; \xi_2 \in \partial B_1.$$ 

Hence, if we let $\sup_{x \in B_{a/2}, \xi_1 \in \partial B_1} K_a(x, a\xi_1) = C_a < \infty$, then, for $x \in B_{a/2}, \; \xi_2 \in \partial B_1$,

$$K_r(x, r\xi_2) \leq C_a \int_{\partial B_1} K_r(a\xi_1, r\xi_2)\Pi(d\xi_1) = C_aK_r(0, r\xi_2) = C_a,$$

and it follows from (3.7) that

$$|w(x)| \leq C_a \int_{\partial B_1} |w(r\xi)|\Pi(d\xi), \quad x \in B_{a/2}, \; r > a.$$

Recall that $w = u_{\theta_{n-1}}$. We multiply the both hand side of the above inequality by $r^{n-3}\eta(r)$, integrate in $r$ from $a$ to $R$, apply the Schwarz inequality and finally use the expression (3.3) to get

$$|u_{\theta_{n-1}}(x)| \leq \frac{C_a}{\sqrt{\sigma_n}} \left[ \int_a^R r^{n-3}\eta(r)dr \right]^{-1/2} \cdot \sqrt{\mathcal{E}^\eta(u, u)}, \quad x \in B_{a/2},$$

which tends to $0$ as $R \to \infty$ by (3.2). Since $a > 0$ is arbitrary, we arrive at (3.6).

It also holds that

$$u_{\theta_k} = 0 \quad \text{for any} \quad 1 \leq k \leq n - 1. \quad (3.8)$$

In fact, if we let $\xi_i = \frac{\xi_i}{r}, \; 1 \leq i \leq n, \; \xi = (\xi_1, \cdots, \xi_n) \in \partial B_1$, then $\theta_k, \; 1 \leq k \leq n - 1$, is an angle of two $n$-vectors $\xi^{(k)} = (0, \cdots 0, \xi_k, \cdots, \xi_n), \; \xi^0 = (0, \cdots 0, 1, 0, \cdots, 0)$. Consider the subspace $V$ of $\mathbb{R}^n$ spanned by $\xi^k$ and $e_k$ and take a unit vector $\hat{e}$ in $V$ orthogonal to $e_k$. Let $O$ be an orthogonal matrix whose $(n-1)$-th and $n$-th column vectors are $e_k$ and $\hat{e}$, respectively. We make the orthogonal transformation $y = \check{O}x$. Then $\theta_k$ equals an angle of two vectors on the $(y_{n-1}, y_n)$-plane in the new coordinate system $y$ and (3.6) applies.

Thus $u$ depends only on $r$ and, in terms of a scale function $ds(r) = \frac{dr}{\eta(r)r^{n-1}}$ on $(0, \infty)$, (3.3) and (3.6) are reduced, respectively, to

$$\mathcal{E}^\eta(u, u) = \sigma_n \int_0^\infty \left( \frac{du(r)}{ds(r)} \right)^2 ds(r), \quad Lu(r) = \frac{1}{r^{n-1}} \frac{d}{dr} \cdot \frac{du(r)}{ds(r)}.$$ 

By (3.4), $Lu = 0$ so that $u(r) = C_1 + C_2 s(r), \; r > 0$, for some constant $C_1, \; C_2$. Since $\mathcal{E}^\eta(s, s) = \sigma_n \cdot s(0, \infty)$ is finite if and only if $n = 1$, we get the desired expressions from (3.4). $\square$

It is conjectured that the energy form $\mathcal{E}^\rho$ satisfies the Liouville property for any $\rho$ prescribed in the above of (3.1) when $n \geq 2$.

The diffusion process $Z$ on $\mathbb{R}^n$ associated with $\mathcal{E}^\rho$ is called the distorted Brownian motion. Let $X$ be its time changed process defined as (1.1) by means of $m(dx) = f(x)dx$ for a strictly positive bounded function $f \in L^1(\mathbb{R}^n)$. Let $\mathbb{R}^n \cup \{\Delta\}$ be the one point compactification of $\mathbb{R}^n$. If $\mathcal{E}^\rho$ satisfies the Liouville property, then it can be shown as [CF1, Theorem 3.4] that any $m$-symmetric proper diffusion extension of $X$ shares the same finite dimensional distribution with the one-point reflection of $X$ at $\Delta$. See [F2] for more details on these points.
References


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