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京都大学
Many point reflections at infinity
of a time changed reflecting diffusion

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1 Introduction

The boundary problem of a Markov process $X$ concerns all possible Markovian prolongations of $X$ beyond its life time $\zeta$ whenever $\zeta$ is finite. Let $Z = (Z_t, Q_z)$ be a conservative right process on a locally compact separable metric space $E$ and $\Delta$ be the point at infinity of $E$. Suppose $Z$ is transient relative to an excessive measure $m$: for the 0-order resolvent $R$ of $Z$, $Rf(z) < \infty$, $m$-a.e. for some strictly positive function (or equivalently, for any non-negative function) $f \in L^1(E; m)$. Then

$$Q_z(\lim_{t \to \infty} Z_t = \Delta) = 1 \quad \text{for q.e. } x \in E,$$

if $Rf$ is lower semicontinuous for any non-negative Borel function $f$ ([FTa]). The last condition is not needed when $X$ is $m$-symmetric ([CF2]).

Take any strictly positive bounded function $f \in L^1(E; m)$. Then $A_t = \int_0^t f(Z_s)ds$, $t \geq 0$ is a strictly increasing PCAF of $Z$ with $E^Q_z[A_{\infty}] = Rf(x) < \infty$ for q.e. $x \in E$. The time changed process $X = (X_t, \zeta, P_x)$ of $Z$ by means of $A$ is defined by

$$X_t = Z_{\tau_t}, \quad t \geq 0, \quad \tau = A^{-1}, \quad \zeta = A_{\infty}, \quad P_x = Q_z, \quad x \in E. \quad (1.1)$$

Since $P_x(\zeta < \infty, \lim_{t \to \infty} X_t = \Delta) = P_x(\zeta < \infty) = 1$, the boundary problem for $X$ at $\Delta$ makes perfect sense. For different choices of $f$, the corresponding processes $X$ have the same geometric shapes related each other only by time changes. Thus a study of the boundary problem for $X$ is a good way to make a close look at a geometric picture of a conservative transient process $Z$ around $\Delta$.

When a right process $Z$ is $m$-symmetric, we can work with the associated Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(E; m)$. Let $\mathcal{F}_e$ and $\mathcal{F}^{ref}$ be its extended Dirichlet space and its reflected Dirichlet space ([CF2]). Then $\mathcal{F} \subset \mathcal{F}_e \subset \mathcal{F}^{ref}$ and the inner product $\mathcal{E}$ is extended from $\mathcal{F}$ to both spaces. Define the subspace $\mathcal{H}^*$ of $\mathcal{F}^{ref}$ by

$$\mathcal{H}^* = \{u \in \mathcal{F}^{ref} : \mathcal{E}(u, v) = 0 \quad \text{for any} \quad v \in \mathcal{F}_e\}. \quad (1.2)$$

The stated boundary problem for $Z$ is closely related to dim$(\mathcal{H}^*)$. The process $Z$ or the associated Dirichlet form $(\mathcal{E}, \mathcal{F})$ is said to satisfy a Liouville property if dim$(\mathcal{H}^*) = 1$. We will be concerned with the cases where $Z$ are the reflecting Brownian motion on an unbounded domain of $\mathbb{R}^n$ and the distorted Brownian motion on the whole space $\mathbb{R}^n$.

We first consider the reflecting Brownian motion (RBM) $Z$ on the closure $\overline{D}$ of a Lipschitz domain $D \subset \mathbb{R}^n$ that is a special case of the reflecting diffusion process constructed in [FTo]. $Z$ is always conservative. $Z$ is symmetric with respect to the Lebesgue measure on $D$ and the Dirichlet form $\mathcal{E}$ of $Z$ on $L^2(D)$ is given by

$$\mathcal{E} = \frac{1}{2} \mathcal{D}, \quad \mathcal{D}(\mathcal{E}) = H^1(D) = BL(D) \cap L^2(D),$$
where
\[
\mathbf{D}(u, v) = \int_{D} \nabla u(x) \cdot \nabla v(x) dx, \quad \text{BL}(D) = \{u \in L^{2}_{\text{loc}}(D) : |\nabla u| \in L^{2}(D)\}.
\]
BL(D) is the reflected Dirichlet space of Z.

We require that
(A.1) \ Z is transient,
and accordingly it must be that \( n \geq 3 \) and \( D \) is unbounded. When \( d \geq 3 \), an infinite cone \( D \) satisfies (A.1) but an infinite cylinder does not. Under (A.1), the extended Sobolev space \( H^{1}_{e}(D) \) is a Hilbert space with inner product \( \frac{1}{2} \mathbf{D} \) so that it does not contain any non-zero constant, while BL(D) does. Hence \( H^{1}_{e}(D) \) is a proper subspace of BL(D) and the space \( \mathcal{H}^{*}(D) \) defined by
\[
\mathcal{H}^{*}(D) = \{u \in \text{BL}(D) : \mathbf{D}(u, v) = 0 \text{ for every } v \in H^{1}_{e}(D)\},
\]
is a non-trivial family of harmonic functions on \( D \).

In what follows, we assume that \( n \geq 3 \). A domain \( D \subset \mathbb{R}^{d} \) is called a uniform domain if there exists \( C > 0 \) such that, for every \( x, y \in D \), there is a rectifiable curve \( \gamma \) in \( D \) connecting \( x \) and \( y \) with length(\( \gamma \)) \( \leq C|x - y| \), and moreover
\[
\min\{|x - z|, |z - y|\} \leq C \text{dist}(z, D^{c}) \quad \text{for every } z \in \gamma.
\]
A typical example of an unbounded uniform domain is an infinite cone.

According to [CF1],

- a domain \( D \) containing an unbounded uniform domain satisfies (A.1).  
- \( Z \) satisfies the Liouville property \( \dim(\mathcal{H}^{*}(D)) = 1 \) whenever \( D \setminus \overline{B_{r}(0)} \) is a unbounded uniform domain, for some \( r > 0 \).

The proof used the two facts that

- for an unbounded uniform domain \( D \), any \( u \in \text{BL}(D) \) admits a bounded linear extension to BL(\( \mathbb{R}^{d} \)) ([HK]).
- any harmonic function on \( \mathbb{R}^{d} \) with finite Dirichlet integral is constant, namely, the RBM on \( \mathbb{R}^{n} \) satisfies the Liouville property \( \dim(\mathcal{H}^{*}(\mathbb{R}^{n})) = 1 \) ([B]).

On the other hand, \( \dim(\mathcal{H}^{*}(D)) = 2 \) for a domain with two symmetric cone branches ([CF2]):
\[
D = B_{1}(0) \cup \left\{ x \in \mathbb{R}^{n} : x_{n}^{2} > \left( \sum_{k=1}^{n-1} x_{k}^{2} \right)^{1/2} \right\}, \quad n \geq 3.
\]
This domain is not uniform because of the presence of a bottleneck.

2 RBM on a domain with \( N \) unbounded uniform branches

In this section, we consider a Lipschitz domain \( D \) of \( \mathbb{R}^{n}, n \geq 3 \), such that

(A.2) \( D \setminus \overline{B_{r}(0)} = \bigcup_{j=1}^{N} C_{j} \)
for some $r > 0$ and an integer $N$, where $C_1, \cdots, C_N$ are unbounded uniform domains whose closures are mutually disjoint.

Obviously $D$ has the property (A.1).

Let $\partial_j$ be the point at infinity of the unbounded closed set $\overline{C}_j$ for each $1 \leq j \leq N$. Denote the $N$-points set $\{\partial_1, \cdots, \partial_N\}$ by $F$ and put $D^* = D \cup F$. $D^*$ can be made to be a compact Hausdorff space if we employ as a local base of neighborhoods of each point $\partial_j \in F$ the neighborhoods of $\partial_j$ in $\overline{C}_j \cup \{\partial_j\}$. $D^*$ may be called the $N$-points compactification of $D$.

For the RBM $Z = (Z_t, Q_x)$ on $\overline{D}$, define the approaching probabilities $\varphi_j(x)$ by

$$\varphi_j(x) = Q_x \left( \lim_{t \to \infty} Z_t = \partial_j \right), \quad x \in \overline{D}, \quad 1 \leq j \leq N.$$  

**Theorem 2.1.** It holds that

$$\begin{cases} \sum_{j=1}^{N} \varphi_j(x) = 1, & \varphi_j(x) > 0, \quad 1 \leq j \leq N, \quad \text{for every } x \in \overline{D}, \\ \dim(H^*(D)) = N, & \mathcal{H}^*(D) = \{C_j \varphi_j : c_j \in \mathbb{R}\}. \end{cases}$$

We fix a strictly positive $f \in L^1(D)$ and let $X = (X_t, \zeta, P_x)$ be the time changed process of $Z$ by the PCAF $A_t = \int_0^t f(Z_s)ds$. $X$ is then symmetric with respect to $m(dx) = f(x)dx$ and its Dirichlet form $(\mathcal{E}^X, \mathcal{F}^X)$ on $L^2(D;m)$ is given by $\mathcal{E}^X = \frac{1}{2}D$, $\mathcal{F}^X = H^1(D) \cap L^2(D;m)$. The reflected Dirichlet space of $X$ is still $BL(D)$. $\varphi_j(x)$ can be rewritten as

$$\varphi_j(x) = P_x(\zeta < \infty, X_{\zeta-} = \partial_j), \quad x \in \overline{D}, \quad 1 \leq j \leq N.$$  

A map $\Pi$ from the boundary set $F = \{\partial_1, \cdots, \partial_N\}$ onto a finite set $\hat{F} = \{\hat{\partial}_1, \cdots, \hat{\partial}_{\ell}\}$ with $\ell \leq N$ is called a partition of $F$. We let $D^{\Pi,*} = D \cup \hat{F}$. We extend the map $\Pi$ from $F$ to $D^*$ by setting $\Pi x = x, \quad x \in \overline{D}$, and introduce the quotient topology on $D^{\Pi,*}$ by $\Pi$, in other words,

$$U_\Pi = \{U \subset D^{\Pi,*} : \Pi^{-1}(U) \text{ is an open subset of } \overline{D}^*\}$$

is taken to be the family of open subsets of $D^{\Pi,*}$.

$D^{\Pi,*}$ is a compact Hausdorff space and may be called an $\ell$-points compactification of $D$ obtained from $D^*$ by identifying the points in the set $\Pi^{-1}(\hat{\partial}_i) \subset F$ as a single point $\hat{\partial}_i$ for each $1 \leq i \leq \ell$.

Given a partition $\Pi$ of $F$, the approaching probabilities $\hat{\varphi}_i$ of the time changed RBM $X = (X_t, \zeta, P_x)$ to $\hat{\partial}_i \in \hat{F}$ are defined by

$$\hat{\varphi}_i(x) = \sum_{j \in \Pi^{-1}(\hat{\partial}_i)} \varphi_j(x), \quad x \in \overline{D}, \quad 1 \leq i \leq \ell.$$  

The measure $m(dx) = f(x)dx$ is extended from $D$ to $D^{\Pi,*}$ by setting $m(\hat{F}) = 0$.

- $\hat{\varphi}_i$ is strictly positive on $\overline{D}$ for every $1 \leq i \leq N$,
- $m$ is a finite measure on $\overline{D}$
- $G^X g = G^Z(fg)$ is lower semicontinuous for the $0$-order resolvent $G^X$ (resp. $G^Z$) of $X$ (resp. $Z$) and any non-negative Borel function $g$ on $\overline{D}$.

Thus all requirements for the unique existence of $\ell$-point extension of $X$ from $D$ to $D^{\Pi,*}$ in Section 7.7 of [CF2] are fulfilled.
Theorem 2.2. There exists a unique $m$-symmetric recurrent diffusion extension $X^{\Pi,*}$ of $X$ from $\overline{D}$ to $\overline{D}^{\Pi,*}$. The Dirichlet form $(\mathcal{E}^{\Pi,*}, \mathcal{F}^{\Pi,*})$ of $X^{\Pi,*}$ on $L^{2}(\overline{D}^{\Pi,*}; m)$ ($=L^{2}(D;m))$ admits the extended Dirichlet space expressed as

$$
\begin{align*}
\mathcal{F}^{\Pi,*}_{e} &= \mathcal{H}^{1}_{e}(D) \oplus \{\sum_{i=1}^{\ell} c_{i} \varphi_{i} : c_{i} \in \mathbb{R}\} \subset BL(D), \\
\mathcal{E}^{\Pi,*}(u, v) &= \frac{1}{2} D(u, v), \quad u, v \in \mathcal{F}^{\Pi,*}_{e}.
\end{align*}
$$

Actually the family $\{X^{\Pi,*} : \Pi \text{ is a partition of } F\}$ exhausts all possible $m$-symmetric conservative diffusion extensions of the time changed RBM $X$ on $\overline{D}$ as will be formulated below. Let $E$ be a Lusin space into which $\overline{D}$ is homeomorphically embedded as an open subset. The measure $m(dx) = f(x)dx$ on $\overline{D}$ is extended to $E$ by setting $m(E \backslash \overline{D}) = 0$. Let $Y = (Y_{t}, P_{x}^{Y})$ be an $m$-symmetric conservative diffusion process on $E$ whose part process on $D$ is identical in law with $X$. The following theorem extends Theorem 3.4 in [CF1] (the case that $N=1$).

Theorem 2.3. There exists a partition $\Pi$ of $F$ such that $E$ is quasi-homeomorphic with $\overline{D}^{\Pi,*}$ and $Y$ is a quasi-homeomorphic image of $X^{\Pi,*}$.

Outline of a proof of Theorem 2.3

Let $\mathcal{E}^{Y}$ be the Dirichlet form of $Y$ on $L^{2}(E; m)$. Since $\mathcal{E}^{Y}$ is quasi-regular, we can use a quasi homeomorphism to assume

- $E$ is a locally compact separable metric space,
- $\mathcal{E}^{Y}$ is a regular Dirichlet form on $L^{2}(E; m)$,
- $Y$ is an associated Hunt process on $E$,
- $\tilde{F} := E \backslash \overline{D}$ is quasi-closed.

As $Y$ is a conservative extension of the non-conservative process $X$, $\tilde{F}$ is not $\mathcal{E}^{Y}$-polar. Every function in $\mathcal{F}^{Y}_{e}$ will be taken to be $\mathcal{E}^{Y}$-quasi continuous. By Theorem 7.1.6 of [CF2], one can conclude that

$$
\begin{align*}
\mathcal{F}^{Y}_{e} \subset BL(D), \quad \mathcal{H}^{Y} := \{Hu : u \in \mathcal{F}^{Y}_{e}\} \subset \mathcal{H}^{*}, \\
\mathcal{E}^{Y}(u, u) &= \frac{1}{2} D(u, u) + \frac{1}{2} \mu_{\langle Hu\rangle}^{c}(\tilde{F}), \quad u \in \mathcal{F}^{Y}_{e},
\end{align*}
$$

where $Hu(x) = \mathbb{E}_{x}^{Y}[u(Y_{\sigma_{\tilde{F}}})], \quad x \in E$. We show that

$$
\mu_{(u)}^{c}(\tilde{F}) = 0 \quad u \in \mathcal{H}^{Y}. \tag{2.1}
$$

Take any $u \in \mathcal{H}^{Y}$. Theorem 2.1 and the above inclusion imply that $u = \sum_{j=1}^{N} c_{j} \varphi_{j}$ for some constants $c_{j}$. As $u$ is continuous along the sample path of $Y$, $u$ takes only the values $\{c_{1}, \cdots, c_{N}\}$ on the boundary $\tilde{F}$ $\nu$-almost everywhere where

$$
\nu(B) = \int_{\overline{D}} P_{x}^{Y} \left(Y_{\sigma_{\tilde{F}}} \in B, \sigma_{\tilde{F}} < \infty \right) m(dx), \quad B \in \mathcal{B}(E).
$$

Since $\tilde{F}$ is a quasi-support of $\nu$, $u$ takes only the values $\{c_{1}, \cdots, c_{N}\}$ quasi-everywhere on $\tilde{F}$. (2.1) then follows from the image measure density property of $\mu_{(u)}^{c}$ due to Bouleau-Hirsch.

Define a partition $\Pi$ of $F$ by means of the values taken by functions in $\mathcal{H}^{Y}$ along the path of $X$ to obtain

$$
(\mathcal{F}^{Y}_{e}, \mathcal{E}^{Y}) = (\mathcal{F}^{\Pi,*}_{e}, \mathcal{E}^{\Pi,*}).
$$

Both being quasi-regular, they are related by a quasi-homeomorphism of their underlying spaces.
Remark 2.4. Given measurable functions $a_{ij}(x), 1 \leq i, j \leq n,$ on $D$ such that

$$a_{ij}(x) = a_{ji}(x), \quad \Lambda^{-1} |\xi|^2 \leq \sum_{1 \leq i,j \leq n} a_{ij}(x) \xi_i \xi_j \leq \Lambda |\xi|^2, \quad x \in D, \xi \in \mathbb{R}^n,$$

for some constant $\Lambda \geq 1,$ we define a Dirichlet form $(\mathcal{A}, H^1(D))$ on $L^2(D)$ by

$$\mathcal{A}(u, v) = \int_D \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial u}{\partial x_i}(x) \frac{\partial v}{\partial x_j}(x) dx, \quad u, v \in H^1(D).$$

If we replace the Dirichlet form $(\mathcal{D}, H^1(D))$ on $L^2(D)$ and the associated RBM $Z$ on $\overline{D}$, respectively, by $(\mathcal{A}, H^1(D))$ and the associated reflecting diffusion process on $\overline{D}$ constructed in [FTo], all assertions stated above remain valid with no essential change.

By this replacement, the extended Dirichlet space and the reflected Dirichlet space are still $H^1_+(D)$ and $BL(D)$, respectively, although the inner product $\mathcal{D}$ is replaced by $\mathcal{A}$. It suffices to notice that any function in $BL(\mathbb{R}^n)$ is a sum of a function in $H^1_+(\mathbb{R}^n)$ and a constant $c$ and $\mathcal{A}(c, c) = 0$.

3 Liouville property of energy forms on $\mathbb{R}^n$

In this section, we consider a positive Borel function $\rho$ on $\mathbb{R}^n$ that is locally bounded above and locally uniformly bounded away from 0, and an associated form

$$\mathcal{E}^\rho(u, v) = \int_{\mathbb{R}^n} \nabla u(x) \cdot \nabla v(x) \rho(x) dx. \quad (3.1)$$

$(\mathcal{E}^\rho, C^1_0(\mathbb{R}^n))$ is closable on $L^2(\mathbb{R}^n) = L^2(\mathbb{R}^n, dx)$ and the closure $(\mathcal{E}^\rho, \mathcal{F}^\rho)$ (called an energy form) is a strongly local regular Dirichlet form on $L^2(\mathbb{R}^n)$. It is irreducible ([FOT, Theorem 4.6.4]). In general, an irreducible recurrent Dirichlet form enjoys the Liouville property in view of [CF2, Lemma 6.7.3]. It therefore suffices to consider only the transient case in order to study the Liouville property of $\mathcal{E}^\rho$. We shall examine this property when $\rho(x)$ is a positive smooth function depending only on the radial part of the variable $x \in \mathbb{R}^n$.

**Theorem 3.1.** For any positive smooth function $\eta$ on $[0, \infty)$, let $\rho(x) = \eta(|x|), \ x \in \mathbb{R}^n$. Then $\mathcal{E}^\rho$ satisfies the Liouville property when $n \geq 2$.

When $n = 1$, $\mathcal{E}^\rho$ satisfies the Liouville property in recurrent case but $\text{dim}(\mathcal{H}^*) = 2$ in transient case.

**Proof.** According to Theorem 1.6.7 in the first edition of [FOT], $\mathcal{E}^\rho$ is transient if and only if

$$(T) \quad \int_1^\infty \frac{1}{\eta(r)r^{n-1}} dr < \infty.$$

In what follows, we assume that $\eta$ satisfies condition (T).

It then follows from $1/r = (r^{n-3}\eta(r))^{1/2}(\eta(r)r^{n-1})^{-1/2}$ and the Schwarz inequality that

$$\int_1^\infty r^{n-3}\eta(r) dr = \infty. \quad (3.2)$$
We use the polar coordinate

\[
\begin{align*}
  x_1 &= r \cos \theta_1 \\
  x_2 &= r \sin \theta_1 \cos \theta_2 \\
  x_3 &= r \sin \theta_1 \sin \theta_2 \cos \theta_3 \\
  &\vdots \\
  x_{n-1} &= r \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{n-2} \cos \theta_{n-1} \\
  x_n &= r \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{n-2} \sin \theta_{n-1}.
\end{align*}
\]

Then, for \( u, v \in C_0^1(\mathbb{R}^n) \),

\[
\mathcal{E}^\rho(u, v) = \int_{[0, \infty) \times [0, \pi]^{n-2} \times [0, 2\pi]} [u_r v_r + \frac{u_{\theta_1} v_{\theta_1}}{r^2} + \frac{u_{\theta_2} v_{\theta_2}}{r^2 \sin^2 \theta_1} + \cdots + \frac{u_{\theta_{n-1}} v_{\theta_{n-1}}}{r^2 \sin^2 \theta_1 \cdots \sin^2 \theta_{n-2}}] \times \eta(r) r^{n-1} \sin^{n-2} \theta_1 \cdots \sin \theta_{n-2} dr d\theta_1 \cdots d\theta_{n-1}.
\]

For a smooth function \( u \) on \( \mathbb{R}^n \), we denote by \( \mathcal{E}^\eta(u, u) \) the value of the integral of the right hand side of (3.3) for \( v = u \).

As in the case that \( \rho = 1 \), the reflected Dirichlet space of \( \mathcal{E}^\rho \) is given by

\[
\mathcal{F}^{\rho, \text{ref}} = \{ u \in L_{1\text{oc}}^2(\mathbb{R}^n) : \int_{\mathbb{R}^n} |\nabla u(x)|^2 \eta(|x|) dx < \infty \}.
\]

Since \( \mathcal{H}^* = \{ u \in \mathcal{F}^{\rho, \text{ref}} : \mathcal{E}^\rho(u, v) = 0 \text{ for every } v \in C_0^\infty(\mathbb{R}^n) \} \), it follows from (3.3) that \( u \in \mathcal{H}^* \) if and only if

\[
\text{u is smooth, } \mathcal{E}^\eta(u, u) < \infty \text{ and } \mathcal{L}u(x) = 0, \quad x \in \mathbb{R}^n,
\]

where

\[
\mathcal{L}u(r, \theta_1, \ldots, \theta_{n-1}) = \frac{1}{r^{n-1}}(u_r \cdot \eta(r)r^{n-1})_r + \frac{\eta(r)}{r^2 \sin^{n-2} \theta_1}(u_{\theta_1} \sin^{n-2} \theta_1)_{\theta_1} + \frac{\eta(r)}{r^2 \sin^2 \theta_1 \sin^{n-3} \theta_2}(u_{\theta_2} \sin^{n-3} \theta_2)_{\theta_2}
\]

\[
+ \cdots + \frac{\eta(r)}{r^2 \sin^2 \theta_1 \cdots \sin^2 \theta_{n-3} \sin \theta_{n-2}}(u_{\theta_{n-2}} \sin \theta_{n-2})_{\theta_{n-2}} + \frac{\eta(r)}{r^2 \sin^2 \theta_1 \cdots \sin^2 \theta_{n-2}}(u_{\theta_{n-1}})_{\theta_{n-1}}.
\]

Now take any function \( u \in \mathcal{H}^* \). We claim that

\[
\eta_{\theta_{n-1}} = 0.
\]

Put \( w = u_{\theta_{n-1}} \). Due to the expression (3.5) of \( \mathcal{L}, \mathcal{L}w = (\mathcal{L}u)_{\theta_{n-1}} = 0 \), namely, \( w \) is \( \mathcal{L} \)-harmonic. For \( B_r = \{ x \in \mathbb{R}^n ; |x| < r \} \) and the uniform probability measure \( \Pi(d\xi) \) on \( \partial B_1 \), \( w \) therefore admits the Poisson integral formula

\[
w(x) = \int_{\partial B_1} K_r(x, r\xi) w(r\xi) \Pi(d\xi). \quad x \in B_r,
\]

where \( K_r(x, r\xi) \) is the Poisson kernel for \( B_r \) with respect to \( \mathcal{L} \), which is known to be continuous in \( (x, \xi) \in B_r \times \partial B_1 \). We also note that \( K_r(0, r\xi) = 1 \) for any \( \xi \in \partial B_1 \) by the rotation invariance of \( \mathcal{L} \) around the origin 0.
Fix $a > 0$. It then holds for any $r > a$ that
\[ K_r(x, r\xi^2) = \int_{\partial B_a} K_a(x, a\xi_1) K_r(a\xi_1, r\xi^2) \Pi(d\xi_1), \quad x \in B_a, \xi_2 \in \partial B_1. \]
Hence, if we let
\[ \sup_{x \in B_{a/2}, \xi_2 \in \partial B_1} K_a(x, a\xi_1) = C_a < \infty, \text{then, for } x \in B_{a/2}, \xi_2 \in \partial B_1, \]
\[ K_r(x, r\xi^2) \leq C_a \int_{\partial B_1} K_r(a\xi_1, r\xi^2) \Pi(d\xi_1) = C_a K_r(0, r\xi^2) = C_a, \]
and it follows from (3.7) that
\[ |w(x)| \leq C_a \int_{\partial B_1} |w(r\xi)| \Pi(d\xi), \quad x \in B_{a/2}, \quad r > a. \]
Recall that $w = u_{\theta_{n-1}}$. We multiply both hand side of the above inequality by $r^{n-3}\eta(r)$, integrate in $r$ from $a$ to $R$, apply the Schwarz inequality and finally use the expression (3.3) to get
\[ |u_{\theta_{n-1}}(x)| \leq C_a \sigma_n^{-1/2} \sqrt{\mathcal{E}^\rho(u, u)}, \quad x \in B_{a/2}, \]
which tends to 0 as $R \to \infty$ by (3.2). Since $a > 0$ is arbitrary, we arrive at (3.6).

It also holds that
\[ u_{\theta_k} = 0 \quad \text{for any } 1 \leq k \leq n - 1. \quad (3.8) \]
In fact, if we let $\xi_i = \frac{x_i}{r}$, $1 \leq i \leq n$, $\xi = (\xi_1, \cdots, \xi_n) \in \partial B_1$, then $\theta_k$, $1 \leq k \leq n - 1$, is an angle of two $n$-vectors $\xi^{(k)} = (0, \cdots, 0, \xi_{k}, \cdots, \xi_n)$, $e_k = (0, \cdots, 0, 1, \cdots, 0)$. Consider the subspace $V$ of $\mathbb{R}^n$ spanned by $\xi^{(k)}$ and $e_k$ and take a unit vector $\hat{e}$ in $V$ orthogonal to $e_k$. Let $O$ be an orthogonal matrix whose $(n - 1)$-th and $n$-th column vectors are $e_k$ and $\hat{e}$, respectively.

We make the orthogonal transformation $y = {}^{t}Ox$. Then $\theta_k$ equals an angle of two vectors on the $(y_{n-1}, y_n)$-plane in the new coordinate system $y$ and (3.6) applies.

Thus $u$ depends only on $r$ and, in terms of a scale function $ds(r) = \frac{dr}{\eta(r)r^{n-1}}$ on $(0, \infty)$, (3.3) and (3.6) are reduced, respectively, to
\[ \mathcal{E}^\rho(u, u) = \sigma_n \int_{0}^{\infty} \left( \frac{du(r)}{ds(r)} \right)^2 ds(r), \quad \mathcal{L}u(r) = \frac{1}{r^{n-1}} \frac{d}{dr} \cdot \frac{du(r)}{ds(r)}. \]
By (3.4), $\mathcal{L}u = 0$ so that $u(r) = C_1 + C_2 s(r)$, $r > 0$, for some constant $C_1$, $C_2$. Since $\mathcal{E}^\rho(s, s) = \sigma_n \cdot s(0, \infty)$ is finite if and only if $n = 1$, we get the desired estimates from (3.4). \(\square\)

It is conjectured that the energy form $\mathcal{E}^\rho$ satisfies the Liouville property for any $\rho$ prescribed in the above of (3.1) when $n \geq 2$.

The diffusion process $Z$ on $\mathbb{R}^n$ associated with $\mathcal{E}^\rho$ is called the distorted Brownian motion. Let $X$ be its time changed process defined as (1.1) by means of $m(dx) = f(x)dx$ for a strictly positive bounded function $f \in L^1(\mathbb{R}^n)$. Let $\mathbb{R}^n \cup \{\Delta\}$ be the one point compactification of $\mathbb{R}^n$. If $\mathcal{E}^\rho$ satisfies the Liouville property, then it can be shown as [CF1, Theorem 3.4] that any $m$-symmetric proper diffusion extension of $X$ shares the same finite dimensional distribution with the one-point reflection of $X$ at $\Delta$. See [F2] for more details on these points.
References


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