数理解析研究所講究録
第 1952 巻 2015 年 92-99

Many point reflections at infinity of a time changed reflecting diffusion

Masatoshi Fukushima
Osaka University

1 Introduction

The boundary problem of a Markov process X concerns all possible Markovian prolongations of X beyond its life time ζ whenenver ζ is finite. Let $Z=(Z_t, \mathbf{Q}_z)$ be a *conservative* right process on a locally compact separable metric space E and Δ be the point at infinity of E. Suppose Z is *transient* relative to an excessive measure m: for the 0-order resolvent R of Z, $Rf(z)<\infty$, m-a.e. for some strictly positive function (or equivalently, for any non-negative function) $f\in L^1(E;m)$. Then

$$\mathbf{Q}_z(\lim_{t\to\infty} Z_t = \Delta)=1 \quad \text{for q.e. } x\in E,$$

if Rf is lower semicontinuous for any non-negative Borel function f ([FTa]). The last condition is not needed when X is m-symmetric ([CF2]).

Take any strictly positive bounded function $f\in L^1(E;m)$. Then $A_t=\int_0^t f(Z_s)ds,\ t\geq 0$ is a strictly increasing *PCAF* of Z with $\mathbf{E}_z^{\mathbf{Q}}[A_\infty]=Rf(x)<\infty$ for q.e. $x\in E$. The *time changed process* $X=(X_t, \zeta, \mathbf{P}_x)$ of Z by means of A is defined by

$$X_t=Z_{\tau_t},\ t\geq 0, \quad \tau=A^{-1}, \quad \zeta=A_\infty, \quad \mathbf{P}_x=\mathbf{Q}_x,\ x\in E. \tag{1.1}$$

Since $\mathbf{P}_x(\zeta<\infty, \lim_{t\to\zeta} X_t=\Delta)=\mathbf{P}_x(\zeta<\infty)=1$, the boundary problem for X at Δ makes perfect sense. For different choices of f, the corresponding processes X have the same geometric shapes related each other only by time changes. Thus a study of the boundary problem for X is a good way to make a close look at a geometric picture of a conservative transient process Z around Δ.

When a right process Z is m-symmetric, we can work with the associated Dirichlet form $(\mathcal{E},\mathcal{F})$ on $L^2(E;m)$. Let $\mathcal{F}_e$ and $\mathcal{F}^{\rm ref}$ be its extended Dirichlet space and its reflected Dirichlet space ([CF2]). Then $\mathcal{F}\subset\mathcal{F}_e\subset\mathcal{F}^{\rm ref}$ and the inner product $\mathcal{E}$ is extended from $\mathcal{F}$ to both spaces. Define the subspace $\mathcal{H}^*$ of $\mathcal{F}^{\rm ref}$ by

$$\mathcal{H}^*=\{u\in\mathcal{F}^{\rm ref}:\mathcal{E}(u,v)=0 \quad \text{for any} \quad v\in\mathcal{F}_e\}. \tag{1.2}$$

The stated boundary problem for Z is closely related to $\dim(\mathcal{H}^*)$. The process Z or the associated Dirichlet form $(\mathcal{E},\mathcal{F})$ is said to satisfy a *Liouville property* if $\dim(\mathcal{H}^*)=1$. We will be concerned with the cases where Z are the reflecting Brownian motion on an unbounded domain of $\mathbb{R}^n$ and the distorded Brownian motion on the whole space $\mathbb{R}^n$.

We first consider the *reflecting Brownian motion*(RBM) Z on the closure $\overline{D}$ of a Lipschitz domain $D\subset\mathbb{R}^n$ that is a special case of the *reflecting diffusion process* constructed in [FTo]. Z is always conservative. Z is symmetric with respect to the Lebesgue measure on D and the Dirichlet form $\mathcal{E}$ of Z on $L^2(D)$ is given by

$$\mathcal{E}=\frac{1}{2}\mathbf{D}, \quad \mathcal{D}(\mathcal{E})=H^1(D)={\rm BL}(D)\cap L^2(D),$$

where

$$\mathbf{D}(u,v)=\int_D \nabla u(x)\cdot\nabla v(x)dx,\quad \mathrm{BL}(D)=\{u\in L^2_{\mathrm{loc}}(D):\ |\nabla u|\in L^2(D)\}.$$

BL(D) is the reflected Dirichlet space of Z.

We requires that

(A.1) Z is transient,

and accordingly it must be that $n\geq 3$ and D is unbounded. When $d\geq 3$, an infinite cone D satisfies **(A.1)** but an infinite cylinder does not. Under **(A.1)**, the *extended Sobolev space* $H^1_e(D)$ is a Hilbert space with inner product $\frac{1}{2}\mathbf{D}$ so that it does not contain any non-zero constant, while BL(D) does. Hence $H^1_e(D)$ is a proper subspace of BL(D) and the space $\mathcal{H}^*(D)$ defined by

$$\mathcal{H}^*(D)=\{u\in \mathrm{BL}(D):\ \mathbf{D}(u,v)=0 \text{ for every } v\in H^1_e(D)\},$$

is a non-trivial family of harmonic functions on D.

In what follows, we assume that $n\geq 3$. A domain $D\subset\mathbb{R}^d$ is called a *uniform domain* if there exists $C>0$ such that, for every $x,y\in D$, there is a rectifiable curve γ in D connecting x and y with length$(\gamma)\leq C|x-y|$, and moreover

$$\min\{|x-z|,\ |z-y|\}\leq C\mathrm{dist}(z,D^c)\quad \text{for every } z\in\gamma.$$

A typical example of a unbounded uniform domain is an infinite cone.

According to [CF1],

- a domain D containing a unbouded uniform domain satisfies **(A.1)**.
- Z satisfies the Liouville property $\dim(\mathcal{H}^*(D))=1$ whenever $D\setminus\overline{B_r(\mathbf{0})}$ is a unbounded uniform domain, for some $r>0$.

The proof used the two facts that

- for an unbouded uniform domain D, any $u\in$ BL(D) admits a bounded linear extension to BL$(\mathbb{R}^d)$ ([HK]).
- any harmonic function on $\mathbb{R}^d$ with finite Dirichlet integral is constant, namely, the RBM on $\mathbb{R}^n$ satisfies the Liouville property $\dim(\mathcal{H}^*(\mathbb{R}^n))=1$ ([B]).

On the other hand, $\dim(\mathcal{H}^*(D))=2$ for a domain with two symmetric cone branches ([CF2]):

$$D=B_1(\mathbf{0})\cup\left\{x\in\mathbb{R}^n: x_n^2>(\sum_{k=1}^{n-1}x_k^2)^{1/2}\right\},\quad n\geq 3.$$

This domain is not uniform because of the presence of a bottleneck.

2 RBM on a domain with N unbounded uniform branches

In this section, we consider a Lipschitz domain D of $\mathbb{R}^n$, $n\geq 3$, such that

(A.2) $D\setminus\overline{B_r(\mathbf{0})}=\bigcup_{j=1}^N C_j$

for some $r > 0$ and an integer N, where $C_1, \cdots, C_N$ are unbounded uniform domains whose closures are mutually disjoint.

Obviously D has the property **(A.1)**.

Let ∂_j be the point at infinity of the unbounded closed set $\overline{C}_j$ for each $1 \le j \le N$. Denote the N-points set $\{\partial_1, \cdots, \partial_N\}$ by F and put $\overline{D}^* = \overline{D} \cup F$. $\overline{D}^*$ can be made to be a compact Hausdorff space if we employ as a local base of neighborhoods of each point $\partial_j \in F$ the neighborhoods of ∂_j in $\overline{C}_j \cup \{\partial_j\}$. $\overline{D}^*$ may be called the *N-points compactification of $\overline{D}$.*

For the RBM $Z = (Z_t, \mathbf{Q}_z)$ on $\overline{D}$, define the *approaching probabilities* $\varphi_j(x)$ by

$$\varphi_j(x) = \mathbf{Q}_x\left(\lim_{t\to\infty} Z_t = \partial_j\right), \quad x \in \overline{D}, \quad 1 \le j \le N.$$

Theorem 2.1. *It holds that*

$$\begin{cases} \sum_{j=1}^N \varphi_j(x) = 1. \quad \varphi_j(x) > 0, \quad 1 \le j \le N, \quad \textit{for every } x \in \overline{D}, \\ \dim(\mathcal{H}^*(D)) = N. \quad \mathcal{H}^*(D) = \{\sum_{j=1}^N c_j \varphi_j : c_j \in \mathbb{R}\}. \end{cases}$$

We fix a strictly positive $f \in L^1(D)$ and let $X = (X_t, \zeta, \mathbf{P}_x)$ be the time changed process of Z by the PCAF $A_t = \int_0^t f(Z_s)ds$. X is then symmetric with respect to $m(dx) = f(x)dx$ and its Dirichlet form $(\mathcal{E}^X, \mathcal{F}^X)$ on $L^2(D; m)$ is given by $\mathcal{E}^X = \frac{1}{2}\mathbf{D}$, $\mathcal{F}^X = H_e^1(D) \cap L^2(D; m)$. The reflected Dirichlet space of X is still $\mathrm{BL}(D)$. $\varphi_j(x)$ can be rewritten as

$$\varphi_j(x) = \mathbf{P}_x(\zeta < \infty, \ X_{\zeta-} = \partial_j), \quad x \in \overline{D}, \quad 1 \le j \le N.$$

A map Π from the boundary set $F = \{\partial_1, \cdots, \partial_N\}$ onto a finite set $\widehat{F} = \{\widehat{\partial}_1, \cdots, \widehat{\partial}_\ell\}$ with $\ell \le N$ is called a *partition* of F. We let $\overline{D}^{\Pi,*} = \overline{D} \cup \widehat{F}$. We extend the map Π from F to $\overline{D}^*$ by setting $\Pi x = x$, $x \in \overline{D}$, and introduce the *quotient topology on* $\overline{D}^{\Pi,*}$ by Π, in other words,

$$\mathcal{U}_\Pi = \{U \subset \overline{D}^{\Pi,*} : \Pi^{-1}(U) \text{ is an open subset of } \overline{D}^*\}$$

is taken to be the family of open subsets of $\overline{D}^{\Pi,*}$.

$\overline{D}^{\Pi,*}$ is a compact Hausdorff space and may be called an *ℓ-points compactification of $\overline{D}$* obtained from $\overline{D}^*$ by identifying the points in the set $\Pi^{-1}\widehat{\partial}_i \subset F$ as a single point $\widehat{\partial}_i$ for each $1 \le i \le \ell$.

Given a partition Π of F, the approaching probabilities $\widehat{\varphi}_i$ of the time changed RBM $X = (X_t, \zeta, \mathbf{P}_x)$ to $\widehat{\partial}_i \in \widehat{F}$ are defined by

$$\widehat{\varphi}_i(x) = \sum_{j \in \Pi^{-1}\widehat{\partial}_i} \varphi_j(x), \quad x \in \overline{D}, \quad 1 \le i \le \ell.$$

The measure $m(dx) = f(x)dx$ is extended from $\overline{D}$ to $\overline{D}^{\Pi,*}$ by setting $m(\widehat{F}) = 0$.

- $\widehat{\varphi}_i$ is strictly positive on $\overline{D}$ for every $1 \le i \le N$,
- m is a finite measure on $\overline{D}$
- $G^X g = G^Z(fg)$ is lower semicontinuous for the 0-order resolvent G^X (resp. G^Z) of X (resp. Z) and any non-negaitve Borel function g on $\overline{D}$.

Thus all requirements for the unique existence of ℓ-point extension of X from $\overline{D}$ to $\overline{D}^{\Pi,*}$ in Section 7.7 of [CF2] are fulfilled.

Theorem 2.2. *There exists a unique m-symmetric rcurrent diffusion extension* $X^{\Pi,*}$ *of* X *from* $\overline{D}$ *to* $\overline{D}^{\Pi,*}$. *The Diriclet form* $(\mathcal{E}^{\Pi,*},\mathcal{F}^{\Pi,*})$ *of* $X^{\Pi,*}$ *on* $L^2(\overline{D}^{\Pi,*};m)$ $(=L^2(D;m))$ *admits the extended Dirichlet space expressed as*

$$\begin{cases}\mathcal{F}_e^{\Pi,*}=H_e^1(D)\oplus\{\sum_{i=1}^{\ell}c_i\widehat{\varphi}_i\ :\ c_i\in\mathbb{R}\}\subset \mathrm{BL}(D),\\ \mathcal{E}^{\Pi,*}(u,v)=\frac{1}{2}\,\mathbf{D}(u,v),\qquad u,v\in\mathcal{F}_e^{\Pi,*}.\end{cases}$$

Actually the family $\{X^{\Pi,*}:\Pi$ is a partition of $F\}$ exhausts all possible *m-symmetric conservative diffusion extensions* of the time changed RBM X on $\overline{D}$ as will be formulated below. Let E be a Lusin space into which $\overline{D}$ is homeomorpically embedded as an open subset. The measure $m(dx)=f(x)dx$ on $\overline{D}$ is extended to E by setting $m(E\setminus\overline{D})=0$. Let $Y=(Y_t,\mathbf{P}_x^Y)$ be an m-symmetric conservative diffusion process on E whose part process on $\overline{D}$ is identical in law with X. The following theorem extends Theorem 3.4 in [CF1] (the case that $N=1$).

Theorem 2.3. *There exists a partition* Π *of* F *such that* E *is quasi-homeomorphic with* $\overline{D}^{\Pi,*}$ *and* Y *is a quasi-homeomorphic image of* $X^{\Pi,*}$.

Outline of a proof of Theorem 2.3

Let $\mathcal{E}^Y$ be the Dirichlet form of Y on $L^2(E;m)$. Since $\mathcal{E}^Y$ is quasi-regular, we can use a quasi homeomorphism to assume

- E is a locally compact separable metric space,
- $\mathcal{E}^Y$ is a regular Dirichlet form on $L^2(E;m)$,
- Y is an associated Hunt process on E,
- $\widetilde{F}:=E\setminus\overline{D}$ is quasi-closed.

As Y is a conservative extension of the non-conservative process X, $\widetilde{F}$ is not $\mathcal{E}^Y$-polar. Every function in $\mathcal{F}_e^Y$ will be taken to be $\mathcal{E}^Y$-quasi continuous. By Theorem 7.1.6 of [CF2], one can coclude that

$$\begin{cases}\mathcal{F}_e^Y\subset\mathrm{BL}(D),\quad \mathcal{H}^Y:=\{\mathbf{H}u:u\in\mathcal{F}_e^Y\}\subset\mathcal{H}^*,\\ \mathcal{E}^Y(u,u)=\frac{1}{2}\mathbf{D}(u,u)+\frac{1}{2}\mu^c_{\langle \mathbf{H}u\rangle}(\widetilde{F}),\quad u\in\mathcal{F}_e^Y,\end{cases}$$

where $\mathbf{H}u(x)=\mathbf{E}_x^Y[u(Y_{\sigma_{\widetilde{F}}})]$, $x\in E$. We show that

$$\mu^c_{\langle u\rangle}(\widetilde{F})=0\qquad u\in\mathcal{H}^Y. \tag{2.1}$$

Take any $u\in\mathcal{H}^Y$. Theorem 2.1 and the above inclusion imply that $u=\sum_{j=1}^N c_j\varphi_j$ for some constants c_j. As u is continuous along the sample path of Y, u takes only the values $\{c_1,\cdots,c_N\}$ on the boundary $\widetilde{F}$ ν-almost everywhre where

$$\nu(B)=\int_{\overline{D}}\mathbf{P}_x^Y\left(Y_{\sigma_{\widetilde{F}}}\in B,\ \sigma_{\widetilde{F}}<\infty\right)m(dx),\ B\in\mathcal{B}(E).$$

Since $\widetilde{F}$ is a *quasi-support* of ν, u takes only the values $\{c_1,\cdots,c_N\}$ quasi-everywhere on $\widetilde{F}$. (2.1) then follows from the *image measure density property* of $\mu^c_{\langle u\rangle}$ due to Bouleau-Hirsch.

Define a partition Π of F by means of the values taken by functions in $\mathcal{H}^Y$ along the path of X to obtain

$$(\mathcal{F}_e^Y,\mathcal{E}^Y)\ =\ (\mathcal{F}_e^{\Pi,*},\mathcal{E}^{\Pi,*}).$$

Both being quasi-regular, they are related by a quasi-homeomorphism of their underlying spaces.

Remark 2.4. Given measurable functions $a_{ij}(x)$, $1 \le i,j \le n$, on D such that

$$a_{ij}(x) = a_{ji}(x), \quad \Lambda^{-1}|\xi|^2 \le \sum_{1\le i,j\le n} a_{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2, \quad x \in D,\ \xi \in \mathbb{R}^n,$$

for some constant $\Lambda \ge 1$, we define a Dirichlet form $(\mathcal{A}, H^1(D))$ on $L^2(D)$ by

$$\mathcal{A}(u,v) = \int_D \sum_{i,j=1}^n a_{ij}(x)\frac{\partial u}{\partial x_i}(x)\frac{\partial v}{\partial x_j}(x)dx, \quad u,v \in H^1(D).$$

If we replace the Dirichlet form $(\frac{1}{2}\mathbf{D}, H^1(D))$ on $L^2(D)$ and the assoicated RBM Z on $\overline{D}$, respectively, by $(\mathcal{A}, H^1(D))$ and the associated reflecting diffusion process on $\overline{D}$ constructed in [FTo], all assertions stated above remain valid with no essential change.

By this replacement, the extended Dirichlet space and the reflected Dirichlet space are still $H^1_e(D)$ and $\mathrm{BL}(D)$, respectively, although the inner product $\frac{1}{2}\mathbf{D}$ is replaced by $\mathcal{A}$. It suffices to notice that any function in $\mathrm{BL}(\mathbb{R}^n)$ is a sum of a function in $H^1_e(\mathbb{R}^n)$ and a constant c and $\mathcal{A}(c,c) = 0$.

3 Liouville property of energy forms on $\mathbb{R}^n$

In this section, we consider a positive Borel function ρ on $\mathbb{R}^n$ that is locally bounded above and locally uniformly bounded away from 0, and an associated form

$$\mathcal{E}^\rho(u,v) = \int_{\mathbb{R}^n} \nabla u(x)\cdot\nabla v(x)\rho(x)dx. \tag{3.1}$$

$(\mathcal{E}^\rho, C_0^1(\mathbb{R}^n))$ is closable on $L^2(\mathbb{R}^n) = L^2(\mathbb{R}^n, dx)$ and the closure $(\mathcal{E}^\rho, \mathcal{F}^\rho)$ (called an *energy form*) is a strongly local regular Dirichlet form on $L^2(\mathbb{R}^n)$. It is irreducible ([FOT, Theorem 4.6.4]). In general, an irreducible recurrent Dirichlet form enjoys the Liouville property in view of [CF2, Lemma 6.7.3]. It therefore suffices to consider only the transient case in order to study the Liouville property of $\mathcal{E}^\rho$. We shall examine this property when $\rho(x)$ is a positive smooth function depending only on the radial part of the variable $x \in \mathbb{R}^n$.

Theorem 3.1. *For any positive smooth function η on $[0,\infty)$, let $\rho(x) = \eta(|x|)$, $x \in \mathbb{R}^n$. Then $\mathcal{E}^\rho$ satisfies the Liouville property when $n \ge 2$.*
When $n = 1$, $\mathcal{E}^\rho$ satisfies the Liouville property in recurrent case but $\dim(\mathcal{H}^*) = 2$ *in transient case.*

Proof. According to Theorem 1.6.7 in the first edition of [FOT], $\mathcal{E}^\rho$ is transient if and only if

(T) $\displaystyle\int_1^\infty \frac{1}{\eta(r)r^{n-1}}dr < \infty.$

In what follows, we assume that η satisfies condition **(T)**.

It then follows from $1/r = (r^{n-3}\eta(r))^{1/2}(\eta(r)r^{n-1})^{-1/2}$ and the Schwarz inequality that

$$\int_1^\infty r^{n-3}\eta(r)dr = \infty. \tag{3.2}$$

We use the polar coordinate

$$\begin{cases} x_1 = r\cos\theta_1 \\ x_2 = r\sin\theta_1\cos\theta_2 \\ x_3 = r\sin\theta_1\sin\theta_2\cos\theta_3 \\ \cdots \\ x_{n-1} = r\sin\theta_1\sin\theta_2\cdots\sin\theta_{n-2}\cos\theta_{n-1} \\ x_n = r\sin\theta_1\sin\theta_2\cdots\sin\theta_{n-2}\sin\theta_{n-1}. \end{cases}$$

Then, for $u, v \in C_0^1(\mathbb{R}^n)$,

$$\begin{aligned} &\mathcal{E}^\rho(u,v) \qquad (3.3)\\ = &\int_{[0,\infty)\times[0,\pi]^{n-2}\times[0,2\pi]} \left[u_r v_r + \frac{u_{\theta_1} v_{\theta_1}}{r^2} + \frac{u_{\theta_2} v_{\theta_2}}{r^2\sin^2\theta_1} + \cdots + \frac{u_{\theta_{n-1}} v_{\theta_{n-1}}}{r^2\sin^2\theta_1\cdots\sin^2\theta_{n-2}} \right] \\ &\times\eta(r) r^{n-1}\sin^{n-2}\theta_1\cdots\sin\theta_{n-2}\,dr\,d\theta_1\cdots d\theta_{n-1}. \end{aligned}$$

For a smooth function u on $\mathbb{R}^n$, we denote by $\mathcal{E}^\eta(u,u)$ the value of the integral of the right hand side of (3.3) for $v = u$.

As in the case that $\rho = 1$, the reflected Dirichlet space of $\mathcal{E}^\rho$ is given by

$$\mathcal{F}^{\rho,\mathrm{ref}} = \{u \in L^2_{\mathrm{loc}}(\mathbb{R}^n) : \int_{\mathbb{R}^n} |\nabla u(x)|^2 \eta(|x|)dx < \infty\}.$$

Since $\mathcal{H}^* = \{u \in \mathcal{F}^{\rho,\mathrm{ref}} : \mathcal{E}^\rho(u,v) = 0 \quad \text{for every} \quad v \in C_0^\infty(\mathbb{R}^n)\}$, it follows from (3.3) that $u \in \mathcal{H}^*$ if and only if

$$u \text{ is smooth}, \quad \mathcal{E}^\eta(u,u) < \infty \quad \text{and} \quad \mathcal{L}u(x) = 0,\ x \in \mathbb{R}^n, \qquad (3.4)$$

where

$$\begin{aligned} &\mathcal{L}u(r,\theta_1,\cdots,\theta_{n-1}) \qquad (3.5)\\ = &\frac{1}{r^{n-1}}(u_r\cdot\eta(r)r^{n-1})_r + \frac{\eta(r)}{r^2\sin^{n-2}\theta_1}(u_{\theta_1}\sin^{n-2}\theta_1)_{\theta_1} + \frac{\eta(r)}{r^2\sin^2\theta_1\sin^{n-3}\theta_2}(u_{\theta_2}\sin^{n-3}\theta_2)_{\theta_2} \\ + &\cdots + \frac{\eta(r)}{r^2\sin^2\theta_1\cdots\sin^2\theta_{n-3}\sin\theta_{n-2}}(u_{\theta_{n-2}}\sin\theta_{n-2})_{\theta_{n-2}} + \frac{\eta(r)}{r^2\sin^2\theta_1\cdots\sin^2\theta_{n-2}}(u_{\theta_{n-1}})_{\theta_{n-1}} \end{aligned}$$

Now take any function $u \in \mathcal{H}^*$. We claim that

$$u_{\theta_{n-1}} = 0. \qquad (3.6)$$

Put $w = u_{\theta_{n-1}}$. Due to the expression (3.5) of $\mathcal{L}$, $\mathcal{L}w = (\mathcal{L}u)_{\theta_{n-1}} = 0$, namely, w is $\mathcal{L}$-harmonic. For $B_r = \{x \in \mathbb{R}^n; |x| < r\}$ and the uniform probability measure $\Pi(d\xi)$ on ∂B_1, w therefore admits the Poisson integral formula

$$w(x) = \int_{\partial B_1} K_r(x, r\xi)w(r\xi)\Pi(d\xi). \quad x \in B_r, \qquad (3.7)$$

where $K_r(x, r\xi)$ is the Poisson kernel for B_r with respect to $\mathcal{L}$, which is known to be continuous in $(x,\xi) \in B_r \times \partial B_1$. We also note that $K_r(0, r\xi) = 1$ for any $\xi \in \partial B_1$ by the rotation invariance of $\mathcal{L}$ around the origin 0.

Fix $a > 0$. It then holds For any $r > a$ that

$$K_r(x, r\xi_2) = \int_{\partial B_a} K_a(x, a\xi_1) K_r(a\xi_1, r\xi_2) \Pi(d\xi_1), \quad x \in B_q,\ \xi_2 \in \partial B_1.$$

Hence, if we let $\sup_{x \in B_{a/2}, \xi_1 \in \partial B_1} K_a(x, a\xi_1) = C_a < \infty$, then, for $x \in B_{a/2}$, $\xi_2 \in \partial B_1$,

$$K_r(x, r\xi_2) \le C_a \int_{\partial B_1} K_r(a\xi_1, r\xi_2) \Pi(d\xi_1) = C_a K_r(0, r\xi_2) = C_a,$$

and it follows from (3.7) that

$$|w(x)| \le C_a \int_{\partial B_1} |w(r\xi)| \Pi(d\xi), \quad x \in B_{a/2}, \quad r > a.$$

Recall that $w = u_{\theta_{n-1}}$. We multiply the both hand side of the above inequality by $r^{n-3}\eta(r)$, integrate in r from a to R, apply the Schwarz inequality and finally use the expression (3.3) to get

$$|u_{\theta_{n-1}}(x)| \le \frac{C_a}{\sqrt{\sigma_n}} \left[\int_a^R r^{n-3} \eta(r) dr \right]^{-1/2} \cdot \sqrt{\mathcal{E}^\eta(u, u)}, \quad x \in B_{a/2},$$

which tends to 0 as $R \to \infty$ by (3.2). Since $a > 0$ is arbitrary, we arrive at (3.6).

It also holds that

$$u_{\theta_k} = 0 \quad \text{for any} \quad 1 \le k \le n-1. \tag{3.8}$$

In fact, if we let $\xi_i = \frac{x_i}{r}$, $1 \le i \le n$, $\xi = (\xi_1, \cdots, \xi_n) \in \partial B_1$, then θ_k, $1 \le k \le n-1$, is an angle of two n-vectors $\xi^{(k)} = (\underbrace{0, \cdots 0}_{k-1}, \xi_k, \cdots, \xi_n)$, $\mathbf{e}_k = (\underbrace{0, \cdots 0}_{k-1}, 1, 0, \cdots, 0)$. Consider the subspace V of $\mathbb{R}^n$ spanned by ξ^k and $\mathbf{e}_k$ and take a unit vector $\widehat{\mathbf{e}}$ in V orthogonal to $\mathbf{e}_k$. Let O be an orthognal matrix whose $(n-1)$-th and n-th column vectors are $\mathbf{e}_k$ and $\widehat{\mathbf{e}}$, respectively. We make the orthogonal transformation $\mathbf{y} = {}^tO\mathbf{x}$. Then θ_k equals an angle of two vectors on the (y_{n-1}, y_n)-plane in the new coordinate system $\mathbf{y}$ and (3.6) applies.

Thus u depends only on r and, in terms of a scale function $ds(r) = \dfrac{dr}{\eta(r) r^{n-1}}$ on $(0, \infty)$, (3.3) and (3.6) are reduced, respectively, to

$$\mathcal{E}^\eta(u, u) = \sigma_n \int_0^\infty \left(\frac{du(r)}{ds(r)} \right)^2 ds(r), \qquad \mathcal{L}u(r) = \frac{1}{r^{n-1}} \frac{d}{dr} \cdot \frac{du(r)}{ds(r)}.$$

By (3.4), $\mathcal{L}u = 0$ so that $u(r) = C_1 + C_2\ s(r)$, $r > 0$, for some constant C_1, C_2. Since $\mathcal{E}^\eta(s, s) = \sigma_n \cdot s(0, \infty)$ is finite if and only if $n = 1$, we get the desired conclusions from (3.4). □

It is conjectured that the energy form $\mathcal{E}^\rho$ satisfies the Liouville property for any ρ prescribed in the above of (3.1) when $n \ge 2$.

The diffusion process Z on $\mathbb{R}^n$ associated with $\mathcal{E}^\rho$ is called the *distorted Brownian motion.* Let X be its time changed process defined as (1.1) by means of $m(dx) = f(x)dx$ for a strictly positive bounded function $f \in L^1(\mathbb{R}^n)$. Let $\mathbb{R}^n \cup \{\Delta\}$ be the one point compactification of $\mathbb{R}^n$. If $\mathcal{E}^\rho$ satisfies the Liouville property, then it can be shown as [CF1, Theorem 3.4] that any m-symmetric proper diffusion extension of X shares the same finite dimensional distribution with the one-point reflection of X at Δ. See [F2] for more details on these points.

References

[B] M. Brelot, Étude et extensions du principe de Dirichlet, *Ann. Inst. Fourier* **3**(1953/54), 371-419

[CF1] Z.-Q. Chen and M. Fukushima, On unique extension of time changed reflecting Brownian motions, *Ann. Inst. Henri Poincaré Probab. Statist.* **45** (2009), 864-875

[CF2] Z.-Q. Chen and M. Fukushima, *Symmetric Markov Processes, Time Change and Boundary Theory*, Princeton University Press, 2011

[CF3] Z.-Q. Chen and M. Fukushima, One-point reflections, *Stochastic Process Appl.* **125**(2015), 1368-1393 f Dirichlet forms, *Nagoya Math. J* **136**(1994), 1-15

[F1] M. Fukushima, From one dimensional diffusions to symmetric Markov processes, *Stochastic Process Appl.* **120**(2010), 590-604

[F2] M. Fukushima, A note on Liouville property of harmonic functions of finite energy, at Workshop on Probability at Kansai University, January 31, 2015

[FOT] M. Fukushima, Y. Oshima and M. Takeda, *Dirichlet forms and symmetric Markov processes*, de Gruyter, 1994, 2nd revised Edition, 2010

[FTa] M. Fukushima and M. Takeda, *Markov Processes* (in Japanese), Baifukan, Tokyo, 2008, Chinese translation by P.He, ed. by J. Ying, Science Press, Beijing, 2011

[FTo] M. Fukushima and M. Tomisaki, Construction and decomposition of reflecting diffusions on Lipschitz domains with Hölder cusps, *Probab. Theory Relat. Fields* **106**(1996), 521-557

[HK] D.A. Herron and P. Koskela, Uniform Sobolev extension and quasiconformal circle domains, *J. Anal. Math.* **57**(1991), 172-202

[MR] Z.-M. Ma and M. Röckner, *Introduction to the Theory of (non-symmetric) Dirichlet forms*, Springer, 1992

[P] Ross G. Pinsky, Transience/recurence for normally reflected Brownian motion in unbounded domains, *Ann. Probab.* **37**(2009), 676-686

Osaka University
E-mail address: fuku2@mx5.canvas.ne.jp

大阪大学　福島　正俊