数理解析研究所講究録
第 1952 巻 2015 年 55-62

Application of a Regime-switching Diffusion Process Model to Transport Phenomena in Surface Water Bodies

(邦題：地表水における輸送現象に対する状態遷移拡散過程モデルの応用)

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1. Introduction

Assessing transport phenomena in surface water bodies, such as advection-dispersion-deposition phenomena of suspended sediment (SS) particles in drainage canals and migration of fishes in stream networks, is an important research topic of hydro-environmental and ecological research areas because they are closely linked to a wide variety of real problems. Fluid flows in surface water bodies are inherently stochastic due to hydrodynamic disturbances such as turbulence, which make the transport phenomena of solute particles be stochastic. Movements of fishes can be regarded as stochastic transport phenomena as well not only due to the hydrodynamic disturbances but also due to ecological and biological disturbances. Despite the recognition of inherent stochasticity involved in the transport phenomena, most of the conventional researches have approached the problems using deterministic mathematical models, such as the lumped ordinary differential equation models and the Fickian-type models. In principle, these models cannot consistently handle the above-mentioned stochasticity, which on the other hand can be effectively handled with some of the stochastic process models. One of such models that have successfully been applied to assessing transport phenomena is the diffusion process model based on the stochastic differential equation (SDE) (Øksendal, 2007).

This paper presents a stochastic process model for approaching transport phenomena in surface water bodies with multiple regimes. The concept of Regime-Switching Diffusion Process (RSDP) model (Yin and Zhu, 2010) is presented for dealing with the transport phenomena. An RSDP model is identified with an SDE whose coefficients switch the regimes depending on a continuous time Markov chain valued in the finite and discrete state space. The SDE associates linear systems of parabolic partial differential equations (PDEs) governing the time-forward and time-backward evolution of the conditional probability density functions (PDFs), which are the systems of extended Kolmogorov's forward equations (KFEs) and those of extended Kolmogorov's backward equations (KBEs). One significant advantage of using the RSDP model instead of using the conventional diffusion process models is to closely link the system of extended KBEs with statistical quantities characterizing dynamics of the transport phenomena involving the multiple regimes. Statistical quantities conditioned on the backward variables, which are referred to as the spatially-distributed statistics (Yoshioka et al., 2012), can be calculated with the equations deduced from the system of extended KBEs. Examples of the important spatially-distributed statistics in applications are statistical moments of residence time, purification probability (deposition probability for suspended sediment dynamics), and escape probability. Calculating these statistics facilitates assessing transport phenomena in hydro-environments. So far, applications of the RSDP models have been mostly limited to the problems in mathematical finance and socio-economics, and only a few researches have focused on the transport phenomena in water bodies (Yoshioka et al., 2014a).

This article provides a theoretical framework of applying an RSDP model to analytical assessment of transport phenomena in surface water bodies. The problems considered in this paper are (1) advection-dispersion-deposition phenomena of SS particles in surface water bodies, which is a linear problem and (2) migrations of individual fishes in agricultural drainage systems, which on the other hand is a non-linear problem based on an energy minimization principle (Yoshioka et al., 2014b). The core of the latter problem is solving a system of extended Hamilton-Jacobi-Bellman equations (HJBEs) that governs the optimal ascending velocity of individual fishes. In this paper, only basic ideas for approaching these problems using the RSDP model are presented. Engineering applications of the model will be presented elsewhere.

2. Mathematical Models

The conventional diffusion process model and the RSDP model are presented in this section. Some fundamentals of the stochastic control problem focused on in the later section are also presented.

2.1 Conventional Diffusion Process Model

Assume that the domain of water flow Ω is given by a spatially n-dimensional connected open set in the real space $\mathbb{R}^n$. For dealing with transport phenomena in surface water bodies, the domain Ω is typically given by a three-dimensional domain, a vertically two-dimensional domain, a horizontally two-dimensional domain, or a locally one-dimensional open channel network (Yoshioka and Unami, 2013). This paper exclusively focuses on the one- and horizontally two- dimensional cases, which are of main interests in practical hydro-environmental researches.

The instantaneous position of a virtual particle, which represents either an SS particle or an individual fish in this paper, at the time t is denoted by $\mathbf{X}_t = \left[X_{i,t}\right]$. The process $\mathbf{X}_t$ is an n-dimensional stochastic process governed by the conventional single-regime Itô's SDE

$$\mathrm{d}\mathbf{X}_t = \mathbf{A}(t,\mathbf{X}_t)\mathrm{d}t + \sqrt{2\mathbf{B}(t,\mathbf{X}_t)}\mathrm{d}\mathbf{W}_t \tag{1}$$

where $\mathbf{W}_t$ is the n-dimensional standard Brownian motion, $\mathbf{A} = \left[a_i\right]$ is the n-dimensional drift coefficient vector, and $\mathbf{B} = \left[b_{i,j}\right]$ is the $n \times n$-dimensional diffusivity matrix. The SDE(1) governs the Lagrangian movements of individual particles where the stochasticity involved in the phenomena is lumped into the fluctuation term: the second term in the right-hand side of Eq.(1). The stochasticity is assumed to be Brownian whose magnitude is modulated by the diffusivity matrix $\mathbf{B}$.

2.2 Regime-switching Diffusion Process (RSDP) Model

The continuous time Markov chain taking values in the finite and discrete state space $M = \{0,1,2\ldots,K\}$ with the non-negative integer K is denoted by α_t. There exist totally $K+1$ regimes, which are referred to as the regimes 0 through K. The process α_t is assumed to be independent of the Brownian motion $\mathbf{W}_t$ driving the SDE(1). The process α_t satisfies the probabilistic equation

$$\Pr\left\{\alpha_{t+\Delta t} = l \middle| \alpha_t = k, \mathbf{X}_t = \mathbf{x}\right\} = \Delta_{k,l} + q_{k,l}(t,\mathbf{x})\Delta t + o(\Delta t) \tag{2}$$

for $0 \le k,l \le K$ where $\Delta_{k,l}$ is the Kronecker's Delta ($\Delta_{k,l} = 1$ if $k = l$ and $\Delta_{k,l} = 0$ otherwise), $q_{k,l}$ is the transition rate from the regime k to the regime l, and $o(\cdot)$ represents the Landau symbol. The $(K+1)$-dimensional square transition matrix $Q = \left[q_{k,l}\right]$, which is the generator of the stochastic process α_t, is assumed to consist of negative diagonal components and positive non-diagonal components, such that

$$\sum_{l=1}^{K} q_{k,l} = 0 \tag{3}$$

for each k (q-property) (Yin and Zhu, 2010). Utilizing the process α_t, the SDE(1) is extended to the regime-switching SDE as

$$\mathrm{d}\mathbf{X}_t = \mathbf{A}(t,\mathbf{X}_t,\alpha_t)\mathrm{d}t + \sqrt{2\mathbf{B}(t,\mathbf{X}_t,\alpha_t)}\mathrm{d}\mathbf{W}_t = \mathbf{A}^{(\alpha_t)}(t,\mathbf{X}_t)\mathrm{d}t + \sqrt{2\mathbf{B}^{(\alpha_t)}(t,\mathbf{X}_t)}\mathrm{d}\mathbf{W}_t \tag{4}$$

where the coefficients $\mathbf{A}$ and $\mathbf{B}$ now depend on the process α_t and the superscript $(\cdot)$ represents the regime. For each $k \in M$, the generator $L^{(k)}$ associated with the triplet process $Y_t = (t,\mathbf{X}_t,\alpha_t)$ is analytically given by

$$L^{(k)} f^{(k)} = \frac{\partial f^{(k)}}{\partial s} + \sum_{i=1}^{n} a_i^{(k)} \frac{\partial f^{(k)}}{\partial x_i} + \sum_{i,j=1}^{n} b_{i,j}^{(k)} \frac{\partial^2 f^{(k)}}{\partial x_i \partial x_j} + \sum_{m=1}^{K} q_{k,m} f^{(m)} \tag{5}$$

for generic sufficiently regular functions $f^{(k)} = f^{(k)}(s,\mathbf{x})$ ($0 \le k \le K$) conditioned on $Y_s = (s,\mathbf{x},k)$. The system of extended KBEs associated with the SDE(4) is expressed with Eq.(5) as

$$L^{(k)} p^{(k,l)} = 0 \quad (0 \le k,l \le K) \tag{6}$$

where $p^{(k,l)} = p(s,\mathbf{x},k,t,\mathbf{z},l)$ is the conditional PDF such that $Y_t = (t,\mathbf{z},l)$ conditioned on $Y_s = (s,\mathbf{x},k)$ for $s < t$. Eq.(6) is a parabolic system of PDEs. Although not presented in this paper, another parabolic system of PDEs governing time forward evolution of the PDFs, which is referred to as the system of extended KFEs, associates to the SDE(4) (Yin and Zhu, 2010).

2.3 Stochastic control problem with the RSDP model

When applying the RSDP model to a stochastic control problem, known functions in the model would depend on control variables taking values in an admissible set. A value function to be maximized has to be then specified, so that the control variables are optimized on the basis of the dynamic programing principle (Zhu, 2011). In this paper, the control variable is denoted by u and its admissible set by U. The control variable u is assumed to be a Markov control. The value function, which is denoted by $J^u(s,x,k)$, is set as

$$J^u(s,x,k) = \mathrm{E}^{s,x,k}\left[\int_s^T \alpha(s,x,\alpha_t)\mathrm{d}t + \beta(T,X_T,\alpha_T)\right] \tag{7}$$

for each k where $\mathrm{E}^{s,x,k}[\cdot]$ represents the expectation conditioned on $Y_s = (s,\mathbf{x},k)$, $T(>s)$ is the terminal time, α and β are functions to be appropriately determined for each problem. The maximized value function for each k is defined with Eq.(7) as

$$\Phi(s,x,k) = \Phi^{(k)} = \max_{u \in U} J^u(s,x,k) = J^{u^*}(s,x,k) \tag{8}$$

where u^* is the optimal control variable maximizing the value function. The dynamic programming principle states that the stochastic control problem reduces to solving the system of extended HJBEs governing the maximized value functions (Zhu, 2011), which is analytically given by

$$\max_{u \in U}\left(L^{(k)}\Phi^{(k)} + \alpha^{(k)}\right) = 0 \quad (0 \le k \le K) \tag{9}$$

that has to be equipped with appropriate terminal and boundary conditions for its well-posedness. Solving the system of extended HJBE(9) yields the optimal control variable as a function of $\Phi^{(k)}$.

3. Applications

Two hydro-environmental applications of the RSDP model are presented in this section. The first problem is advection-dispersion-deposition phenomena of SS particles in surface water bodies where the domain of water flows is given by either a horizontally two-dimensional shallow water body or a locally one-dimensional open channel network. In this problem, the regimes considered are the water column (regime 0) and the bed of the water body (regime 1). The deposition and re-suspension dynamics of SS particles, both of which play pivotal roles of driving the phenomena, are considered in the model. The second problem is migration of individual fishes in an agricultural drainage system associated with paddy fields, which is identified with a locally one-dimensional open channel network. In this problem, the regimes considered are the channel network (regime 0) and the plots of a paddy field (regimes 1 through K). It is shown that the governing system of extended HJBEs can be analytically reduced to an extended HJBE having singular source terms.

3.1 Advection-dispersion-deposition phenomena of SS particles

Advection-dispersion-deposition phenomena of SS particles are described with the RSDP model. Although the governing equations are derived for the one-dimensional case for the sake of simplicity, their extension to the horizontally two-dimensional case does not encounter technical difficulties (Takagi et al., 2014). The domain Ω in the present case is a locally one-dimensional open channel network. Two distinct regimes of vertical particle positions are considered, which are the regimes 0 and 1 where the former corresponds to the water column and the latter corresponds to the channel bed. The one-dimensional coordinate taken along each reach of the channel network is denoted by x. The instantaneous position of an SS particle at the time t is denoted by X_t. The flow velocity of water in the channel is denoted by V and the dispersivity for the SS particle by $D(>0)$, namely $a^{(0)} = V$ and $b^{(0)} = D$. The SS particle is assumed not to horizontally move on the bed, namely $a^{(1)} = 0$ and $b^{(1)} = 0$, which reduces Eq.(4) to the trivial equation

$$\mathrm{d}X_t = 0 \tag{10}$$

when $\alpha_t = 1$.

There are two physically important variables determining vertical SS particles movements, which are the deposition rate $R_{\mathrm{W}\to\mathrm{B}}$ and the re-suspension rate $R_{\mathrm{B}\to\mathrm{W}}$. The variables $R_{\mathrm{W}\to\mathrm{B}}$ and $R_{\mathrm{B}\to\mathrm{W}}$ are inverses of the mean required times of SS particles from the regimes 0 to 1 and from the regimes 1 to 0, respectively. Following Thonon et al. (2007), the deposition rate is specified as $R_{\mathrm{W}\to\mathrm{B}} = \lambda w_{\mathrm{S}} h^{-1}(>0)$ where $\lambda(>0)$ is a constant, $w_{\mathrm{S}}(>0)$ is the particle settling velocity (Rubey, 1933) determined from the particle geometry, and $h(>0)$ is the water depth. The re-suspension rate is defined as $R_{\mathrm{B}\to\mathrm{W}} = g_0 F\left(\max\left(\sigma_{\mathrm{B}} - \sigma_{\mathrm{c}}, 0\right)\right)(\geq 0)$ where g_0 is a positive coefficient, F is a non-decreasing function with $F(0) = 0$, $\sigma_{\mathrm{B}}(>0)$ is the magnitude of the shear stress on the channel bed, and $\sigma_{\mathrm{c}}(>0)$ is the magnitude of the critical shear stress. By the definition, $R_{\mathrm{B}\to\mathrm{W}} = 0$ if and only if $\sigma_{\mathrm{B}} \leq \sigma_{\mathrm{c}}$. Based on the deposition and re-suspension rates, the transition rates are determined as $q_{00} = -q_{01} = -R_{\mathrm{W}\to\mathrm{B}}(<0)$ and $q_{11} = -q_{10} = -R_{\mathrm{B}\to\mathrm{W}}(\leq 0)$. The generators $L^{(k)}$ ($k = 0,1$) for the triplet process $Y_t = (t, X_t, \alpha_t)$ conditioned on $Y_s = (s, x, k)$ are expressed as

$$L^{(k)}f^{(k)} = \begin{cases} \dfrac{\partial f^{(0)}}{\partial s} + V\dfrac{\partial f^{(0)}}{\partial x} + D\dfrac{\partial^2 f^{(0)}}{\partial x^2} - R_{\mathrm{W}\to\mathrm{B}} f^{(0)} + R_{\mathrm{B}\to\mathrm{W}} f^{(1)} & (k=0) \\ \dfrac{\partial f^{(1)}}{\partial s} + R_{\mathrm{W}\to\mathrm{B}} f^{(0)} - R_{\mathrm{B}\to\mathrm{W}} f^{(1)} & (k=1) \end{cases} \tag{11}$$

for generic sufficiently regular functions $f^{(0)} = f^{(0)}(s,x)$ and $f^{(1)} = f^{(1)}(s,x)$.

The concept deposition probability is introduced for quantifying the stochasticity involved in the phenomena. For a sub-domain $G \subset \Omega$, the deposition probability $P_{\mathrm{d}} = P_{\mathrm{d}}(s,\mathbf{x},G)$ is defined as

$$\begin{aligned} P_{\mathrm{d}}(s,x,G) &= \Pr\left\{ X_{\tau^{s,x}} \in G, \alpha_{\tau^{s,x}} = 1, \sigma_{\mathrm{B}} < \sigma_{\mathrm{c}} \middle| X_s = x, \alpha_s = 0 \right\} \\ &= \Pr\left\{ X_{\tau^{s,x}} \in G, \alpha_{\tau^{s,x}} = 1, \mathrm{R}_{\mathrm{B}\to\mathrm{W}} = 0 \middle| X_s = x, \alpha_s = 0 \right\} \end{aligned} \tag{12}$$

with the stopping time $\tau^{s,x}$ given by

$$\tau^{s,x} = \inf\left\{ t \middle| t > s, \alpha_t = 1, X_s = x, \alpha_s = 0 \right\}. \tag{13}$$

The deposition probability P_{d} is the probability that an SS particles at the position x in the water column (regime 0) at the time s finally deposits to the bed (regime 1) in the sub-domain $G \subset \Omega$. By the definition, an alternative expression of P_{d} is deduced as (Yoshioka et al., 2014a)

$$P_{\mathrm{d}}(s,x,G) = \int_s^{+\infty} \int_G p(s,x,0,t,z,0)\, q_{0,1}(t,z)\,\mathrm{d}z\mathrm{d}t = \int_s^{+\infty} \int_G R_{\mathrm{W}\to\mathrm{B}}\, p^{(0,0)}\,\mathrm{d}z\mathrm{d}t \tag{14}$$

because the quantity $R_{\mathrm{W}\to\mathrm{B}}\, p^{(0,0)}$ represents the conditional PDF that an SS particle at the position x in the water body (regime 0) at the time s deposits to the bed (regime 1) at the position z at the time t. Application of the generator $L^{(0)}$ to the both-hand sides of Eq.(14) yields

$$L^{(0)} P^{(0)} + \chi_G \chi_{(R_{\mathrm{B}\to\mathrm{W}}=0)} R_{\mathrm{W}\to\mathrm{B}} = 0 \tag{15}$$

because of the equality

$$p(s,x,k,s,z,l) = \Delta_{k,l}\,\delta(x-z) \tag{16}$$

where δ represents the Dirac Delta and χ_θ represents the characteristic function for generic set θ. Eq.(16) means that a particle at a point at an instance cannot occupy more than one regimes, which is a physically obvious assumption. Eq.(15) has to be equipped with appropriate terminal and boundary conditions for well-posedness. For time-independent cases where the considered system is autonomous, Eq.(15) reduces to

$$V\frac{\partial P_{\mathrm{d}}}{\partial x} + D\frac{\partial^2 P_{\mathrm{d}}}{\partial x^2} + R_{\mathrm{W}\to\mathrm{B}}\left(\chi_G \chi_{(R_{\mathrm{B}\to\mathrm{W}}=0)} - P_{\mathrm{d}} \right) = 0, \tag{17}$$

which is a linear elliptic differential equation having a discontinuous source term. Solving Eq.(17) achieves a probabilistic assessment of SS capturing efficiency of the water body.

3.2 Migration of individual fishes

A surface agricultural drainage system that drains waters from plots of a paddy field is considered. The drainage system is identified with a locally 1-D open channel network, which is denoted by Ω. In total $K+1$ regimes are involved in this problem where the regime 0 corresponds to the water column in the channel network and the regimes 1 through K to the distinct plots of the paddy field serving as still waters. It is assumed that individual fishes migrate from the channel network to one of the plots and that they do not descend down from the plots to the channel network. The coefficients $a^{(k)}$ and $b^{(k)}$ for the regimes 1 through K are thus set as 0. It is reasonable to assume that there is no direct transition between different plots. Transition from the channel network to each plot occurs only through its outlet, which is modeled with the Deltaic transition rates as

$$q_{0,0} = -\sum_{l=1}^{M} q_{0,l}, \quad q_{0,l} = \delta_l R_l \quad (1 \le l \le K), \text{ and } q_{k,l} = 0 \quad (1 \le k \le K, 0 \le l \le K) \tag{18}$$

where δ_l is the Dirac's Delta concentrated at the point x_l at which the outlet of the l th plot of the paddy field is located and $R_l (\ge 0)$ is the ascending rate from the channel to the l th plot. The drift $a^{(0)}$ of the SDE(4) is specified as $V - u$ where u represents the migration velocity of individual fish and the positive direction of u is taken to be same with that of $-x$. The migration velocity u is the control variable of the model, which is assumed to be constrained in the admissible set U given by

$$U = \{u \mid |u| \le u_{\mathrm{M}}\} \tag{19}$$

with $u_{\mathrm{M}} (> 0)$, which is the maximum swimming speed of the fishes that would vary in both space and time depending on local hydraulic and biological conditions. The dispersivity $b^{(0)}$ of the fish is denoted by $D(> 0)$. The coefficients V and D are assumed not to involve the control variable u. The generators $L^{(k)}$ ($1 \le k \le K$) for the triplet process $Y_t = (t, X_t, \alpha_t)$ conditioned on $Y_s = (s, x, k)$ are expressed as

$$L^{(k)} f^{(k)} = \begin{cases} \dfrac{\partial f^{(0)}}{\partial s} + (V - u)\dfrac{\partial f^{(0)}}{\partial x} + D\dfrac{\partial^2 f^{(0)}}{\partial x^2} - \left(\sum_{j=1}^{K} \delta_j R_j\right) f^{(0)} + \sum_{j=1}^{K} \delta_j R_j f^{(j)} & (k = 0) \\ \dfrac{\partial f^{(k)}}{\partial s} & (1 \le k \le K) \end{cases} . \tag{20}$$

for generic sufficiently regular functions $f^{(k)} = f^{(k)}(s, x)$ ($0 \le k \le K$).

Assuming that each fish conditioned on $Y_s = (s, x, 0)$ strategically migrates from each point x in the channel network to one of the plots based on a minimization principle of the physiological energy consumption (Yoshioka et al., 2014b), the value function J^u to be maximized is proposed as

$$J^u(s, x, k) = \mathrm{E}^{s,x,k}\left[\int_s^{\bar{T}} \left(-\frac{1}{2}u^2\right) \chi_{\{\alpha_t = 0\}} \mathrm{d}t + G(\bar{T}, Y_{\bar{T}})\right] \tag{21}$$

with

$$\bar{T} = \min(T, \tau), \quad \tau = \min_{1 \le l \le K} \tau_l, \text{ and } \tau_l = \inf\{t \mid t > s, X_t = x_l, \alpha_t = l, X_s = x, \alpha_s = k\} \tag{22}$$

where τ is the first exit time of the process X_t from the open channel network to one of the plots, τ_l is the first exit time of the process X_t from the channel network to the l th plot, and $G(\ge 0)$ is the profit specified on the boundary $\partial\Xi$ of the space-time domain $\Xi = (s, T) \times \Omega$. The profit G is specified on $\partial\Xi$ as

$$G(s, x_{\mathrm{D}}) = G(s, x_{\mathrm{U}}) = 0 \text{ and } G(T, x) = 0 \text{ for } k = 0 \tag{23}$$

and

$$G(s, x_{\mathrm{D}}) = G(s, x_{\mathrm{U}}) = P \text{ and } G(T, x) = P \text{ for } 1 \le k \le K \tag{24}$$

where $P(> 0)$ is a constant, and x_{U} and x_{D} represent the points at the upstream- and the downstream- ends of the domain Ω, respectively. Eqs.(23) and (24) mean that the fish gains the profit if and only if it approaches one of the plots. The optimal control variable maximizing the value function J^u is denoted by u^*. On the basis of the dynamic programming principle, the extended HJBE governing the maximized value function

$$\Phi^{(k)} = J^{u^*}(s, x, k) \tag{25}$$

for $k=0$ is derived as

$$\sup_{u\in U}\left(L^{(0)}\Phi^{(0)}-\frac{1}{2}u^2\right)=\frac{\partial\Phi^{(0)}}{\partial s}+D\frac{\partial^2\Phi^{(0)}}{\partial x^2}+\left(\sum_{k=1}^{K}\delta_k R_k\right)\left(P-\Phi^{(0)}\right)+\left((V-u)\frac{\partial\Phi^{(0)}}{\partial x}-\frac{1}{2}u^2\right)_{u=u^*}=0 \quad (26)$$

because $\Phi^{(k)}=P$ for $1\le k\le K$. The optimal control variable u^* is analytically expressed with the solution $\Phi^{(0)}$ as

$$u^*=-\chi\frac{\partial\Phi^{(0)}}{\partial x}-(1-\chi)u_{\mathrm{M}}\,\mathrm{sgn}\left(\frac{\partial\Phi^{(0)}}{\partial x}\right) \quad (27)$$

with the abbreviation $\chi=\chi_{\left\{\left|\frac{\partial\Phi^{(0)}}{\partial x}\right|\le u_{\mathrm{M}}\right\}}$. Substituting Eq.(27) into Eq.(26) yields

$$\frac{\partial\Phi^{(0)}}{\partial s}+(V-v)\frac{\partial\Phi^{(0)}}{\partial x}+D\frac{\partial^2\Phi^{(0)}}{\partial x^2}+\left(\sum_{j=1}^{K}\delta_j R_j\right)\left(P-\Phi^{(0)}\right)-\frac{1-\chi}{2}u_{\mathrm{M}}^2=0 \quad (28)$$

with the auxiliary variable v defined by

$$v=\frac{\chi}{2}\frac{\partial\Phi^{(0)}}{\partial x}+(1-\chi)u_{\mathrm{M}}\,\mathrm{sgn}\left(\frac{\partial\Phi^{(0)}}{\partial x}\right). \quad (29)$$

The extended HJBE(28) is a non-linear and non-conservative parabolic PDE that can be numerically solved with a finite element method utilizing an appropriate stabilization and regularization techniques (Yoshioka et al., 2015). Analytical assessment of the optimal migration strategy of fishes can be performed by solving the extended HJBE(28) that provides u^*.

Once the optimal control variable u^* is calculated from the extended HJBE(28), a variety of spatially-distributed statistics characterizing the migration of fishes can be assessed. One of such examples is the ascending probability A_{P} defined as

$$A_{\mathrm{P}}(s,x)=\Pr\left\{X_\tau\in\omega,\alpha_\tau=1,\tau<T\,\middle|\,X_s=x,\alpha_s=0\right\}, \quad (30)$$

which is the probability that a fish at the time s at the position x in the drainage system reaches one of the plots of the paddy field by the terminal time T. By the help of Dynkin's formula (Øksendal, 2007), the governing equation of the ascending probability A_{P} is derived as

$$\frac{\partial A_{\mathrm{P}}}{\partial s}+\left(V-u^*\right)\frac{\partial A_{\mathrm{P}}}{\partial x}+D\frac{\partial^2 A_{\mathrm{P}}}{\partial x^2}+\left(\sum_{j=1}^{K}\delta_j R_j\right)\left(1-A_{\mathrm{P}}\right)=0, \quad (31)$$

which is a linear parabolic PDE having deltaic source terms.

4. Conclusions

An RSDP model for analytically assessing transport phenomena in surface water bodies was proposed and its applications to the advection-dispersion-deposition phenomena of SS particles and the migration of individual fishes were presented. Although the former problem may also be approached based on the deterministic mathematical models assuming the Fick's type laws, such models cannot consistently handle the stochasticity involved in the phenomena. This consistency issue is not encountered in the RSDP model because it can quantify the stochasticity using the system of extended KBEs, which leads to the governing equations of the spatially-distributed statistics. Similarly, the latter problem cannot be dealt with using deterministic models. The present mathematical framework for analytically assessing the migration of fishes, which utilizes the extended KBEs and the extended HJBEs, is attractive because of its potential ability in describing the phenomena considering the biological and ecological feedback mechanisms.

Future researches will address applications of the RSDP model to the other transport phenomena, such as advection-dispersion-reaction phenomena of colloidal particles and migration of individual fishes with moving and resting regimes. Detailed mathematical analysis on the RSDP model is also an important research topic to be addressed in future researches.

Acknowledgements

This research is supported by JSPS under grant No. 25·2731. The first author received financial supports from RIMS for attending the conference "Probability Symposium". The authors thank to the attendees of the conference for their valuable suggestions and comments on our research.

References

[1] Øksendal, B. (2007) *Stochastic Differential Equations*, Springer-Verlag, Berlin, pp. 1-167.

[2] Rubey, W.W. (1933) Settling velocities of gravel, sand and silt particles. *American Journal of Science*, Ser. 5, Vol. 25, pp. 325-338.

[3] Takagi, K., Yoshioka, H., Unami, K., and Fujihara, M. (2014) Analytical assessment of water purification ability of freshwater lagoons receiving agricultural runoff, *Proc. PAWEES2014*, p. 185.

[4] Thonon, I., De Jong, K., Van der Perk, M., and Middelkoop, H. (2007) Modeling floodplain sedimentation using particle tracking, *Hydrological Processes*, Vol. 21, No. 11, pp. 1401-1412.

[5] Unami, K., Ishida, K., Kawachi, T., Maeda, S., and Takeuchi, J. (2010) A stochastic model for behavior of fish ascending an agricultural drainage system, *Paddy and Water Environment*, Vol. 8, No. 2, pp. 105-111.

[6] Yin, G.G. and Zhu, C. (2010) *Hybrid Switching Diffusions*, Springer Science+Business Media, LLC, pp. 1-67.

[7] Yoshioka, H., Unami, K., and Kawachi, T. (2012) Partial differential equation model for spatially distributed statistics of contaminant particles in locally one-dimensional open channel networks, *Proc. The Tenth International Conference on Hydroscience and Engineering*, 24 pp.

[8] Yoshioka, H. and Unami, K. (2013) A cell-vertex finite volume scheme for solute transport equations in open channel networks, *Probabilistic Engineering Mechanics*, Vol. 31, pp. 30-38.

[9] Yoshioka, H., Unami, K., and Fujihara, M. (2014a) A regime-switching diffusion process model for longitudinal dispersion phenomena in vegetated open channels, *Proc. The 63rd National Congress of Theoretical and Applied Mechanics*, OS 15-02-03, pp. 1-2.

[10] Yoshioka, H., Unami, K., and Fujihara, M. (2014b) Mathematical and numerical analyses on a Hamilton-Jacobi-Bellman equation governing ascending behaviour of fishes, *RIMS Kôkyûroku: Mechanisms and Mathematical Aspects of Nonlinear Wave Phenomena*. (to appear).

[11] Yoshioka, H., Unami, K., and Fujihara, M. (2015) A conforming finite element scheme for Hamilton-Jacobi-Bellman equations defined on connected graphs, *Proc. Computational Engineering Conference*, Vol. 20. (to appear).

[12] Zhu, C. (2011) Optimal control of the risk process in a regime-switching environment, *Automatica*, Vol. 47, No. 8, pp. 1570-1579.