Affine-invariant quadruple systems

By

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§ 1. Introduction

Let \( t, v, k, \lambda \) be positive integers satisfying \( v > k > t \). A \( t-(v, k, \lambda) \) design is an ordered pair \((V, B)\), where \( V \) is a finite set of \( v \) points, \( B \) is a collection of \( k \)-subsets of \( V \), say blocks, such that every \( t \)-subset of \( V \) occurs in exactly \( \lambda \) blocks in \( B \). In what follows we simply write \( t \)-designs. A 3-\((v, 4, 1)\) design is called a Steiner quadruple system and denoted by \( \text{SQS}(v) \). It is known that an \( \text{SQS}(v) \) exists if and only if \( v \equiv 2, 4 \mod 6 \) (see [9]). For \( \lambda > 1 \), a 3-\((v, 4, \lambda)\) design is called a \( \lambda \)-fold quadruple system and denoted by \( \lambda \)-fold \( \text{QS}(v) \) for short.

An automorphism group \( G \) of a \( t \)-design \((V, B)\) is a permutation group defined on \( V \) which leaves \( B \) invariant. For a fixed block \( B \in B \), the orbit of \( B \) under \( G \) is \( \mathcal{O}_G(B) = \{ B^g | g \in G \} \). Thus, \( B \) can be partitioned into orbits under \( G \), say \( G \)-orbits. Moreover, if the cardinality of an orbit \( \mathcal{O} \) equals to the order of \( G \), then \( \mathcal{O} \) is said to be full, otherwise, short. Any block in \( \mathcal{O} \) can be regarded as a base block of the orbit.

In particular, a \( t-(v, k, \lambda) \)-design is said to be cyclic if it admits a cyclic group \( C_v \) of order \( v \) as its automorphism. A \( C_v \)-orbit is called a cyclic orbit. Without loss of generality, we identify the point set of a cyclic \( t \)-design with the additive group of \( \mathbb{Z}_v = \mathbb{Z}/v\mathbb{Z} \), the integers modulo \( v \). Furthermore, a cyclic \( t \)-design is said to be strictly cyclic, if all cyclic orbits are full. In what follows, we denote a cyclic \( \text{SQS} \) by \( \text{CSQS} \), a strictly cyclic \( \text{SQS} \) by \( \text{sSQS} \). The necessary conditions for the existence of a \( \text{CSQS}(v) \) and an \( \text{sSQS}(v) \) are \( v \equiv 2, 4 \mod 6 \) and \( v \equiv 2, 10 \mod 24 \) respectively (see [12]).

The work on \( \text{sSQS} \) by Köhler [12] established a connection between \( \text{sSQS} \) and 1-factors of “Köhler graphs” named after him. Some approaches to Köhler’s work by Siemon [23] [24] checked the existence of 1-factors of “Köhler graphs” for quite a few admissible parameters. Piotrowski [22] constructed \( \text{sSQS}(2p) \) admitting the dihedral
group $D_{2p}$ as automorphism. For more information on CSQS and SQS with other specified automorphism groups, the reader may refer to Lindner and Rosa [17], Grannel and Griggs [8], Hartman and Phelps [10], Munemasa and Sawa [21].

Let $(\mathbb{Z}_v, \mathcal{B})$ be an sSQS. For any $B \in \mathcal{B}$ and $\tau : x \mapsto \alpha x, \alpha \in \mathbb{Z}_v^\times$ if $B^\tau \in \mathcal{B}$, then $\alpha$ is called a multiplier of $(\mathbb{Z}_v, \mathcal{B})$, where $\mathbb{Z}_v^\times$ is the multiplicative group of $\mathbb{Z}_v$, i.e. the group of all units of $\mathbb{Z}_v$.

**Definition 1.1.** For an sSQS $(\mathbb{Z}_v, \mathcal{B})$, if all the units of $\mathbb{Z}_v$ are multipliers, then $(\mathbb{Z}_v, \mathcal{B})$ is said to be affine-invariant.

In another words, an affine-invariant sSQS $(\mathbb{Z}_v, \mathcal{B})$ admits the affine group $A$ as an automorphism, where $A$ is defined by $A = \{(i, \alpha) \mid i \in \mathbb{Z}_v, \alpha \in \mathbb{Z}_v^\times \} \cong \mathbb{Z}_v \times \mathbb{Z}_v^\times$. Given a quadruple $B$, denote the orbit of $B$ under the affine group $A$ by $\mathcal{O}_A(B)$, say an affine orbit.

**Example 1.2.** The unique (up to isomorphism) SQS(10) is affine-invariant strictly cyclic. Let $\mathbb{Z}_{10}$ be its point set. Let $B_1 = \{0, 1, 5, 9\}$, $B_2 = \{0, 2, 5, 8\}$, $B_3 = \{0, 1, 3, 4\}$

be base blocks of the cyclic orbits. We have $B_1 \times 3+5 = \{0, 3, 5, 7\} + 5 = \{5, 8, 0, 2\} = B_2$ over $\mathbb{Z}_{10}$. Hence, the cyclic orbits of $B_1$ and $B_2$ are contained in the same affine orbit. In fact, there are two affine orbits having $B_1$ or $B_2$ and $B_3$ as base blocks respectively.

In general, for $3-(v, 4, \lambda)$ designs admitting the affine group, we also say they are affine-invariant. Affine-invariant $3-(p, 4, \lambda)$ designs were first proposed by Köhler [14] for odd primes $p$ and admissible $\lambda$ by means of some graph $KG(p)$. Along this direction, Brand and Sutinuntopas [4] generalized Köhler’s results to finite fields. In particular, we denote a 2-fold quadruple system of order $v$ by $2QS(v)$ for short.

**Theorem 1.3 (Köhler [14]).** If the graph $KG(p)$ has a 1-factor, then

(i) an affine-invariant $3-(p, 4, 2)$ designs exists, for $p \equiv 1, 5 \pmod{12}$ and

(ii) an affine-invariant $3-(p, 4, 4)$ designs exists, for $p \equiv 7, 11 \pmod{12}$.

Approach on sSQS (i.e., $\lambda = 1$) is less known. Yoshikawa [29] presented the following results in his master thesis.

**Theorem 1.4 (Yoshikawa [29]).** There exists an affine-invariant sSQS(2p), for prime $p \equiv 1, 5 \pmod{12}$ and $5 \leq p < 200$, i.e., $p \in \{5, 17, 29, 37, 41, 53, 61, 73, 89, 97, 101, 109, 113, 137, 149, 157, 173, 181, 193, 197\}$.
§ 2. A family of graphs associated with PSL(2,p)

In this section, we introduce a family of graphs which play important roles in our constructions. Suppose $p$ is a prime with $p \equiv 1,5 \pmod{12}$. Let $\mathbb{F}_p$ denote the finite field of order $p$. Denote the 1-dimensional projective line by $\mathcal{P}(\mathbb{F}_p)$ which can be identified with $\mathbb{F}_p \cup \{\infty\}$.

Let $\sigma_A : x \mapsto 1-x, \sigma_B : x \mapsto \frac{1}{x}, \sigma_C : x \mapsto \frac{1-x}{1-2x}$ be mapping in $PSL(2,p)$. Let $x \in \mathcal{P}(\mathbb{F}_p)$. Denote the orbit of $x$ under the subgroups $\langle \sigma_A, \sigma_B \rangle$ by $C(x)$, i.e.,

$$C(x) = \{x^{\sigma} | \sigma \in \langle \sigma_A, \sigma_B \rangle\} = \left\{x, \frac{1}{x}, \frac{x-1}{x}, \frac{x}{x-1}, \frac{1}{1-x}, 1-x\right\}.$$

Thus $\mathcal{P}(\mathbb{F}_p)$ can be partitioned into $\{C(x) | x \in \mathcal{P}(\mathbb{F}_p)\}$. In projective geometry, $C(x)$ is also called the cross-ratio class with respect to $x$. The cardinality of $C(x)$ is established as follows.

$$|C(x)| = \begin{cases} 3 & \text{if } x \in \{0,1,\infty\} \cup \{-1, 2, 2^{-1}\}; \\ 2 & \text{if } x \in \{\xi, 1-\xi\}; \\ 6 & \text{otherwise}, \end{cases}$$

where $\xi = \frac{1+\sqrt{-3}}{2}$ is a root of $x^2 - x + 1 = 0$, when $p \equiv 1 \pmod{3}$.

**Definition 2.1.** Let $CG(\mathcal{P}(\mathbb{F}_p))$ be a graph (multigraph with loops) with vertex set $V = \{C(x) | x \in \mathcal{P}(\mathbb{F}_p)\}$. For any pair of vertices (not necessarily distinct) $C, C' \in V$, let $C$ be adjacent to $C'$ by $r_{C,C'}$ edges, where $r_{C,C'} = \frac{1}{2} |\{x | x \in C, x^{\sigma} \in C\}|$.

Let $\Omega_p = \mathbb{F}_p \setminus \{0,1,-1,2,2^{-1}\}$. Let $CG(\Omega_p)$ denote the induced subgraph on $\{C(x) | x \in \Omega_p\}$ of $CG(\mathcal{P}(\mathbb{F}_p))$. In another word, by removing the vertices $C(0)$ and $C(2)$ from $CG(\mathcal{P}(\mathbb{F}_p))$, the resulting graph is $CG(\Omega_p)$. Let $CG^*(\Omega_p)$ denote the resulting graph (possibly having multiple edges) obtained by removing all loops from $CG(\Omega_p)$.

**Lemma 2.2 ([19],[18]).** For $p > 17$, in $CG^*(\Omega_p)$, all the vertices have degree 3 except the following.

(i) $C(3)$ has degree 2;

(ii) For $p \equiv 1 \pmod{12}$, $C(\xi)$ has degree 1, where $\xi = \frac{1+\sqrt{-3}}{2}$ is a root of $x^2 - x + 1 = 0$;

(iii) $C(\chi)$ has degree 2, where $\chi = \frac{1+\sqrt{-1}}{2}$ is a root of $2x^2 - 2x + 1 = 0$;
For $p \equiv 1, 29, 41, 49 \pmod{60}$, $C(\mu)$ has degree 1, where $\mu = \frac{3+\sqrt{5}}{2}$ is a root of $x^2 - 3x + 1 = 0$.

\textbf{Theorem 2.3} ([19],[18]). For $p \equiv 1, 5 \pmod{12}$ and $p \not\equiv 1, 49 \pmod{60}$, $CG(\Omega_p)$ has a 1-factor if it has no bridge besides its pendant edge.

\section{Direct Constructions of affine-invariant $s$SQS($2p$)}

Suppose $(\mathbb{Z}_{2p}, \mathcal{B})$ is an affine-invariant sSQS, where $p \equiv 1, 5 \pmod{12}$ is prime, which satisfies the necessary condition for the existence of an sSQS(2p) (see [12]). Denote the set of nonzero elements of the finite field $\mathbb{Z}_p$ by $\mathbb{Z}_p^*$. We identify the point set $\mathbb{Z}_{2p}$ with $\mathbb{Z}_p \times \mathbb{Z}_2$, and denote the point $(x, y)$ by $x_y$ for convenience. Additions and multiplications over $\mathbb{Z}_p \times \mathbb{Z}_2$ are defined as follows:

\begin{align*}
  x_y + x'_y &= (x + x')(y + y') \\
  xx' &= (xx')(yy')
\end{align*}

where $x + x'$, $xx'$ are addition and multiplication modulo $p$, and $y + y'$, $yy'$ are addition and multiplication modulo 2. For an sSQS $(\mathbb{Z}_p \times \mathbb{Z}_2, \mathcal{B})$, we classify all blocks (quadruples) in $\mathcal{B}$ into three types.

Type I contains all the quadruples of form $\{a_0, b_0, c_1, d_1\}$, simply denoted by $\{a, b; c, d\}$, where $a \neq b$ and $c \neq d$.

Type II contains all the quadruples of form $\{a_0, b_0, c_0, d_1\}$ or $\{a_1, b_1, c_1, d_0\}$ simply denoted by $\{a, b, c; d\}$, where $a, b, c$ are pairwise distinct.

Type III contains all the quadruples of form $\{a_0, b_0, c_0, d_0\}$ or $\{a_1, b_1, c_1, d_1\}$, simply denoted by $\{a, b, c, d\}$, where $a, b, c, d$ are pairwise distinct.

Similarly, the triples of form $\{a_0, b_0, c_0\}$ or $\{a_1, b_1, c_1\}$ are called \emph{pure triples}, simply denoted by $\{a, b, c\}$, and the triples of form $\{a_0, b_0, c_1\}$ or $\{a_1, b_1, c_0\}$ are called \emph{mixed triples} simply denoted by $\{a, b; c\}$. Clearly, pure triples are contained in Type II and (or) III quadruples, and mixed triples are contained in Type I and (or) II quadruples.

\textbf{Construction 3.1} ([19]). If $CG(\Omega_p)$ has a 1-factor, let $a_1, a_2, \ldots, a_{\lceil \frac{p}{2} \rceil}$ be elements in $\Omega_p$, such that

\begin{equation*}
  E(F) = \left\{ \{C(a_1), C(a_1^{\sigma_C})\}, \{C(a_2), C(a_2^{\sigma_C})\}, \ldots, \{C(a_{\lceil \frac{p}{2} \rceil}), C(a_{\lceil \frac{p}{2} \rceil}^{\sigma_C})\} \right\},
\end{equation*}

is the edge set of $F$. 

Let $b_1, b_2, \ldots, b_{\frac{p-1}{4}}$ be elements in $\mathbb{Z}_p \setminus \{0, 1, 2^{-1}\}$, such that

$$\{\text{orb}_{AC}(b_i) \mid i = 1, 2, \ldots, \frac{p-1}{4}\} = \{\text{orb}_{AC}(b) \mid b \in \mathbb{Z}_p \setminus \{0, 1, 2^{-1}\}\},$$

where $\text{orb}_{AC}(b) = \{b, 1-b, \frac{b}{2b-1}, \frac{1-b}{1-2b}\}$.

All base blocks of affine-invariant sSQS(2p) are shown as follows.

(i) For $p \equiv 1 \pmod{12}$,

- Type I, $\{0, 1; b_i, 1-b_i\}$, for $i = 1, 2, \ldots, \frac{p-1}{4}$,
- Type II', $\{0, 1, -1; 0\}$,
- Type III', $\{0, 1, a_i, 1-a_i\}$, for $i = 1, 2, \ldots, \frac{p-13}{12}$, $a_i \not\in C(\xi) \cup C(\xi^{\sigma c})$,

where $\xi = \frac{1 + \sqrt{-3}}{2}$ is a root of $x^2 - x + 1 = 0$ over $\mathbb{Z}_p$.

(ii) For $p \equiv 5 \pmod{12}$,

- Type I, $\{0, 1; b_i, 1-b_i\}$, for $i = 1, 2, \ldots, \frac{p-1}{4}$,
- Type II', $\{0, 1, -1; 0\}$,
- Type III, $\{0, 1, a_i, 1-a_i\}$, for $i = 1, 2, \ldots, \frac{p-5}{12}$,

§ 4. Recursive Constructions of affine-invariant sSQS(2p^m)

Let $p \equiv 5 \pmod{12}$. We begin by giving the recursive construction of an affine-invariant sSQS over $\mathbb{Z}_{2p^2} \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_2$ from the affine-invariant sSQS over $\mathbb{Z}_{2p} \cong \mathbb{Z}_p \times \mathbb{Z}_2$.

Construction 4.1 ([20]). For prime $p \equiv 5 \pmod{12}$, the base blocks of the affine-invariant sSQS(2p^2) are

Type I' $\{0, 1; \alpha, \beta\}$

Type I $\{0, 1; b_i + sp, 1 - (b_i + sp)\}$, for $i = 1, 2, \ldots, \frac{p-1}{4}$, $s = 0, 1, \ldots, p-1$;

Type II' $\{0, 1, -1 + sp; sp\}$, for $s = 0, 1, \ldots, \frac{p-1}{2}$;

Type III $\{0, 1, a_i + sp, 1 - (a_i + sp)\}$, for $i = 1, 2, \ldots, \frac{p-5}{12}$, $s = 0, 1, \ldots, p-1$;

Type IV $\{0, p, s; \alpha s + 2^{-1} \beta p\}$, for $s = g^0, g^1, \ldots, g^{\frac{p-3}{2}}$, $g$ is a generator of $\mathbb{Z}_p^\times$;

Type V $pB \pmod{p^2}$, for all base blocks $B$ of the affine-invariant sSQS(2p),

where $\alpha, \beta$ are roots of $2x^2 - 2x + 1 = 0$ over $\mathbb{Z}_{p^2}$. 
Furthermore, the recursive construction can be generalized to affine-invariant $sSQS(2p^m)$.

**Construction 4.2 ([20]).** For prime $p \equiv 5 \pmod{12}$, if the affine-invariant $sSQS(2p)$ and $sSQS(2p^{m-1})$ are constructed, then the base blocks of the affine-invariant $sSQS(2p^m)$ can be obtained as follows.

Type I' $\{0, 1; \alpha, \beta\}$

Type I $\{0, 1; b_i + sp^{m-1}, 1 - (b_i + sp^{m-1})\}$, for $i = 1, 2, \ldots, \frac{p-5}{4}$, $s = 0, 1, \ldots, p - 1$;

Type II' $\{0, 1, -1 + sp^{m-1}, 1 - (b_i + sp^{m-1})\}$, for $s = 0, 1, \frac{p-5}{2}$;

Type III $\{0, 1, a_i + sp^{m-1}, 1 - (a_i + sp^{m-1})\}$, for $i = 1, 2, \ldots, \frac{p-5}{12}$, $s = 0, 1, \ldots, p - 1$;

Type IV $\{0, p^t, s_t; \alpha s_t + (2s_t - p^t)^{-1} \beta p^t s_t\}$, for $t = 1, 2, \ldots, m-1$, $s_t = g_t^0, g_t^1, \ldots, g_t^{\frac{x(p^{m-1})-1}{4}}$, $g_t$ is a generator of $\mathbb{Z}_{p^{2(m-t)}}^\times$;

Type V $pB \pmod{p^m}$, for all base blocks $B$ of the affine-invariant $sSQS(2p^{m-1})$.

where $\alpha, \beta$ are roots of $2x^2 - 2x + 1 = 0$ over $\mathbb{Z}_p$.

§ 5. Direct Constructions of affine-invariant $2QS(p)$

We denote a 2-fold quadruple system of order $v$ $(3-(v, 4, 2)$ design) by $2QS(v)$ for short. Suppose $p$ is a prime with $p \equiv 5 \pmod{12}$. We can again use the graph $CG(\Omega_p)$ to obtain the base blocks. Roughly speaking, by removing a 1-factor from $CG(\Omega_p)$, the resulting graph leads to the base blocks of an affine-invariant $2QS(p)$.

**Construction 5.1 ([18]).** Suppose $CG(\Omega_p)$ has a 1-factor, say $F$. For every edge $e_i = \{C(x), C(x^\sigma_c)\}$ in $CG(\Omega_p) - F$ with $C(x) \neq C(x^\sigma_c)$, let $a_i = x$, where $i = 1, 2, \ldots, l_p$ and $l_p = \begin{cases} \frac{p-17}{6} & \text{if } p \equiv 29, 41 \pmod{60}, \\ \frac{p-11}{6} & \text{otherwise} \end{cases}$ denote the number of edges (excluding loops) in $CG(\Omega_p) - F$. Then the base blocks of the affine-invariant $2QS(p)$ are

Type I $\{0, 1, a_i, 1 - a_i\}$, for $i = 1, 2, \ldots, l_p$;

Type II $\{0, 1, a_{i_p} + 1, 1 - a_{i_p} + 1\}$, where $a_{i_p} + 1 = -1$;

Type III $\{0, 1, a_{i_p} + 2, 1 - a_{i_p} + 2\}$, where $a_{i_p} + 2 = \chi$;

Type IV $\{0, 1, a_{i_p} + 3, 1 - a_{i_p} + 3\}$, where $a_{i_p} + 3 = \mu$, if $p \equiv 29, 41 \pmod{60}$, where $\mu = \frac{3 + \sqrt{2}}{2}$ is a root of $x^2 - 3x + 1 = 0$ and $\chi = \frac{1 + \sqrt{-1}}{2}$ is a root of $2x^2 - 2x + 1 = 0$. 


It is remarkable that this construction can be naturally generalized to finite fields $\mathbb{F}_q$ for prime power $q \equiv 5 \pmod{12}$. Accordingly, the graph $\text{CG}(\mathcal{P}(\mathbb{F}_q))$ is also a natural generalization of $\text{CG}(\mathcal{P}(\mathbb{F}_p))$.

§ 6. Recursive Constructions of affine-invariant $2\text{QS}(p^m)$

We begin by constructing an affine-invariant $2\text{QS}(p^2)$ from the affine-invariant $2\text{QS}(p)$. Let $\chi_1$ and $\chi_2$ denote a root of $2x^2 - 2x + 1 = 0$ over $\mathbb{Z}_p$ and $\mathbb{Z}_{p^2}$ respectively. Let $\mu_1$ and $\mu_2$ denote a root of $x^2 - 3x + 1 = 0$ over $\mathbb{Z}_p$ and $\mathbb{Z}_{p^2}$ respectively. Denote $B_s^{(1)}(a) = \{0, 1, a + sp, 1 - (a + sp)\}$.

Construction 6.1 ([18]). For prime $p \equiv 5 \pmod{12}$, by using the same notation with Construction 5.1, the base blocks of the affine-invariant $2\text{QS}(p^2)$ are

Type I $B_s^{(1)}(a_i)$, for $i = 1, \ldots, l_p$ and $s = 0, \ldots, p - 1$;

Type II $B_s^{(1)}(-1)$, and $s = 0, 1, \ldots, p - 1$;

Type III $B_s^{(1)}(\chi_2)$, and $s = 0, 1, \ldots, \frac{p-1}{2}$;

Type IV $B_s^{(1)}(\mu_2)$, and $s = 0, 1, \ldots, p - 1$, if $p \equiv 29, 41 \pmod{60}$;

Type V $\{0, p, s, s+p\}$, for $s = g^0, g^1, \ldots, g^\frac{p-3}{2}$, $g$ is a generator of $\mathbb{Z}_{p^2}^\times$;

Type VI $pB \pmod{p^2}$, for all base blocks $B$ of the affine-invariant $2\text{QS}(p)$.

Construction 6.1 can be naturally generalized to construct affine-invariant $2\text{QS}(p^m)$ for any positive integer $m$. Generally, $|\mathbb{Z}_{p^m}^\times| = p(p^{m-1} - 1)$ and

$$\mathbb{Z}_{p^m}^\times = \mathbb{Z}_{p^m} \setminus p\mathbb{Z}_{p^{m-1}} = (\mathbb{Z}_p \setminus \{0\}) + p\mathbb{Z}_{p^{m-1}},$$

where

$$p\mathbb{Z}_{p^{m-1}} = p\mathbb{Z}/p^m\mathbb{Z} = \{p, 2p, \ldots, p(p^{m-1} - 1)\}.$$

Let $\chi_t$ and $\mu_t$ denote a root of $2x^2 - 2x + 1 = 0$ and $x^2 - 3x + 1 = 0$ respectively over $\mathbb{Z}_{p^t}$. Denote $B_s^{(1)}(a) = \{0, 1, a + sp^{m-1}, 1 - (a + sp^{m-1})\}$.

Construction 6.2 ([18]). For prime $p \equiv 5 \pmod{12}$, by using the same notation with Construction 5.1, the base blocks of the affine-invariant $2\text{QS}(p^m)$ are

Type I $B_s^{(1)}(a_i)$, for $i = 1, \ldots, l_p$ and $s = 0, \ldots, p - 1$;

Type II $B_s^{(1)}(-1)$, and $s = 0, 1, \ldots, p - 1$;

Type III $B_s^{(1)}(\chi_m)$, and $s = 0, 1, \ldots, \frac{p-1}{2}$;
Type IV $B_{s}^{(1)}(\mu_{m})$, and $s = 0, 1, \ldots, p - 1$, if $p \equiv 29, 41 \pmod{60}$;

Type V $\{0, p^{t}, s_{t}, \mathcal{S}_{t}+p^{t}\}$, for $t = 1, \ldots, m - 1$ and $s_{t} = g_{t}^{0}, g_{t}^{1}, \ldots, g_{t}^{\frac{\varphi(p^{m-t})}{2}}$, where $g_{t}$ is a generator of $\mathbb{Z}_{p^{2(m-t)}}^\times$;

Type VI $pB \pmod{p^{m}}$, for all base blocks $B$ of the affine-invariant $2QS(p^{m-1})$.

§ 7. Related unsolved problems

The studies on designs admitting affine groups are less known. We present some natural problems related to affine-invariant designs.

**Problem 7.1.** Does there exist affine-invariant $sSQS(2n)$ or $2QS(n)$ when $n$ is not a prime power?

**Problem 7.2.** If we relax the condition of block size, does there exist affine-invariant 3BD?

**Problem 7.3.** Does there exist affine-invariant $t-(v, k, \lambda)$ design with larger $k$ or $t$?

References


AFFINE-INVARIANT QUADRUPLE SYSTEMS


