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Godsil–McKay switching and twisted Grassmann graphs

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1 Introduction

The twisted Grassmann graphs are the first family of non-vertex-transitive distance-regular graphs with unbounded diameter. We refer the reader to [2, 3, 5] for an extensive discussion of distance-regular graphs, to [9] for a characterization of Grassmann graphs, and to [1, 6] for more information on the twisted Grassmann graphs.

Let V be a $(2e+1)$-dimensional vector space over $\mathrm{GF}(q)$. If W is a subset of V closed under multiplication by the elements of $\mathrm{GF}(q)$, then we denote by $[W]$ the set of 1-dimensional subspaces (projective points) contained in W. We also denote by $\begin{bmatrix} W \\ k \end{bmatrix}$ the set of k-dimensional subspaces of W, when W is a vector space. The Grassmann graph $J_q(2e+1, e+1)$ is the graph with vertex set $\begin{bmatrix} V \\ e+1 \end{bmatrix}$, where two vertices W_1, W_2 are adjacent whenever $\dim W_1 \cap W_2 = e$.

Let H be a fixed hyperplane of V. The twisted Grassmann graph $\tilde{J}_q(2e+1, e)$ (see [4]) has $\mathcal{A} \cup \mathcal{B}$ as the set of vertices, where

$$\mathcal{A} = \{W \in \begin{bmatrix} V \\ e+1 \end{bmatrix} \mid W \not\subset H\},$$
$$\mathcal{B} = \begin{bmatrix} H \\ e-1 \end{bmatrix}.$$

The adjacency is defined as follows:

$$W_1 \sim W_2 \iff \begin{cases} \dim W_1 \cap W_2 = e & \text{if } W_1 \in \mathcal{A},\ W_2 \in \mathcal{A}, \\ W_1 \supset W_2 & \text{if } W_1 \in \mathcal{A},\ W_2 \in \mathcal{B}, \\ \dim W_1 \cap W_2 = e-2 & \text{if } W_1 \in \mathcal{B},\ W_2 \in \mathcal{B}. \end{cases}$$

Let σ be a polarity of H. That is, σ is an inclusion-reversing permutation of the set of subspaces of H, such that σ^2 is the identity. The pseudo-geometric design constructed by Jungnickel and Tonchev [8] has $[V]$ as the set of points, and $\mathcal{A}' \cup \mathcal{B}'$ as the set of blocks, where

$$\mathcal{A}' = \{[\sigma(W \cap H) \cup (W \setminus H)] \mid W \in \mathcal{A}\},$$
$$\mathcal{B}' = \{[W] \mid W \in \begin{bmatrix} H \\ e+1 \end{bmatrix}\}.$$

It is shown in [8] that the incidence structure $([V], \mathcal{A}' \cup \mathcal{B}')$ is a 2-(v,k,λ) design, where

$$v = \frac{q^{2e+1}-1}{q-1},\ k = \frac{q^{e+1}-1}{q-1},\ \lambda = \frac{(q^{2e-1}-1)\cdots(q^{e+1}-1)}{(q^{e-1}-1)\cdots(q-1)}.$$

The block graph of the design $([V], \mathcal{A}' \cup \mathcal{B}')$ is isomorphic to the twisted Grassmann graph $\tilde{J}_q(2e+1,e)$ (see [10]). In this report, we show that this block graph is obtained from the Grassmann graph $J_q(2e+1,e+1)$ via Godsil–McKay switching. The following diagram illustrates the situation.

$$\begin{array}{ccc}
\mathrm{PG}_d(2d,q) & \xrightarrow{\text{block graph}} & J_q(2d+1,d+1) \\
{\scriptstyle\text{distort}}\downarrow & & \downarrow{\scriptstyle\text{GM switching}} \\
\text{pseudo-geometric design} & \xrightarrow{\text{block graph}} & \tilde{J}_q(2d+1,d+1)
\end{array}$$

2 Godsil–McKay switching

Let Γ be a graph with vertex set X, and let $\{C_1,\dots,C_t,D\}$ be a partition of X such that $\{C_1,\dots,C_t\}$ is an equitable partition of $X \setminus D$. This means that the number of neighbors in C_i of a vertex x depends only on j for which $x \in C_j$ holds, and independent of the choice of x as long as $x \in C_j$. Assume also that for any $x \in D$ and $i \in \{1,\dots,t\}$, x has either $0, \frac{1}{2}|C_i|$ or $|C_i|$ neighbors in C_i. The graph $\tilde{\Gamma}$ obtained by interchanging adjacency and nonadjacency between $x \in D$ and the vertices in C_i whenever x has $\frac{1}{2}|C_i|$ neighbors in C_i, is cospectral with Γ (see [7]). The operation of constructing $\tilde{\Gamma}$ from Γ is called the Godsil–McKay switching.

In the next section, we take Γ to be the Grassmann graph $J_q(2e+1,e+1)$, and define an equitable partition $\tilde{\mathcal{C}}$ of $\begin{bmatrix} V \\ e+1 \end{bmatrix} \setminus D$ for an appropriate D.

3 An equitable partition of the Grassmann graph derived from a polarity

We keep the same notation as in Section 1. Let

$$
\begin{aligned}
C_U &= \{W \in \mathcal{A} \mid W \cap H = U\} \quad (U \in \begin{bmatrix} H \\ e \end{bmatrix}), \\
D &= \begin{bmatrix} H \\ e+1 \end{bmatrix}, \\
\mathcal{C} &= \{C_U \cup C_{\sigma(U)} \mid U \in \begin{bmatrix} H \\ e \end{bmatrix}\}.
\end{aligned} \tag{1}
$$

Then

$$
\mathcal{A} = \bigcup_{U \in \left[\begin{smallmatrix} H \\ e \end{smallmatrix}\right]} C_U \quad \text{(disjoint)},
$$

Lemma 1. *For $U \in \left[\begin{smallmatrix} H \\ e \end{smallmatrix}\right]$ and $W_2 \in D$,*

$$
|\{W_1 \in C_U \cup C_{\sigma(U)} \mid \dim W_1 \cap W_2 = e\}| \in \{|C_U \cup C_{\sigma(U)}|, \frac{1}{2}|C_U \cup C_{\sigma(U)}|, 0\}.
$$

Proof. Since

$$
\{W_1 \in C_U \mid \dim W_1 \cap W_2 = e\} = \begin{cases} C_U & \text{if } W_2 \supset U, \\ \emptyset & \text{otherwise,} \end{cases}
$$

we have

$$
\begin{aligned}
&|\{W_1 \in C_U \cup C_{\sigma(U)} \mid \dim W_1 \cap W_2 = e\}| \\
&= \begin{cases} |C_U \cup C_{\sigma(U)}| & \text{if } W_2 \supset U + \sigma(U), \\ |C_U| & \text{if } W_2 \supset U \text{ and } W_2 \not\supset \sigma(U), \\ |C_{\sigma(U)}| & \text{if } W_2 \not\supset U \text{ and } W_2 \supset \sigma(U), \\ 0 & \text{otherwise.} \end{cases} \\
&\in \{|C_U \cup C_{\sigma(U)}|, \frac{1}{2}|C_U \cup C_{\sigma(U)}|, 0\}.
\end{aligned}
$$

□

Lemma 2. *Let $\{C_1, C_2, \ldots, C_t\}$ be an equitable partition of the graph $J_q(2e, e)$ with vertex set $\left[\begin{smallmatrix} H \\ e \end{smallmatrix}\right]$. Let*

$$
\tilde{C}_i = \{W \in \begin{bmatrix} V \\ e+1 \end{bmatrix} \mid W \cap H \in C_i\} \quad (1 \le i \le t).
$$

Then $\{\tilde{C}_1, \tilde{C}_2, \ldots, \tilde{C}_t\}$ is an equitable partition of the subgraph $J_q(2e+1, e+1)$ induced by $\mathcal{A}$.

Proof. By the assumption, for $1 \le i, j \le t$, there exists an integer m_{ij} such that

$$|\{U \in C_j \mid \dim U \cap U' = e-1\}| = m_{ij} \quad (\forall U' \in C_i).$$

For $W' \in \tilde{C}_i$, we have $U' = W' \cap H \in C_i$, so

$$\begin{aligned}
&|\{W \in \tilde{C}_j \mid \dim W \cap W' = e\}| \\
&= \sum_{U \in C_j} |\{W \in \begin{bmatrix} V \\ e+1 \end{bmatrix} \mid W \cap H = U,\ \dim W \cap W' = e\}| \\
&= \sum_{\substack{U \in C_j \\ \dim U \cap U' = e-1}} |\{W \in \begin{bmatrix} V \\ e+1 \end{bmatrix} \mid W \cap H = U,\ W \cap W' \not\subset H\}| \\
&= \sum_{\substack{U \in C_j \\ \dim U \cap U' = e-1}} \frac{q^e - |U \cap U'|}{q^e - q^{e-1}} |\{W \in \begin{bmatrix} V \\ e+1 \end{bmatrix} \mid W \cap H = U,\ W \cap W' \not\subset H\}| \\
&= \frac{1}{q^e - q^{e-1}} \sum_{\substack{U \in C_j \\ \dim U \cap U' = e-1}} |\{(x, W) \in (W' \setminus H) \times \begin{bmatrix} V \\ e+1 \end{bmatrix} \mid W \cap H = U,\ x \in W\}| \\
&= \frac{1}{q^e - q^{e-1}} \sum_{\substack{U \in C_j \\ \dim U \cap U' = e-1}} |W' \setminus H| \\
&= \frac{q^{e+1} - q^e}{q^e - q^{e-1}} |\{U \in C_j \mid \dim U \cap U' = e-1\}| \\
&= q m_{ij}.
\end{aligned}$$

Therefore, every vertex in $\tilde{C}_i$ has exactly qm_{ij} neighbors in $\tilde{C}_j$. $\square$

Lemma 3. *Let σ be a polarity of H. Then the partition*

$$\{\{U, \sigma(U)\} \mid U \in \begin{bmatrix} H \\ e \end{bmatrix}\}$$

of the graph $J_q(2e, e)$ with vertex set $\begin{bmatrix} H \\ e \end{bmatrix}$, is equitable.

Proof. This is immediate since $\dim U \cap U' = \dim \sigma(U) \cap \sigma(U')$ for any $U, U' \in \begin{bmatrix} H \\ e \end{bmatrix}$. $\square$

Lemma 4. *The partition $\mathcal{C}$ defined in* (1) *is an equitable partition of the subgraph of $J_q(2e+1, e+1)$ induced by $\mathcal{A}$.*

Proof. Immediate from Lemmas 2 and 3. $\square$

4 The isomorphism

By Lemmas 1 and 4, we can apply the Godsil–McKay switching to the Grassmann graph $J_q(2e+1, e+1)$. Let $\tilde{\Gamma}$ be the Godsil-McKay switching of $J_q(2e+1, e+1)$ with respect to $\mathcal{C}$. We claim that $\phi : \begin{bmatrix} V \\ e+1 \end{bmatrix} \to \mathcal{A}' \cup \mathcal{B}'$ defined by

$$\phi(W) = \begin{cases} [\sigma(W \cap H) \cup (W \setminus H)] & \text{if } W \in \mathcal{A}, \\ [W] & \text{otherwise.} \end{cases}$$

is an isomorphism from $\tilde{\Gamma}$ to the block graph of the design $([V], \mathcal{A}' \cup \mathcal{B}')$.

Let $W_1, W_2 \in \begin{bmatrix} V \\ e+1 \end{bmatrix}$. First suppose $W_1, W_2 \in \mathcal{A}$. Since

$$\begin{aligned} |[W_1 \cap W_2]| &= |[W_1 \cap W_2 \cap H]| + |[(W_1 \cap W_2) \setminus H]| \\ &= |[W_1 \cap H] \cap [W_2 \cap H| + |[W_1 \setminus H] \cap [W_2 \setminus H]| \\ &= |[\sigma(W_1 \cap H)] \cap [\sigma(W_2 \cap H)| + |[W_1 \setminus H] \cap [W_2 \setminus H]| \\ &= |[\sigma(W_1 \cap H) \cup (W_1 \setminus H)] \cap [\sigma(W_2 \cap H) \cup (W_2 \setminus H)]| \\ &= |\phi(W_1) \cap \phi(W_2)|, \end{aligned}$$

we have

$$\begin{aligned} W_1 \sim W_2 \text{ in } \tilde{\Gamma} &\iff W_1 \sim W_2 \text{ in } \Gamma \\ &\iff \dim W_1 \cap W_2 = e \\ &\iff |[W_1 \cap W_2]| = \frac{q^e - 1}{q - 1} \\ &\iff |\phi(W_1) \cap \phi(W_2)| = \frac{q^e - 1}{q - 1} \\ &\iff \phi(W_1) \sim \phi(W_2). \end{aligned}$$

Next suppose $W_1 \in \mathcal{A}$, $W_2 \in D$. Then there exists $U \in \begin{bmatrix} H \\ e \end{bmatrix}$ such that $W_1 \in C_U$. Since

$$\begin{aligned} |[\sigma(U)] \cap [W_2]| &= |[\sigma(U)] \cap [W_2]| \\ &= |[\sigma(U) \cup (W_1 \setminus H)] \cap [W_2]| \\ &= |[\sigma(W_1 \cap H) \cup (W_1 \setminus H)] \cap [W_2]| \\ &= |\phi(W_1) \cap \phi(W_2)|, \end{aligned}$$

we have

$$\begin{aligned} W_1 \sim W_2 \text{ in } \tilde{\Gamma} &\iff W_2 \supset U \text{ and } W_2 \supset \sigma(U) \text{ or } W_2 \not\supset U \text{ and } W_2 \supset \sigma(U) \\ &\iff W_2 \supset \sigma(U) \\ &\iff [W_2] \supset [\sigma(U)] \\ &\iff [\sigma(U)] \cap [W_2] = [\sigma(U)] \end{aligned}$$

$$\begin{aligned}
&\iff |[\sigma(U)] \cap [W_2]| = |[\sigma(U)]| \\
&\iff |[\sigma(W_1 \cap H)] \cap [W_2]| = |[\sigma(U)]| \\
&\iff |\phi(W_1) \cap \phi(W_2)| = \frac{q^e - 1}{q - 1} \\
&\iff \phi(W_1) \sim \phi(W_2).
\end{aligned}$$

Finally, suppose $W_1, W_2 \in D$. Since

$$\begin{aligned}
|[W_1 \cap W_2]| &= |[W_1] \cap [W_2]| \\
&= |\phi(W_1) \cap \phi(W_2)|,
\end{aligned}$$

we have

$$\begin{aligned}
W_1 \sim W_2 &\iff \dim W_1 \cap W_2 = e \\
&\iff |[W_1 \cap W_2]| = \frac{q^e - 1}{q - 1} \\
&\iff |\phi(W_1) \cap \phi(W_2)| = \frac{q^e - 1}{q - 1} \\
&\iff \phi(W_1) \sim \phi(W_2).
\end{aligned}$$

Note that the Godsil–McKay switching we have described depends on a polarity of the hyperplane H. One might wonder whether different choice of a polarity gives rise to nonisomorphic graphs. This question has already been addressed in the context of pseudo-geometric designs in [8]. Since the composition of two polarities is a collineation of (the projective space defined by) H, and every collineation of H extends to that of V, the resulting switched graphs are isomorphic. The fact that the resulting graph is not isomorphic to the original Grassmann graph is related to the existence of an extra automorphism (i.e., a polarity) of the Grassmann graph $J_q(2e, e)$ with vertex set $\left[{H \atop e}\right]$, which does not extend to an automorphism of $J_q(2e + 1, e + 1)$.

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