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Godsil–McKay switching and twisted Grassmann graphs

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1 Introduction

The twisted Grassmann graphs are the first family of non-vertex-transitive distance-regular graphs with unbounded diameter. We refer the reader to [2, 3, 5] for an extensive discussion of distance-regular graphs, to [9] for a characterization of Grassmann graphs, and to [1, 6] for more information on the twisted Grassmann graphs.

Let $V$ be a $(2e+1)$-dimensional vector space over $\text{GF}(q)$. If $W$ is a subset of $V$ closed under multiplication by the elements of $\text{GF}(q)$, then we denote by $[W]$ the set of 1-dimensional subspaces (projective points) contained in $W$. We also denote by $[{W}]^k$ the set of $k$-dimensional subspaces of $W$, when $W$ is a vector space. The Grassmann graph $J_q(2e+1, e+1)$ is the graph with vertex set $[V]_{e+1}$, where two vertices $W_1, W_2$ are adjacent whenever $\dim W_1 \cap W_2 = e$.

Let $H$ be a fixed hyperplane of $V$. The twisted Grassmann graph $\tilde{J}_q(2e + 1, e)$ (see [4]) has $\mathcal{A} \cup \mathcal{B}$ as the set of vertices, where
\[ \mathcal{A} = \{ W \in \left[ \begin{array}{c} V \\ e + 1 \end{array} \right] | W \not\subset H \}, \]
\[ \mathcal{B} = \left[ \begin{array}{c} H \\ e - 1 \end{array} \right]. \]

The adjacency is defined as follows:

\[ W_1 \sim W_2 \iff \begin{cases} \dim W_1 \cap W_2 = e & \text{if } W_1 \in \mathcal{A}, W_2 \in \mathcal{A}, \\ W_1 \supset W_2 & \text{if } W_1 \in \mathcal{A}, W_2 \in \mathcal{B}, \\ \dim W_1 \cap W_2 = e - 2 & \text{if } W_1 \in \mathcal{B}, W_2 \in \mathcal{B}. \end{cases} \]
Let $\sigma$ be a polarity of $H$. That is, $\sigma$ is an inclusion-reversing permutation of the set of subspaces of $H$, such that $\sigma^2$ is the identity. The pseudo-geometric design constructed by Jungnickel and Tonchev [8] has $[V]$ as the set of points, and $\mathcal{A}' \cup \mathcal{B}'$ as the set of blocks, where

$$\mathcal{A}' = \{ [\sigma(W \cap H) \cup (W \setminus H)] \mid W \in \mathcal{A} \},$$

$$\mathcal{B}' = \{ [W] \mid W \in \binom{H_{e+1}}{e+1} \}.$$

It is shown in [8] that the incidence structure $([V], \mathcal{A}' \cup \mathcal{B}')$ is a $2-(v, k, \lambda)$ design, where

$$v = \frac{q^{2e+1} - 1}{q - 1}, \quad k = \frac{q^{e+1} - 1}{q - 1}, \quad \lambda = \frac{(q^{2e-1} - 1) \cdots (q^{e+1} - 1)}{(q^{e-1} - 1) \cdots (q - 1)}.$$

The block graph of the design $([V], \mathcal{A}' \cup \mathcal{B}')$ is isomorphic to the twisted Grassmann graph $\tilde{J}_q(2e+1, e)$ (see [10]). In this report, we show that this block graph is obtained from the Grassmann graph $J_q(2e+1, e+1)$ via Godsil–McKay switching. The following diagram illustrates the situation.

$$\begin{align*}
\text{PG}_d(2d, q) & \xrightarrow{\text{block graph}} J_q(2d + 1, d + 1) \\
\text{distort} & \downarrow \quad \text{GM switching} \\
\text{pseudo-geometric design} & \xrightarrow{\text{block graph}} \tilde{J}_q(2d + 1, d + 1)
\end{align*}$$

## 2 Godsil–McKay switching

Let $\Gamma$ be a graph with vertex set $X$, and let $\{C_1, \ldots, C_t, D\}$ be a partition of $X$ such that $\{C_1, \ldots, C_t\}$ is an equitable partition of $X \setminus D$. This means that the number of neighbors in $C_i$ of a vertex $x$ depends only on $j$ for which $x \in C_j$ holds, and independent of the choice of $x$ as long as $x \in C_j$. Assume also that for any $x \in D$ and $i \in \{1, \ldots, t\}$, $x$ has either $0$, $\frac{1}{2}|C_i|$ or $|C_i|$ neighbors in $C_i$. The graph $\tilde{\Gamma}$ obtained by interchanging adjacency and nonadjacency between $x \in D$ and the vertices in $C_i$ whenever $x$ has $\frac{1}{2}|C_i|$ neighbors in $C_i$, is cospectral with $\Gamma$ (see [7]). The operation of constructing $\tilde{\Gamma}$ from $\Gamma$ is called the Godsil–McKay switching.

In the next section, we take $\Gamma$ to be the Grassmann graph $J_q(2e + 1, e + 1)$, and define an equitable partition $\tilde{C}$ of $V_{e+1} \setminus D$ for an appropriate $D$. 
3 An equitable partition of the Grassmann graph derived from a polarity

We keep the same notation as in Section 1. Let

\[ C_U = \{ W \in \mathcal{A} \mid W \cap H = U \} \ (U \in \binom{H}{e}), \]

\[ D = \begin{bmatrix} H \\ e + 1 \end{bmatrix}, \]

\[ C = \{ C_U \cup C_{\sigma(U)} \mid U \in \binom{H}{e} \}. \]

Then

\[ \mathcal{A} = \bigcup_{U \in \binom{H}{e}} C_U \quad \text{(disjoint),} \]

Lemma 1. For \( U \in \binom{H}{e} \) and \( W_2 \in D, \)

\[ |\{ W_1 \in C_U \cup C_{\sigma(U)} \mid \dim W_1 \cap W_2 = e \}| \in \{|C_U \cup C_{\sigma(U)}|, \frac{1}{2}|C_U \cup C_{\sigma(U)}|, 0\}. \]

Proof. Since

\[ \{ W_1 \in C_U \mid \dim W_1 \cap W_2 = e \} = \begin{cases} C_U & \text{if } W_2 \supset U, \\ \emptyset & \text{otherwise,} \end{cases} \]

we have

\[ |\{ W_1 \in C_U \cup C_{\sigma(U)} \mid \dim W_1 \cap W_2 = e \}| \]
\[ = \begin{cases} |C_U \cup C_{\sigma(U)}| & \text{if } W_2 \supset U + \sigma(U), \\ |C_U| & \text{if } W_2 \supset U \text{ and } W_2 \not\supset \sigma(U), \\ |C_{\sigma(U)}| & \text{if } W_2 \not\supset U \text{ and } W_2 \supset \sigma(U), \\ 0 & \text{otherwise.} \end{cases} \]
\[ \in \{|C_U \cup C_{\sigma(U)}|, \frac{1}{2}|C_U \cup C_{\sigma(U)}|, 0\}. \]

Lemma 2. Let \( \{ C_1, C_2, \ldots, C_t \} \) be an equitable partition of the graph \( J_q(2e, e) \) with vertex set \( \binom{H}{e} \). Let

\[ \tilde{C}_i = \{ W \in \begin{bmatrix} V \\ e + 1 \end{bmatrix} \mid W \cap H \in C_i \} \ (1 \leq i \leq t). \]

Then \( \{ \tilde{C}_1, \tilde{C}_2, \ldots, \tilde{C}_t \} \) is an equitable partition of the subgraph \( J_q(2e+1, e+1) \) induced by \( \mathcal{A} \).
Proof. By the assumption, for $1 \leq i, j \leq t$, there exists an integer $m_{ij}$ such that
\[
|\{U \in C_j \mid \dim U \cap U' = e - 1\}| = m_{ij} \quad (\forall U' \in C_i).
\]
For $W' \in \tilde{C}_i$, we have $U' = W' \cap H \in C_i$, so
\[
|\{W \in \tilde{C}_j \mid \dim W \cap W' = e\}|
= \sum_{U \in C_j} |\{W \in \left[\begin{array}{l}V \\ e + 1\end{array}\right] \mid W \cap H = U, \dim W \cap W' = e\}|
= \sum_{U \in C_j} |\{W \in \left[\begin{array}{l}V \\ e + 1\end{array}\right] \mid W \cap H = U, W \cap W' \not\subset H\}|
= \frac{q^e - |U \cap U'|}{q^e - q^{e-1}} \sum_{U \in C_j} |\{W \in \left[\begin{array}{l}V \\ e + 1\end{array}\right] \mid W \cap H = U, W \cap W' \not\subset H\}|
= \frac{1}{q^e - q^{e-1}} \sum_{U \in C_j} |\{(x, W) \in (W' \setminus H) \times \left[\begin{array}{l}V \\ e + 1\end{array}\right] \mid W \cap H = U, x \in W\}|
= \frac{1}{q^e - q^{e-1}} \sum_{U \in C_j} |W' \setminus H|
= \frac{q^{e+1} - q^e}{q^e - q^{e-1}} \sum_{U \in C_j} |\{U \in C_j \mid \dim U \cap U' = e - 1\}|
= qm_{ij}.
\]
Therefore, every vertex in $\tilde{C}_i$ has exactly $qm_{ij}$ neighbors in $\tilde{C}_j$.

Lemma 3. Let $\sigma$ be a polarity of $H$. Then the partition
\[
\{\{U, \sigma(U)\} \mid U \in \left[\begin{array}{l}H \\ e\end{array}\right]\}
\]
of the graph $J_q(2e, e)$ with vertex set $[H]_e$, is equitable.

Proof. This is immediate since $\dim U \cap U' = \dim \sigma(U) \cap \sigma(U')$ for any $U, U' \in [H]_e$. 

Lemma 4. The partition $C$ defined in (1) is an equitable partition of the subgraph of $J_q(2e + 1, e + 1)$ induced by $\mathcal{A}$.

Proof. Immediate from Lemmas 2 and 3. 

4 The isomorphism

By Lemmas 1 and 4, we can apply the Godsil–McKay switching to the Grassmann graph $J_q(2e+1,e+1)$. Let $\Gamma$ be the Godsil-McKay switching of $J_q(2e+1,e+1)$ with respect to $C$. We claim that $\phi : \binom{V}{e+1} \to A' \cup B'$ defined by

$$\phi(W) = \begin{cases} [\sigma(W \cap H) \cup (W \setminus H)] & \text{if } W \in A, \\ [W] & \text{otherwise.} \end{cases}$$

is an isomorphism from $\tilde{\Gamma}$ to the block graph of the design $(\binom{V}{e}, A' \cup B')$.

Let $W_1, W_2 \in \binom{V}{e+1}$. First suppose $W_1, W_2 \in A$. Since

$$||W_1 \cap W_2|| = ||W_1 \cap W_2 \cap H|| + ||(W_1 \cap W_2) \setminus H|| = ||W_1 \cap H|| \cap ||W_2 \cap H|| + ||(W_1 \setminus H) \cap (W_2 \setminus H)|| = ||(W_1 \cap H) \cup (W_1 \setminus H) \cap (W_2 \cap H) \cup (W_2 \setminus H)|| = ||\phi(W_1) \cap \phi(W_2)||,$$

we have

$$W_1 \sim W_2 \text{ in } \tilde{\Gamma} \iff W_1 \sim W_2 \text{ in } \Gamma \iff \dim W_1 \cap W_2 = e \iff ||W_1 \cap W_2|| = \frac{q^e - 1}{q - 1} \iff \phi(W_1) \sim \phi(W_2).$$

Next suppose $W_1 \in A, W_2 \in D$. Then there exists $U \in \binom{H}{e}$ such that $W_1 \in C_U$. Since

$$||\sigma(U) \cap W_2|| = ||\sigma(U) \cap W_2|| = ||\sigma(U) \cup (W_1 \setminus H) \cap W_2|| = ||\sigma(W_1 \cap H) \cup (W_1 \setminus H) \cap W_2|| = ||\phi(W_1) \cap \phi(W_2)||,$$

we have

$$W_1 \sim W_2 \text{ in } \tilde{\Gamma} \iff W_2 \supset U \text{ and } W_2 \supset \sigma(U) \text{ or } W_2 \not\supset U \text{ and } W_2 \supset \sigma(U) \iff W_2 \supset \sigma(U) \iff [W_2] \supset [\sigma(U)] \iff [\sigma(U)] \cap [W_2] = [\sigma(U)].$$
\[ \Leftrightarrow |[\sigma(U) \cap [W_2]| = |[\sigma(U)]| \]
\[ \Leftrightarrow |[\sigma(W_1 \cap H) \cap [W_2]| = |[\sigma(U)]| \]
\[ \Leftrightarrow |\phi(W_1) \cap \phi(W_2)| = \frac{q^e - 1}{q - 1} \]
\[ \Leftrightarrow \phi(W_1) \sim \phi(W_2). \]

Finally, suppose \( W_1, W_2 \in D \). Since
\[ |[W_1 \cap W_2]| = |[W_1]| \cap [W_2]| = |\phi(W_1) \cap \phi(W_2)|, \]
we have
\[ W_1 \sim W_2 \iff \dim W_1 \cap W_2 = e \]
\[ \Leftrightarrow |[W_1 \cap W_2]| = \frac{q^e - 1}{q - 1} \]
\[ \Leftrightarrow |\phi(W_1) \cap \phi(W_2)| = \frac{q^e - 1}{q - 1} \]
\[ \Leftrightarrow \phi(W_1) \sim \phi(W_2). \]

Note that the Godsil–McKay switching we have described depends on a polarity of the hyperplane \( H \). One might wonder whether different choice of a polarity gives rise to nonisomorphic graphs. This question has already been addressed in the context of pseudo-geometric designs in [8]. Since the composition of two polarities is a collineation of (the projective space defined by) \( H \), and every collineation of \( H \) extends to that of \( V \), the resulting switched graphs are isomorphic. The fact that the resulting graph is not isomorphic to the original Grassmann graph is related to the existence of an extra automorphism (i.e., a polarity) of the Grassmann graph \( J_q(2e, e) \) with vertex set \( \left[ e \right]^H \), which does not extend to an automorphism of \( J_q(2e + 1, e + 1) \).

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References


