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<tr>
<td>Author(s)</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2015), 1956: 29-35</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2015-07</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/224062">http://hdl.handle.net/2433/224062</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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Godsil–McKay switching and twisted Grassmann graphs

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July 23, 2014

1 Introduction

The twisted Grassmann graphs are the first family of non-vertex-transitive distance-
regular graphs with unbounded diameter. We refer the reader to [2, 3, 5] for an
extensive discussion of distance-regular graphs, to [9] for a characterization of Grass-
mann graphs, and to [1, 6] for more information on the twisted Grassmann graphs.

Let $V$ be a $(2e+1)$-dimensional vector space over $GF(q)$. If $W$ is a subset of $V$
closed under multiplication by the elements of $GF(q)$, then we denote by $[W]$ the set
of 1-dimensional subspaces (projective points) contained in $W$. We also denote by $[W]_k$ the set of $k$-dimensional subspaces of $W$, when $W$ is a vector space. The Grassmann
graph $J_q(2e+1, e+1)$ is the graph with vertex set $[V]_{e+1}$, where two vertices $W_1, W_2$
are adjacent whenever $\dim W_1 \cap W_2 = e$.

Let $H$ be a fixed hyperplane of $V$. The twisted Grassmann graph $\tilde{J}_q(2e + 1, e)$
(see [4]) has $A \cup B$ as the set of vertices, where
\[
A = \{W \in [V]_{e+1} | W \not\subset H\},
\]
\[
B = [H]_{e-1}.
\]
The adjacency is defined as follows:
\[
W_1 \sim W_2 \iff \begin{cases} 
\dim W_1 \cap W_2 = e & \text{if } W_1 \in A, W_2 \in A, \\
W_1 \supset W_2 & \text{if } W_1 \in A, W_2 \in B, \\
\dim W_1 \cap W_2 = e - 2 & \text{if } W_1 \in B, W_2 \in B.
\end{cases}
\]
Let $\sigma$ be a polarity of $H$. That is, $\sigma$ is an inclusion-reversing permutation of the set of subspaces of $H$, such that $\sigma^2$ is the identity. The pseudo-geometric design constructed by Jungnickel and Tonchev [8] has $[V]$ as the set of points, and $\mathcal{A}' \cup \mathcal{B}'$ as the set of blocks, where

$$\mathcal{A}' = \{[\sigma(W \cap H) \cup (W \setminus H)] | W \in \mathcal{A}\},$$

$$\mathcal{B}' = \{[W] | W \in \left[\begin{array}{l} H \\ e + 1 \end{array}\right]\}.$$ 

It is shown in [8] that the incidence structure $([V], \mathcal{A}' \cup \mathcal{B}')$ is a $2-(v, k, \lambda)$ design, where

$$v = \frac{q^{2e+1} - 1}{q - 1}, \quad k = \frac{q^{e+1} - 1}{q - 1}, \quad \lambda = \frac{(q^{2e-1} - 1) \cdots (q^{e+1} - 1)}{(q^{e-1} - 1) \cdots (q - 1)}.$$ 

The block graph of the design $([V], \mathcal{A}' \cup \mathcal{B}')$ is isomorphic to the twisted Grassmann graph $\tilde{J}_q(2e+1, e)$ (see [10]). In this report, we show that this block graph is obtained from the Grassmann graph $J_q(2e+1, e+1)$ via Godsil–McKay switching. The following diagram illustrates the situation.

$$\begin{array}{c}
PG_d(2d, q) \xrightarrow{\text{block graph}} J_q(2d + 1, d + 1) \\
distort \downarrow \quad \text{GM switching} \\
pseudo-geometric design \xrightarrow{\text{block graph}} \tilde{J}_q(2d + 1, d + 1)
\end{array}$$

## 2 Godsil–McKay switching

Let $\Gamma$ be a graph with vertex set $X$, and let $\{C_1, \ldots, C_t, D\}$ be a partition of $X$ such that $\{C_1, \ldots, C_t\}$ is an equitable partition of $X \setminus D$. This means that the number of neighbors in $C_i$ of a vertex $x$ depends only on $j$ for which $x \in C_j$ holds, and independent of the choice of $x$ as long as $x \in C_j$. Assume also that for any $x \in D$ and $i \in \{1, \ldots, t\}$, $x$ has either $0, \frac{1}{2}|C_i|$ or $|C_i|$ neighbors in $C_i$. The graph $\tilde{\Gamma}$ obtained by interchanging adjacency and nonadjacency between $x \in D$ and the vertices in $C_i$ whenever $x$ has $\frac{1}{2}|C_i|$ neighbors in $C_i$, is cospectral with $\Gamma$ (see [7]). The operation of constructing $\tilde{\Gamma}$ from $\Gamma$ is called the Godsil–McKay switching.

In the next section, we take $\Gamma$ to be the Grassmann graph $J_q(2e+1, e+1)$, and define an equitable partition $\tilde{C}$ of $[V] \setminus D$ for an appropriate $D$. 

3 An equitable partition of the Grassmann graph derived from a polarity

We keep the same notation as in Section 1. Let

\[ C_U = \{ W \in \mathcal{A} \mid W \cap H = U \} \quad (U \in \binom{H}{e}), \]

\[ D = \binom{H}{e+1}, \]

\[ \mathcal{C} = \{ C_U \cup C_{\sigma(U)} \mid U \in \binom{H}{e} \}. \]

Then

\[ \mathcal{A} = \bigcup_{U \in \binom{H}{e}} C_U \quad \text{(disjoint)}, \]

Lemma 1. For \( U \in \binom{H}{e} \) and \( W_2 \in D \),

\[ |\{ W_1 \in C_U \cup C_{\sigma(U)} \mid \dim W_1 \cap W_2 = e \}| \in \{ |C_U \cup C_{\sigma(U)}|, \frac{1}{2} |C_U \cup C_{\sigma(U)}|, 0 \}. \]

Proof. Since

\[ \{ W_1 \in C_U \mid \dim W_1 \cap W_2 = e \} = \begin{cases} C_U & \text{if } W_2 \supset U, \\
\emptyset & \text{otherwise}, \end{cases} \]

we have

\[ |\{ W_1 \in C_U \cup C_{\sigma(U)} \mid \dim W_1 \cap W_2 = e \}| = \begin{cases} |C_U \cup C_{\sigma(U)}| & \text{if } W_2 \supset U + \sigma(U), \\
|C_U| & \text{if } W_2 \supset U \text{ and } W_2 \not\supset \sigma(U), \\
|C_{\sigma(U)}| & \text{if } W_2 \not\supset U \text{ and } W_2 \supset \sigma(U), \\
0 & \text{otherwise.} \end{cases} \]

\[ \in \{ |C_U \cup C_{\sigma(U)}|, \frac{1}{2} |C_U \cup C_{\sigma(U)}|, 0 \}. \]

Lemma 2. Let \( \{ C_1, C_2, \ldots, C_t \} \) be an equitable partition of the graph \( J_q(2e, e) \) with vertex set \( \binom{H}{e} \). Let

\[ \tilde{C}_i = \{ W \in \binom{V}{e+1} \mid W \cap H \in C_i \} \quad (1 \leq i \leq t). \]

Then \( \{ \tilde{C}_1, \tilde{C}_2, \ldots, \tilde{C}_t \} \) is an equitable partition of the subgraph \( J_q(2e+1, e+1) \) induced by \( \mathcal{A} \).
Proof. By the assumption, for $1 \leq i, j \leq t$, there exists an integer $m_{ij}$ such that
\[ |\{ U \in C_j \mid \dim U \cap U' = e - 1 \}| = m_{ij} \quad (\forall U' \in C_i). \]
For $W' \in \tilde{C}_i$, we have $U' = W' \cap H \in C_i$, so
\[
|\{ W \in \tilde{C}_j \mid \dim W \cap W' = e \}| \\
= \sum_{U \in C_j} |\{ W \in \left[ \begin{array}{c} V \\ e + 1 \end{array} \right] \mid W \cap H = U, \dim W \cap W' = e \}| \\
= \sum_{U \in C_j, \dim U \cap U' = e - 1} \frac{q^e - |U \cap U'|}{q^e - q^{e-1}} |\{ W \in \left[ \begin{array}{c} V \\ e + 1 \end{array} \right] \mid W \cap H = U, W \cap W' \notin H \}| \\
= \frac{1}{q^e - q^{e-1}} \sum_{U \in C_j, \dim U \cap U' = e - 1} |\{(x, W) \in (W' \backslash H) \times \left[ \begin{array}{c} V \\ e + 1 \end{array} \right] \mid W \cap H = U, x \in W \}| \\
= \frac{1}{q^e - q^{e-1}} \sum_{U \in C_j, \dim U \cap U' = e - 1} |W' \backslash H| \\
= \frac{q^{e+1} - q^e}{q^e - q^{e-1}} |\{ U \in C_j \mid \dim U \cap U' = e - 1 \}| \\
= qm_{ij}.
\]
Therefore, every vertex in $\tilde{C}_i$ has exactly $qm_{ij}$ neighbors in $\tilde{C}_j$. \done

Lemma 3. Let $\sigma$ be a polarity of $H$. Then the partition
\[ \{ \{ U, \sigma(U) \} \mid U \in \left[ \begin{array}{c} H \\ e \end{array} \right] \} \]
of the graph $J_q(2e, e)$ with vertex set $[H]_e$, is equitable.

Proof. This is immediate since $\dim U \cap U' = \dim \sigma(U) \cap \sigma(U')$ for any $U, U' \in [H]_e$. \done

Lemma 4. The partition $C$ defined in (1) is an equitable partition of the subgraph of $J_q(2e + 1, e + 1)$ induced by $A$.

Proof. Immediate from Lemmas 2 and 3. \done
4 The isomorphism

By Lemmas 1 and 4, we can apply the Godsil–McKay switching to the Grassmann graph $J_q(2e+1,e+1)$. Let $\tilde{\Gamma}$ be the Godsil-McKay switching of $J_q(2e+1,e+1)$ with respect to $C$. We claim that $\phi : [V]_{e+1} \to A' \cup B'$ defined by

$$\phi(W) = \begin{cases} [\sigma(W \cap H) \cup (W \setminus H)] & \text{if } W \in A, \\ [W] & \text{otherwise.} \end{cases}$$

is an isomorphism from $\tilde{\Gamma}$ to the block graph of the design $([V], A' \cup B')$.

Let $W_1, W_2 \in [V]_{e+1}$. First suppose $W_1, W_2 \in A$. Since

$$|[W_1 \cap W_2]| = |(W_1 \cap W_2) \cap H| + |(W_1 \cap W_2) \setminus H| = |(W_1 \cap H) \cap (W_2 \setminus H)| + |(W_1 \setminus H) \cap (W_2 \cap H)| = |(\sigma(W_1 \cap H) \cup (W_1 \setminus H)) \cap (\sigma(W_2 \setminus H) \cup (W_2 \setminus H))| = |\phi(W_1) \cap \phi(W_2)|,$$

we have

$$W_1 \sim W_2 \text{ in } \tilde{\Gamma} \iff W_1 \sim W_2 \text{ in } \Gamma \iff \dim W_1 \cap W_2 = e \iff \frac{q^e - 1}{q - 1} \iff |\phi(W_1) \cap \phi(W_2)| = \frac{q^e - 1}{q - 1} \iff \phi(W_1) \sim \phi(W_2).$$

Next suppose $W_1 \in A$, $W_2 \in D$. Then there exists $U \in [H]_{e}$ such that $W_1 \in C_U$. Since

$$|\sigma(U) \cap [W_2]| = |\sigma(U) \cap [W_2]| = |\sigma(U) \cup (W_1 \setminus H) \cap [W_2]| = |\sigma(W_1 \cap H) \cup (W_1 \setminus H) \cap [W_2]| = |\phi(W_1) \cap \phi(W_2)|,$$

we have

$$W_1 \sim W_2 \text{ in } \tilde{\Gamma} \iff W_2 \supset U \text{ and } W_2 \supset \sigma(U) \text{ or } W_2 \not\supset U \text{ and } W_2 \supset \sigma(U) \iff \frac{q^e - 1}{q - 1} \iff \phi(W_1) \sim \phi(W_2).$$
Finally, suppose $W_1, W_2 \in D$. Since

$$||W_1 \cap W_2|| = ||W_1|| \cap ||W_2|| = |\phi(W_1) \cap \phi(W_2)|,$$

we have

$$W_1 \sim W_2 \iff \dim W_1 \cap W_2 = e$$

$$\iff ||W_1 \cap W_2|| = \frac{q^e - 1}{q - 1}$$

$$\iff |\phi(W_1) \cap \phi(W_2)| = \frac{q^e - 1}{q - 1}$$

$$\iff \phi(W_1) \sim \phi(W_2).$$

Note that the Godsil–McKay switching we have described depends on a polarity of the hyperplane $H$. One might wonder whether different choice of a polarity gives rise to nonisomorphic graphs. This question has already been addressed in the context of pseudo-geometric designs in [8]. Since the composition of two polarities is a collineation of (the projective space defined by) $H$, and every collineation of $H$ extends to that of $V$, the resulting switched graphs are isomorphic. The fact that the resulting graph is not isomorphic to the original Grassmann graph is related to the existence of an extra automorphism (i.e., a polarity) of the Grassmann graph $J_q(2e, e)$ with vertex set $[H_e]$, which does not extend to an automorphism of $J_q(2e + 1, e + 1)$.

Acknowledgements

The author would like to thank Alexander Gavrilyuk for helpful discussions.

References


