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Boundary value problems of differential equations with irregular singularities in microlocal analysis

By

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Abstract

For a boundary value problem with a hyperbolic equation, we construct pure solutions whose singularities are located on the characteristic roots of the principal symbol. In particular, we show a concrete expression of the solutions.

§ 1. Introduction

There are several studies about hyperbolic equations. For example, by Bony-Schapira [5], there exist fundamental solutions of initial value problem for hyperbolic equations in the category of Sato's hyperfunctions (we call them just hyperfunctions from now on).

One typical example of hyperbolic equations is the Airy equation:

$$(\partial_x^2 - x)u(x) = 0,$$

where $\partial_x = d/dx$. Sir George Biddel Airy was an astronomer and the head of Astronomer Royal at the Greenwich Observatory, which is located on the prime meridian. The Airy equation appears in the analysis with respect to optics of rainbows ([1]).

The Airy operator $P = \partial_x^2 - x$ has an irregular singular point at infinity in the Riemann sphere $\mathbb{CP}^1$. This fact leads to the Stokes' phenomena, which was discovered by G. G. Stokes [19]. Namely, solutions of differential equations with irregular singularities in a domain change different forms in another domain.

Several mathematicians define irregularities of such equations. By Komatsu [15], this equation has an irregularity $\sigma = 5/2$ at the infinity in the Riemann sphere, while Malgrange's irregularity is equal to $i_\infty(P) = 3$ ([17]).

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For the Airy equation, it is well known that solutions are constructed by asymptotic expansions, integral representations and so on. As is well known, one can get the integral representation

$$\mathrm{Ai}(x) = \frac{1}{2\pi\sqrt{-1}} \int_{-\sqrt{-1}\infty}^{+\sqrt{-1}\infty} \exp(xt - \frac{t^3}{3})\, dt, \tag{1.1}$$

which is called the Airy function $\mathrm{Ai}(x)$. By the steepest descend method (details are shown in [8], [23], for instance), the saddle point of (1.1) is $t = \pm\sqrt{x}$. When x tends to $+\sqrt{-1}\infty$, we have an asymptotic expansion of (1.1) as follows:

$$\mathrm{Ai}(x) \sim \frac{1}{2\sqrt{\pi}} x^{-1/4} \exp\left(-\frac{2}{3}x^{3/2}\right).$$

On the other hand, by the WKB method (for example, refer [14]), we assume that solutions for differential equations with a large parameter ξ have the following form:

$$\exp\left(\int_{x_0}^{x} \sum_{j=-1}^{\infty} S_j(x)\xi^{-j}\right).$$

By the Taylor expansion of the term with respect to $j \geq 0$, we get the WKB solution. A property of convergence for asymptotic series' depends on the Borel summation.

Though there are several ways of constructing solutions of hyperbolic equations, we will present concrete expressions of solutions for boundary value problems. In [6], we show how to make solutions of boundary value problems with such hyperbolic equations by two transforms: a fractional coordinate transform and the quantised Legendre transform. Moreover, we can get boundary values by the initial data with microdifferential operators with fractional power order.

In this paper, we present some results and surveys about boundary value problems for hyperbolic operators by following the purpose of the conference [1].

§ 2. Preliminaries

In this section, let M be an n dimensional real analytic manifold and X its complexification. In fact, we can regard $M = \mathbb{R}^n$ and $X = \mathbb{C}^n$ since we treat only a local case. To begin with, we define hyperfunctions.

Definition 2.1 (Hyperfunctions). For a positive $R > 0$ and open conic cones Γ_1, $\Gamma_2, \cdots, \Gamma_N$ in $\mathbb{R}^n$, there exist functions $F_j(z)$ $(j = 1, 2, \cdots, N)$ which are holomorphic

[1] RIMS Joint Research "Introductory Workshop on Path Integrals and Pseudo-Differential Operators", from 7 October 2014 to 10.

on a domain $\{z \in \mathbb{C}^n : |z - x_0| < R,\ \mathrm{Im} z \in \Gamma_j\}$ such that we can express

$$f(x) = \sum_{j=1}^{N} F_j(x + \sqrt{-1}\, 0\Gamma_j)$$

in a neighbourhood of x_0. We call F_j $(j = 1, 2, \cdots, N)$ defining functions.

The difference between hyperfunctions and distributions is whether an increasing condition is imposed or not. Furthermore, Kataoka introduces the concept of mildness to hyperfunctions ([12]). By this property, we characterise the boundary value of hyperfunctions from the positive side, i.e., we can obtain $u(+0, x')$ as a hyperfunction $(x' = (x_2, \cdots, x_n))$. Under this concept, Oaku introduces the property of F-mildness. He applied this property to the analysis of Fuchsian partial differential operators ([16]).

We can develop the theory of hyperfunctions into the sheaf theory. In fact, its sheaf theory has a good prospect. In particular, we can explain a construction of microfunctions via sheaf theory. As is well known, microfunctions are defined on the cotangent space on $\mathbb{R}^n$, which are regarded as singularities of hyperfunctions. That is, a set $\mathscr{C}$ of all microfunctions is characterised by a quotient space $\mathscr{B}/\mathscr{A}$, where $\mathscr{B}$ is a sheaf of hyperfunctions and $\mathscr{A}$ is a sheaf of analytic functions. Moreover, we have the following exact sequence:

$$0 \longrightarrow \mathscr{A} \longrightarrow \mathscr{B} \longrightarrow \pi_* \mathscr{C} \longrightarrow 0,$$

where π is a projection map from T^*X to X. The map $\mathscr{B} \to \pi_* \mathscr{C}$ is usually expressed as sp, which is an abbreviation of the spectrum. For a section $u \in \mathscr{B}(M)$, the support of $\mathrm{sp}(u)$ on T^*M is called a singular spectrum and denoted by $\mathrm{SS}\, u$. The precise definitions are referred to several literatures, for instance, [11], [18].

A differential operator with analytic functional coefficients

$$\sum_{j=0}^{m} a_j(x) \partial^j$$

acts on a hyperfunction because we can justify a multiplication of analytic functions by hyperfunctions and a differentiation. Concerning a differentiation of negative order, it is possible to show its operation on a hyperfunction by introducing the estimates below.

Definition 2.2 (Microdifferential operators). For a formal series of operators

$$P(x, \partial_x) = \sum_{k=-\infty}^{\infty} a_k(x) \partial_x^k,$$

we call $P(x, \partial_x)$ a microdifferential operator if the following conditions are satisfied:

(1) For any compact set K in an open set U,

$$\lim_{k\to\infty}\left(\sup_{x\in K}|a_k(x)|k!\right)^{1/k}=0.$$

(2) For $k=-l$ $(l=1,2,\cdots)$,

$$\overline{\lim_{l\to\infty}}\left(\sup_{x\in K}\frac{|a_l(x)|}{l!}\right)^{1/l}<\infty.$$

If $a_k(x)=0$ for all $k>m$, we call P a microdifferential operator of order m. $\mathscr{E}^{\infty}$ stands for a set of all microdifferential operators. $\mathscr{E}^{\infty}(m)$ is a set of all microdifferential operators of order m.

Microdifferential operators can be transformed by quantised contact transformations. See the detail in [18]. Furthermore, we can define pseudodifferential operators, which are the extension of differential operators of infinite order in the cotangent space. The class of pseudodifferential operators is wider than that of microdifferential operators. By the theory of Aoki–Kataoka's symbol calculus ([4], [12]), the class of pseudodifferential operators can be defined as a quotient space. We denote a set of all pseudodifferential operators by $\mathscr{E}^{\mathbb{R}}$.

§ 3. A construction of pure solutions

By employing the theory of microlocal method above, we can study partial differential equations. In [6], we have microlocal solutions for the boundary value problem

$$\begin{cases}P(t,\partial_t,\partial_x)u(t,x)=0, & 0<t<\varepsilon, |x|<\varepsilon,\\ \mathrm{SS}(u)\cap\{t>0\}\subset H_j, & (*)\end{cases}\tag{3.1}$$

where $P(t,\partial_t,\partial_x)$ is a hyperbolic operator with its principal symbol

$$\sigma(P)(t,\tau,\xi)=\prod_{j=1}^{m}(\tau-\sqrt{-1}t^{\lambda}\alpha_j(t)\xi)$$

at $t=0$ and $H_j:=\{(t,*;\sqrt{-1}\tau,\sqrt{-1}\xi);\tau-\sqrt{-1}t^{\lambda}\alpha_j(t)\xi=0\}$. We assume that each $\alpha_j(t)$ is a purely imaginary-valued function and $\alpha_j(0)$ are mutually distinct. We call solutions satisfying the condition $(*)$ j-pure.

There are numerous researches about branching of singularities for such hyperbolic operators ([2], [3], [9], [20], [21]). On the other hand, we give a construction of solutions for hyperbolic operators of general order by using a fractional coordinate transform.

The idea of the construction of j-pure solutions are as follows. To begin with, we multiply t^m by the operator P. We denote the corresponding solution by $\tilde{u}(t,x)$ as a hyperfunction. By using a fractional coordinate transform

$$\tilde{t} = \frac{t^{\lambda+1}}{\lambda+1}, \tag{3.2}$$

a solution $\tilde{u}(t,x)$ corresponds to $v(\tilde{t},x)$ of the equation

$$Q(\tilde{t}, \partial_{\tilde{t}}, \partial_x)v(\tilde{t},x) = 0,$$

where $Q(\tilde{t}, \partial_{\tilde{t}}, \partial_x)$ is a differential operator whose coefficients have fractional power singularities (branch points) with respect to $\tilde{t}$ and $v(\tilde{t},x)$ is a microfunction which is represented by a hyperfunction with support in $\tilde{t} \geq 0$. By virtue of the quantised Legendre transform from $(\tilde{t},x)$-space to (w,y)-space:

$$\beta \circ * \circ \beta^{-1} : \begin{cases} \partial_{\tilde{t}} \mapsto w\partial_y, & \partial_x \mapsto \partial_y, \\ \tilde{t} \mapsto -\partial_w(\partial_y)^{-1}, & x \mapsto y + \partial_w w(\partial_y)^{-1}, \end{cases}$$

the equation becomes $(L + R\circ)\beta[v](w,y) = 0 \mod \mathscr{E}^{\mathbb{R}} \cdot \partial_w$, where L is an ordinary differential operator of m-th order with respect to w and $R\circ$ is a remaining operator, which is also an m-th order.

We introduce an iteration scheme

$$\begin{cases} LU_0 = 0, \\ LU_{k+1} = -R \circ U_k \mod \mathscr{E}^{\mathbb{R}} \cdot \partial_w \quad (k = 0, 1, 2, \cdots) \end{cases} \tag{3.3}$$

for a formal symbol $U(w,\xi) = \sum_{j=0}^{\infty} U_j(w)\xi^{-j/(\lambda+1)}$. Then we can get a solution which has a form $U(w,\partial_x)f(x)$ for an arbitrary microfunction $f(x)$. Therefore we get j-pure solutions in $\mathbb{R}_t \times \mathbb{R}_x^n$ in this manner. Details are shown in [6].

§ 4. Integral representations of hyperbolic equations with a large parameter

In the previous section, we construct j-pure solutions for the equation which is transformed by a fractional coordinate transform and the quantised Legendre transform. Solutions after two transforms can correspond to solutions in the original space $(t,x) \in \mathbb{R} \times \mathbb{R}^n$. We will study the correspondence between 'before transformation' and 'after transformation'.

Theorem 4.1. *For the equation* $Q(\tilde{t}, \partial_{\tilde{t}}, \partial_x)v(\tilde{t},x) = 0$, *we obtain hyperfunction solutions whose defining functions are*

$$V_{\pm}(t,z) = \sum_{l=0}^{\lambda} \left(\sum_{k=0}^{\infty} U_{\pm}^{((\lambda+1)k+l)}(z)t^k \right) \cdot (t)_{+}^{l/(\lambda+1)},$$

where $U_\pm = \sum_{k=0}^\infty U_\pm^{(k)}(z)t^k$ *are defining function of* $u(t,x)$, *where*

$$(t)_+ = \begin{cases} t, & t \geq 0 \\ 0 & t < 0. \end{cases}$$

See more details in [13].

A sketch of the proof. Set

$$P(t, \partial_t, \partial_x) = \sum_{j=0}^{m} \left(\sum_{k=0}^{j} a_{jk}(t)\partial_x^k \right) \partial_t^{m-j}.$$

By multiplying t^m, we get

$$t^m P = \sum_{j=0}^{m} \left(\sum_{k=0}^{j} a_{jk}(t)\partial_x^k \right) t^j \prod_{l=0}^{m-j-1} (t\partial_t - l).$$

By the fractional coordinate transform (3.2), the operator above becomes

$$\sum_{j=0}^{m} \left(\sum_{k=0}^{j} a_{jk}(\{(\lambda+1)\tilde{t}\}^{1/(\lambda+1)})\partial_x^k \right) \{(\lambda+1)\tilde{t}\}^{j/(\lambda+1)} \prod_{l=0}^{m-j-1} (\tilde{t}\partial_{\tilde{t}} - l).$$

Using the Taylor expansion $a_{jk}(t) = \sum_{s=0}^\infty a_{jk}^{(s)} t^s/s!$, we obtain the following form of the operator $t^m P$:

$$Q(\tilde{t}, \partial_{\tilde{t}}, \partial_x) = \sum_{\substack{l' \geq 0 \\ 0 \leq k \leq j \leq m}} \tilde{a}_{jk}^{l'} \tilde{t}^{k+l'/(\lambda+1)} \partial_x^k \tilde{E}_j,$$

where

$$\tilde{a}_{jk}^{l'} = \frac{(\lambda+1)^{k+l'/(\lambda+1)} a_{jk}^{(l'+(\lambda+1)k-j)}(0)}{(l'+(\lambda+1)k-j)!}, \quad \tilde{E}_j = \prod_{l=0}^{m-j-1} \{(\lambda+1)\tilde{t}\partial_{\tilde{t}} - l\}.$$

Then the dominant part of the operator Q is

$$\tilde{L} = \sum_{0 \leq k \leq j \leq m} \tilde{a}_{jk}^0 \tilde{t}^k \partial_x^k \tilde{E}_j,$$

where

$$\tilde{a}_{jk}^0 = \frac{a_{jk}^{((\lambda+1)k-j)}(0)}{((\lambda+1)k-j)!} (\lambda+1)^k$$

for $j/(\lambda+1) \leq k \leq j$. We remark that $\tilde{L}$ does not include the term of fractional power with respect to $\tilde{t}$.

Lemma 4.2. *We obtain*

$$\tilde{E}_j = \sum_{n=0}^{m-j} p_n \tilde{t}^n \partial_{\tilde{t}}^n,$$

where each p_n is a suitable constant with respect to $\tilde{t}$.

Therefore, we can get

$$Q = \sum_{\substack{l' \geq 0 \\ 0 \leq k \leq j \leq m \\ 0 \leq n \leq m-j}} \tilde{t}^{l'/(\lambda+1)} \left(\tilde{a}_{jk}^{l'} p_n \partial_x^k \tilde{t}^{k+n} \partial_{\tilde{t}}^n \right).$$

By this form, we have a representation of defining functions for hyperfunction solutions.

□

Here we note that the inverse quantised Legendre transform as follows:

$$\text{(4.1)} \qquad \begin{cases} w \mapsto \partial_{\tilde{t}}(\partial_x)^{-1}, & y \mapsto x + (\partial_x)^{-1}\tilde{t}\partial_{\tilde{t}} \\ \partial_w \mapsto -\tilde{t}\partial_x, & \partial_y \mapsto \partial_x. \end{cases}$$

This inverse transform induces the correspondence between solutions in $\mathbb{R}_{\tilde{t}} \times \mathbb{R}_x$ and solutions in $\mathbb{R}_w \times \mathbb{R}_y$.

From now on, we consider an ordinary differential equation

$$P(t, \partial_t, \xi)u(t, \xi) = 0$$

with a large parameter ξ instead of a partial differential equation

$$P(t, \partial_t, \partial_x)u(t, x) = 0$$

by following the way of microlocal analysis. We remark that we use the same notation u as a solution.

We suppose that solutions for the equation with a large parameter have the following integral representations:

$$\text{(4.2)} \qquad u_j(t, \xi) = \int_{C_j} \exp(\varphi_j(t)\xi s) U_j(s, \xi)\, ds \quad (j = 1, 2, \cdots, m),$$

where each C_j is a suitable contour and

$$\varphi_j(t) = \int_0^t t^\lambda \alpha_j(t)\, dt.$$

The representation (4.2) is called the Euler transform in [10]. This $u_j(t, \xi)$ is a formal solution so far.

In the WKB analysis, we set $s = 1$ and consider the Borel sum. As for ours, the solutions form integral representations with respect to s. For the sake of brevity, we consider the equation of second order. By the theory of ordinary differential equations of second order, we may set

$$U_j(s,\xi) = \sum_{k=0}^{\infty} c_k(\xi)(s - s_j)^{\rho_j + k}$$

in a neighbourhood of each $s = s_j$ $(j = 1, 2)$, where $\rho_j(\neq 0)$ is a root for the indicial equation.

Setting $\tilde{R}$ as a remaining part of the operator Q, we get the expression $Q = \tilde{L} + \tilde{R}$. If we assume that a solution of the ordinary differential equation $\tilde{L}u = 0$ forms (4.2), we can determine the integrand U_j and the contour C_j by the theory of the Euler transform ([10]).

In the last of this paper, we show some examples about solutions of type (4.2) for hyperbolic equations.

Example 4.3 (Weber's operator). For $P = \partial_t^2 - t^2\xi^2$, we have a pair of solutions for $Pu = 0$ as follows:

$$u_{\pm} = \int_{L_{\pm}} e^{s\tilde{t}\xi}(s \pm 1)^{-3/4} \sum_{k=0}^{\infty} c_{k,\pm}(s \pm 1)^k ds,$$

with a fractional coordinate transform

$$\tilde{t} = \frac{1}{2}t^2,$$

where $L_{\pm}$ are suitable paths and $c_{k,\pm}$ are constants. Using the steepest descent method, u_+ has an asymptotic expansion as

$$u_+ \sim e^{-\xi t^2/2}(\xi t^2)^{-1/4} \sum_{k=0}^{\infty} C_k t^{2k}.$$

On the other hand, for the operator $P = \partial_t^2 + \partial_t - t^2\xi^2$, we obtain $\tilde{L}(t, \partial_t, \xi) + \tilde{R}(t, \partial_t)$ with

$$\tilde{L} = t^2\partial_t^2 + \frac{1}{2}t\partial_t - t^2\xi^2, \quad \tilde{R} = \frac{\sqrt{2}}{2}t^{3/2}\partial_t.$$

We note that fundamental solutions for $Pu = 0$ are

$$\exp(-\frac{t}{2} - \frac{\xi t^2}{2})H_\nu(\sqrt{\xi}t), \quad \exp(-\frac{t}{2} - \frac{\xi t^2}{2})\,{}_1F_1(\frac{1 + 4\xi}{16\xi}, \frac{1}{2}; \xi t^2)$$

with $\nu = (-1 - 4\xi)/(8\xi)$, where $H_\nu(x)$ stands for the Hermite function and ${}_1F_1(\alpha, \gamma; z)$ stands for the confluent hypergeometric function.

We apply our method to the Airy type case.

Example 4.4 (Airy's operator). For $P = \partial_t^2 - t\xi^2$, the WKB solutions which we denote $u_\pm$ are

$$u_\pm = \frac{\sqrt{t}}{\sqrt{t^{3/2}\xi + \frac{5}{32}t^{-3/2}\xi^{-1}}} \exp \pm \left(\frac{2}{3}t^{3/2} + \cdots \right)$$

and one of their Borel sums becomes

$$u_{+,B}(t,y) = \frac{\sqrt{3}}{2\sqrt{\pi}} \frac{1}{t} \left(\frac{3}{4}\frac{y}{t^{3/2}} + \frac{1}{2} \right)^{-1/2} F(\frac{1}{6}, \frac{5}{6}, \frac{1}{2}; \frac{3}{4}\frac{y}{t^{3/2}} + \frac{1}{2})$$

([22]), where $F(\alpha, \beta, \gamma; z)$ is the Gauss hypergeometric function.

On the other hand, we have $Q = \tilde{t}^2\partial_{\tilde{t}}^2 + \frac{1}{3}\tilde{t}\partial_{\tilde{t}} - \tilde{t}^2\xi^2 = \tilde{L}$. Then a solution by our method is as follows:

$$\begin{aligned} u_j(t,\xi) &= \int_{L_j} e^{s\tilde{t}\xi}(s^2-1)^{-5/6}\, ds \sim e^{\tilde{t}}\tilde{t}^{-1/6} \sum_{n=0}^{\infty} c_n(2\tilde{t})^{-n} \\ &= e^{\xi t^{3/2}}(\xi t)^{-1/4} \sum_{n=0}^{\infty} d_n t^{-(3/2)n} \quad (j = 1, 2) \end{aligned}$$

with some constants C_j and suitable contours L_j $(j = 1, 2)$.

We give an example in the case that the Airy operator with a lower order term. For $P = \partial_t^2 + \partial_t - t\xi^2$, fundamental solutions for $Pu = 0$ become

$$e^{-t/2}\mathrm{Ai}(\frac{\frac{1}{4} + \xi^2 t}{\xi^{4/3}}), \quad e^{-t/2}\mathrm{Bi}(\frac{\frac{1}{4} + \xi^2 t}{\xi^{4/3}}),$$

where $\mathrm{Ai}(z)$ and $\mathrm{Bi}(z)$ stand for the Airy functions. We remark that the dominant term of $\mathrm{Ai}(((1/4) + \xi^2 t)/\xi^{4/3})$ is $\mathrm{Ai}(1/(4\xi^{4/3})) + \mathrm{Ai}'(1/(4\xi^{4/3}))\xi^{2/3}t + O(t^2)$.

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