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Inverse and direct bifurcation problems for nonlinear elliptic equations

Tetsutaro Shibata
Graduate School of Engineering
Hiroshima University

1 Elliptic inverse bifurcation problems

We first consider

\[-\Delta u + f(u) = \lambda u \quad \text{in } \Omega,\]
\[u > 0, \quad \text{in } \Omega,\]
\[u(0) = 0 \quad \text{on } \partial\Omega.\] (1.1)

where $\Omega \subset \mathbb{R}^N$ is an appropriately smooth bounded domain, and $\lambda > 0$ is a parameter. We assume that $f(u)$ is unknown to satisfy the conditions (A.1)-(A.3):

(A.1) $f(u)$ is a function of $C^1$ for $u \geq 0$ satisfying $f(0) = f'(0) = 0$.

(A.2) $f(u)/u$ is strictly increasing for $u \geq 0$.

(A.3) $f(u)/u \to \infty$ as $u \to \infty$.

The typical examples of $f(u)$ which satisfy (A.1)-(A.3) are as follows.

\[f(u) = u^p \quad (p > 1),\]
\[f(u) = u^p + u^m \quad (p > m > 1).\]

Our first purpose is to study the inverse bifurcation problems in $L^q$-framework ($1 \leq q \leq \infty$). From mathematical point of view, since (1.1) is regarded as an eigenvalue problem, it seems natural to treat it in $L^2$-framework. Moreover, from biological point of view, it also seems significant to investigate it in $L^1$-framework.

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Now we introduce the notion of $L^q$-bifurcation curve. We know the following fundamental properties of bifurcation diagrams of (1.1).

1. Let $1 \leq q \leq \infty$ be fixed. Let $\| \cdot \|_q$ be $L^q$-norm. For any given $\alpha > 0$, there exists a unique solution pair $(\lambda, u) = (\lambda(q, \alpha), u_\alpha) \in \mathbb{R}_+ \times C^2(\bar{\Omega})$ such that $\|u_\alpha\|_q = \alpha$.

2. The following set gives all the solutions of (1.1):

$$\{(\lambda(q, \alpha), u_\alpha) : \alpha > 0\} \subset \mathbb{R}_+ \times C^2(\bar{\Omega})$$

3. $\lambda(q, \alpha) \to \lambda_1$ ($\alpha \to 0$, $\lambda_1$ : the first eigenvalue of $-\Delta_D$), $\lambda(q, \alpha) \nearrow \infty$ ($\alpha \to \infty$).

Let $f(u) = f_1(u)$ and $f(u) = f_2(u)$ be unknown to satisfy (A.1)-(A.3). Furthermore, let

$$F_j(u) := \int_0^u f_j(s)ds \quad (j = 1, 2).$$

Assume that $F_1$ and $F_2$ satisfy the following condition (B.1).

(B.1) Let $W := \{u \geq 0 : F_1(u) = F_2(u)\}$. Then $W$ consists, at most, of the (finite or infinite numbers of) intervals and the points $\{u_n\}_{n=1}^\infty$ whose accumulation point is only $\infty$.

**Theorem 1.1.** [14] Assume that $f_1$ and $f_2$ are unknown to satisfy (A.1)-(A.3) and (B.1). Furthermore, if $N \geq 2$, then assume that $f_1$ and $f_2$ satisfy the following (A.4).

(A.4) For $u, v \geq 0$,

$$F_j(u + v) \leq C(F_j(u) + F_j(v)) \quad (j = 1, 2).$$

Suppose $\lambda_1(2, \alpha) = \lambda_2(2, \alpha)$ for any $\alpha > 0$. Here, $\lambda_j(2, \alpha)$ is the $L^2$-bifurcation curve associated with $f(u) = f_j(u)$ ($j = 1, 2$). Then $f_1(u) \equiv f_2(u)$ for $u \geq 0$. 

**Fig. 1**
2 Sketch of the Proof of Theorem 1.1

For simplicity, we prove Theorem 1.1 for the case $N = 1$. Let $\Omega = I = (0,1)$. For $j = 1, 2$ and $v \in H_0^1(I)$, let

$$\Phi_j(v) := \frac{1}{2} \|v'\|^2_2 + \int_0^1 F_j(v(t))dt.$$  \hfill (2.1)

For $\alpha > 0$, we put

$$M_\alpha := \{v \in H_0^1(I) : \|v\|_2 = \alpha\}.$$  \hfill (2.2)

By taking a minimizing sequence, Lagrange multiplier theorem and strong maximum principle, there exists a Lagrange multiplier $\lambda_j(\alpha)$ and a unique minimizer $u_{j,\alpha} \in M_\alpha$ which satisfies (1.1) with $f = f_j$. Then by direct calculation, we obtain the following lemma.

Lemma 2.1. $C_1(\alpha) = C_2(\alpha)$ for $\alpha \geq 0$.

Now we give the sketch of the proof of Theorem 1.1.

Sketch of the Proof of Theorem 1.1 for $N = 1$.

Clearly, $0 \in W$, where $W := \{u \geq 0 : F_1(u) = F_2(u)\}$. First, assume that $0 \in W$ is contained in the interval $[0, \epsilon]$ for some constant $0 < \epsilon \ll 1$. This implies that for $0 \leq u \leq \epsilon$,

$$F_1(u) = F_2(u).$$

Let $K$ be a connected component of $W$ satisfying $[0, \epsilon] \subset K$. Then $K = [0, u_1]$. If $u_1 < \infty$, then without loss of generality, by (B.1), there exists a constant $0 < \epsilon \ll 1$ such that

$$F_1(u) = F_2(u) \quad (0 \leq u \leq u_1),$$

$$F_1(u) < F_2(u), \quad (u_1 < u < u_1 + \epsilon).$$

Now we choose $\alpha > 0$ satisfying $\|u_{2,\alpha}\|_\infty = u_1 + \epsilon$. Then

$$C_1(\alpha) \leq \Phi_1(u_{2,\alpha}) = \frac{1}{2} \|u_{2,\alpha}'\|^2_2 + \int_0^1 F_1(u_{2,\alpha}(t))dt$$

$$< \frac{1}{2} \|u_{2,\alpha}'\|^2_2 + \int_0^1 F_2(u_{2,\alpha}(t))dt$$

$$= \Phi_2(u_{2,\alpha}) = C_2(\alpha).$$

This contradicts Lemma 2.1. Therefore, we see that $u_1 = \infty$ and $K = [0, \infty)$. This implies $F_1(u) \equiv F_2(u)$, and consequently, $f_1(u) \equiv f_2(u)$.

We can also treat the case where $0 \in W$ is an isolated point in $W$. Thus the proof is complete.
3 \(L^1\)-inverse bifurcation problems

It seems that the assumption \(\lambda_1(2, \alpha) = \lambda_2(2, \alpha)\) for any \(\alpha > 0\) in Theorem 1.1 seems little bit strong. It seems better to consider the problem under more weaker condition

\[
\lambda_1(q, \alpha) \approx \lambda_2(q, \alpha) \quad \text{in some sense for } \alpha > \alpha_0,
\]

where \(\alpha_0 > 0\) is a constant. To do this, we consider the following inverse problem.

Let \(\lambda_0(1, \alpha)\) be the \(L^1\)-bifurcation curve associated with \(f(u) = u^p\) \((p > 1)\). Furthermore, let \(\lambda(1, \alpha)\) be the \(L^1\)-bifurcation curve associated with \(f(u) = u^p + g(u)\), where \(g(u)\) is an unknown function.

**Problem.** Assume that for \(\alpha \gg 1\)

\[
\lambda(1, \alpha) \approx \lambda_0(1, \alpha)
\]

in some sense. Then can we conclude \(g(u) \equiv 0\)?

To solve this problem, we assume the following conditions on \(g\).

(B.2) \(g(u)\) is \(C^1\) function for \(u \geq 0\) with compact support.

We note that \(\eta_1(x) = \eta_2(x)\) nearly exponentially for \(x \gg 1\) implies that

\[
\eta_1(x) = \eta_2(x) + o(x^{-N}) \quad (x \to \infty)
\]

for any \(N \in \mathbb{N}\).

**Theorem 3.1 [16].** Let \(N = 1\) and consider (1.1). Let \(p > 1\) be a given constant and assume that \(f(u) = u^p + g(u)\) satisfies (A.1)-(A.3) and (B.2), where \(g(u)\) is unknown. Suppose \(\lambda(1, \alpha) = \lambda_0(1, \alpha)\) nearly exponentially. Then \(g(u) \equiv 0\).

The proof of Theorem 3.1 relies on the fact that the equation (1.1) is ODE, and we treat it in \(L^1\)-framework with the aid of the time map.

Now we give the brief sketch of the proof of Theorem 3.1. Without loss of generality, we assume that \(\text{supp } g \subset [a, b] \quad (0 \leq a < b)\). \(C\) denotes arbitrary positive constants independent of \(\lambda \gg 1\).

We know that \((\lambda, u_\lambda) \in \mathbb{R}+ \times C^2(\overline{I})\) : the solution of (1.1) for given \(\lambda > \pi^2\). Therefore, \(\alpha = \|u_\lambda\|_1\). We write \(\lambda = \lambda(\alpha)\) for simplicity. Let

\[
G(u) := \int_0^u g(s)ds.
\]
For two functions $X(\lambda)$ and $Y(\lambda)$,

$$X(\lambda) \sim Y(\lambda)$$

implies

$$C^{-1}Y(\lambda) \leq X(\lambda) \leq CY(\lambda) \quad (\lambda \gg 1).$$

(3.2)

It is well known that for $\lambda \gg 1$,

$$\|u_\lambda\|_{p-1}^\| = \lambda \left(1 + O(e^{-C\sqrt{\lambda}})\right).$$

(3.3)

We know that for $\lambda > \pi^2$

$$u_\lambda(t) = u_\lambda(1-t), \quad 0 \leq t \leq 1,$$

(3.4)

$$u_\lambda \left(\frac{1}{2}\right) = \max_{0 \leq t \leq 1} u_\lambda(t) = \|u_\lambda\|_{\infty},$$

(3.5)

$$u_\lambda'(t) > 0, \quad 0 \leq t < \frac{1}{2}.$$ (3.6)

For $\lambda > \pi^2$ and $0 \leq s \leq 1$, let

$$L_\lambda(s) := 1 - s^2 - \frac{2}{p+1} (1 - s^{p+1}),$$

(3.7)

$$M_\lambda(s) := 1 - s^2 - \frac{2}{p+1} \|u_\lambda\|_{\infty} (1 - s^{p+1}) - \frac{2}{\lambda \|u_\lambda\|_{\infty}^2} (G(\|u_\lambda\|_{\infty}) - G(\|u_\lambda\|_{\infty} s)), $$

(3.8)

$$U_\lambda := \frac{2(\|u_\lambda\|_{\infty} - \lambda)}{(p+1)\lambda} \int_0^1 \frac{(1-s)(1 - s^{p+1})}{\sqrt{M_\lambda(s)}\sqrt{L_\lambda(s)}(\sqrt{M_\lambda(s)} + \sqrt{L_\lambda(s)})} ds,$$

$$V_\lambda := \frac{2}{\lambda \|u_\lambda\|_{\infty}^2} \int_0^1 \frac{(1-s)(G(\|u_\lambda\|_{\infty}) - G(\|u_\lambda\|_{\infty} s))}{\sqrt{M_\lambda(s)}\sqrt{L_\lambda(s)}(\sqrt{M_\lambda(s)} + \sqrt{L_\lambda(s)})} ds.$$ (3.7)

Lemma 3.2. For $\lambda \gg 1$

$$\|u_\lambda\|_{\infty} - \|u_\lambda\|_1 = \frac{1}{\sqrt{\lambda}} \|u_\lambda\|_{\infty}(C(1) + U_\lambda + V_\lambda),$$

(3.9)

where $C(1)$ is a constant determined explicitly.

Lemma 3.3. For $\lambda \gg 1$

$$|U_\lambda| \leq C\sqrt{\lambda} e^{-C\sqrt{\lambda}}.$$ (3.10)
Proposition 3.4. Assume that $V_\lambda = 0$ for $\lambda \gg 1$. That is,
\begin{equation}
\|u_\lambda\|_\infty - \|u_\lambda\|_1 = \frac{1}{\sqrt{\lambda}}\|u_\lambda\|_\infty(C(1) + U_\lambda).
\end{equation}
(3.11)

Then for $\alpha \gg 1$,
\begin{equation}
\lambda(\alpha) = \alpha^{p-1} + C_1\alpha^{(p-1)/2} + \sum_{k=0}^{N} a_k\alpha^{k(1-p)/2} + o(\alpha^{N(1-p)/2}),
\end{equation}
(3.12)
where $C_1, \{a_j\}_{j=0}^{N}$ are constants determined explicitly.

To prove Proposition 3.3, we would like to calculate $V_\lambda$ precisely.

Lemma 3.5. Let $H(\theta) := G(b) - G(\theta)$. Then, for $\lambda \gg 1$,
\begin{equation}
V_\lambda \sim \sum_{k=0}^{\infty} \left( C_k \int_{0}^{b} H(\theta)\theta^k d\theta \right) \|u_\lambda\|_\infty^{-p+2+k},
\end{equation}
where $C_k \neq 0 (k \in N_0 := N \cup \{0\})$ is a constant.

It should be mentioned that, to prove Lemma 3.5, we need the condition $q = 1$.

By using Lemma 3.5 and the assumption that $\lambda(1, \alpha) = \lambda_0(1, \alpha)$ nearly exponentially, we obtain the following Lemma 3.6.

Lemma 3.6. Let $H(\theta) := G(b) - G(\theta)$. Then for any non-negative integer $n$.
\begin{equation}
\int_{0}^{b} H(\theta)\theta^n d\theta = 0.
\end{equation}
(3.13)

We can prove Lemma 3.6, since we treat it in $L^1$-framework. Theorem 3.1 follows from Lemma 3.6. Thus the proof is complete.

4 Direct problems

We consider the semilinear non-autonomous logistic equation of population dynamics
\begin{align}
-u''(t) + k(t)u(t)^p &= \lambda u(t), & t \in I := (-1/2, 1/2), \\
u(t) &> 0 & t \in I, \\
u(-1/2) &= u(1/2) = 0,
\end{align}
(4.1)(4.2)(4.3)
where $p > 1$ is a given constant, and $\lambda > 0$ is a parameter. We assume that $k(t) \in C^2(I)$ satisfies the following conditions.
\begin{align}
k(t) &> 0, & k(t) = k(-t), & t \in I, \\
k'(t) &\geq 0, & 0 \leq t \leq 1/2.
\end{align}
(4.4)(4.5)
The local and global structure of the bifurcation diagrams of (4.1)-(4.3) have been investigated by many authors in $L^\infty$-framework. Especially, the following basic properties are well known.

(a) For each $\lambda > \pi^2$, there exists a unique solution $u_\lambda \in C^2(I)$ such that $(\lambda, u_\lambda)$ satisfies $(4.1)-(4.3)$.

(b) The set $\{(\lambda, u_\lambda) : \lambda > \pi^2\}$ gives all the solutions of $(1.1)-(1.3)$ and is a continuous unbounded curve in $\mathbb{R}_+ \times C(I)$ emanating from $(\pi^2, 0)$.

(c) $\pi^2 < \mu < \lambda$ holds if and only if $u_\mu < u_\lambda$ in $I$.

For a given $\alpha > 0$, we denote by $(\lambda(q, \alpha), u_\alpha) \in \{(\lambda > \pi^2) \times C^2(I)\}$ the solution pair of $(4.1)-(4.3)$ with $\|k^{1/(p-1)}u_\alpha\|_q = \alpha$, which uniquely exists by (c) above. We call the graph $\lambda = \lambda(q, \alpha)$ $(\alpha > 0)$ the $L^q$-bifurcation diagram of $(4.1)-(4.3)$. Then we know that

(d) $\lambda(q, \alpha)$ is increasing for $\alpha > 0$ and $\lambda(q, \alpha) \rightarrow \infty$ as $\alpha \rightarrow \infty$.

We assume the following condition.

(H) Assume that $k(t)$ satisfies (1.4) and (1.5). Furthermore, $K'(t)/K(t)$ and $K''(t)/K(t)$ are non-increasing for $0 \leq t \leq 1/2$, where $K(t) := k(t)^{-1/(p-1)}$.

Comparing to the autonomous case, however, there are no works which obtain precise asymptotic formula in non-autonomous case. By the terms which come from $k, k', k''$ and $u'$, the tools for autonomous case are not useful any more in non-autonomous problems. To overcome this difficulty, we adopt a new parameter $\|k^{1/(p-1)}u_\alpha\|_q = \alpha$ to parameterize the bifurcation curve $\lambda(q, \alpha)$. By the new idea above, the tools for autonomous problems can be available to our non-autonomous case.

**Theorem 4.1** [15]. Let $p > 1$ and $q \geq 1$ be fixed. Assume that $k$ is a given function which satisfies (H). Then as $\alpha \rightarrow \infty$,

$$\lambda(q, \alpha) \geq \alpha^{p-1} + C_1 \alpha^{(p-1)/2} + a_0 + m_0 - r_{p,q} + o(1), \quad (4.6)$$

$$\lambda(q, \alpha) \leq \alpha^{p-1} + C_1 \alpha^{(p-1)/2} + a_0 + M_0 + o(1), \quad (4.7)$$

where $C_1, C_2, C(q), a_0, M_0, M_1, m_0, r_{p,q}, w_{p,q}$ are constants determined explicitly.

The proof of Theorem 4.1 depends on the precise calculation of the time map.

**References**


