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Inverse and direct bifurcation problems for nonlinear elliptic equations

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1 Elliptic inverse bifurcation problems

We first consider

\[-\Delta u + f(u) = \lambda u \quad \text{in} \; \Omega,\]
\[u > 0, \quad \text{in} \; \Omega,\]
\[u(0) = 0 \quad \text{on} \; \partial\Omega.\]

(1.1)

where $\Omega \subset \mathbb{R}^N$ is an appropriately smooth bounded domain, and $\lambda > 0$ is a parameter. We assume that $f(u)$ is unknown to satisfy the conditions (A.1)–(A.3):

(A.1) $f(u)$ is a function of $C^1$ for $u \geq 0$ satisfying $f(0) = f'(0) = 0$.
(A.2) $f(u)/u$ is strictly increasing for $u \geq 0$.
(A.3) $f(u)/u \to \infty$ as $u \to \infty$.

The typical examples of $f(u)$ which satisfy (A.1)–(A.3) are as follows.

\[f(u) = u^p \quad (p > 1),\]
\[f(u) = u^p + u^m \quad (p > m > 1).\]

Our first purpose is to study the inverse bifurcation problems in $L^q$-framework ($1 \leq q \leq \infty$). From mathematical point of view, since (1.1) is regarded as an eigenvalue problem, it seems natural to treat it in $L^2$-framework. Moreover, from biological point of view, it also seems significant to investigate it in $L^1$-framework.

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Now we introduce the notion of $L^q$-bifurcation curve. We know the following fundamental properties of bifurcation diagrams of (1.1).

1. Let $1 \leq q \leq \infty$ be fixed. Let $\| \cdot \|_q$ be $L^q$-norm. For any given $\alpha > 0$, there exists a unique solution pair $(\lambda, u) = (\lambda(q, \alpha), u_\alpha) \in \mathbb{R}_+ \times C^2(\overline{\Omega})$ such that $\|u_\alpha\|_q = \alpha$.

2. The following set gives all the solutions of (1.1):

$$\{(\lambda(q, \alpha), u_\alpha) : \alpha > 0\} \subset \mathbb{R}_+ \times C^2(\overline{\Omega})$$

3. $\lambda(q, \alpha) \to \lambda_1$ ($\alpha \to 0$, $\lambda_1$ : the first eigenvalue of $-\Delta_D$), $\lambda(q, \alpha) \not\to \infty$ ($\alpha \to \infty$).

Let $f(u) = f_1(u)$ and $f(u) = f_2(u)$ be unknown to satisfy (A.1)–(A.3). Furthermore, let

$$F_j(u) := \int_0^u f_j(s)ds \quad (j = 1, 2).$$

Assume that $F_1$ and $F_2$ satisfy the following condition (B.1).

(B.1) Let $W := \{u \geq 0 : F_1(u) = F_2(u)\}$. Then $W$ consists, at most, of the (finite or infinite numbers of) intervals and the points $\{u_n\}_{n=1}^{\infty}$ whose accumulation point is only $\infty$.

**Theorem 1.1.** [14] Assume that $f_1$ and $f_2$ are unknown to satisfy (A.1)–(A.3) and (B.1). Furthermore, if $N \geq 2$, then assume that $f_1$ and $f_2$ satisfy the following (A.4).

(A.4) For $u, v \geq 0$,

$$F_j(u + v) \leq C(F_j(u) + F_j(v)) \quad (j = 1, 2).$$

Suppose $\lambda_1(2, \alpha) = \lambda_2(2, \alpha)$ for any $\alpha > 0$. Here, $\lambda_j(2, \alpha)$ is the $L^2$-bifurcation curve associated with $f(u) = f_j(u)$ ($j = 1, 2$). Then $f_1(u) \equiv f_2(u)$ for $u \geq 0$. 

2 Sketch of the Proof of Theorem 1.1

For simplicity, we prove Theorem 1.1 for the case $N=1$. Let $\Omega = I = (0,1)$. For $j = 1, 2$ and $v \in H_0^1(I)$, let
\[
\Phi_j(v) := \frac{1}{2} \|v'\|_2^2 + \int_0^1 F_j(v(t))dt.
\] (2.1)

For $\alpha > 0$, we put
\[
M_\alpha := \{v \in H_0^1(I) : \|v\|_2 = \alpha\}.
\]

For $j = 1, 2$ and $\alpha > 0$ we put
\[
C_j(\alpha) := \min\{\Phi_j(v) : v \in M_\alpha\}.
\] (2.2)

By taking a minimizing sequence, Lagrange multiplier theorem and strong maximum principle, there exists a Lagrange multiplier $\lambda_j(\alpha)$ and a unique minimizer $u_{j,\alpha} \in M_\alpha$ which satisfies (1.1) with $f = f_j$. Then by direct calculation, we obtain the following lemma.

**Lemma 2.1.** $C_1(\alpha) = C_2(\alpha)$ for $\alpha \geq 0$.

Now we give the sketch of the proof of Theorem 1.1.

**Sketch of the Proof of Theorem 1.1 for $N=1$.**

Clearly, $0 \in W$, where $W := \{u \geq 0 : F_1(u) = F_2(u)\}$. First, assume that $0 \in W$ is contained in the interval $[0, \epsilon]$ for some constant $0 < \epsilon \ll 1$. This implies that for $0 \leq u \leq \epsilon$,
\[
F_1(u) = F_2(u).
\]

Let $K$ be a connected component of $W$ satisfying $[0, \epsilon] \subset K$. Then $K = [0, u_1]$. If $u_1 < \infty$, then without loss of generality, by (B.1), there exists a constant $0 < \epsilon \ll 1$ such that
\[
F_1(u) = F_2(u) \quad (0 \leq u \leq u_1),
\]
\[
F_1(u) < F_2(u) \quad (u_1 < u < u_1 + \epsilon).
\]

Now we choose $\alpha > 0$ satisfying $\|u_{2,\alpha}\|_\infty = u_1 + \epsilon$. Then
\[
C_1(\alpha) \leq \Phi_1(u_{2,\alpha}) = \frac{1}{2} \|u_{2,\alpha}'\|_2^2 + \int_0^1 F_1(u_{2,\alpha}(t))dt
\]
\[
< \frac{1}{2} \|u_{2,\alpha}'\|_2^2 + \int_0^1 F_2(u_{2,\alpha}(t))dt = \Phi_2(u_{2,\alpha}) = C_2(\alpha).
\]

This contradicts Lemma 2.1. Therefore, we see that $u_1 = \infty$ and $K = [0, \infty)$. This implies $F_1(u) \equiv F_2(u)$, and consequently, $f_1(u) \equiv f_2(u)$.

We can also treat the case where $0 \in W$ is an isolated point in $W$. Thus the proof is complete.
3 \textit{L}^1\textit{-inverse bifurcation problems}

It seems that the assumption $\lambda_1(2, \alpha) = \lambda_2(2, \alpha)$ for any $\alpha > 0$ in Theorem 1.1 seems a bit strong. It seems better to consider the problem under more weaker condition

$$\lambda_1(q, \alpha) \approx \lambda_2(q, \alpha) \text{ in some sense for } \alpha > \alpha_0,$$

where $\alpha_0 > 0$ is a constant. To do this, we consider the following inverse problem.

Let $\lambda_0(1, \alpha)$ be the $L^1$-bifurcation curve associated with $f(u) = u^p$ $(p > 1)$. Furthermore, let $\lambda(1, \alpha)$ be the $L^1$-bifurcation curve associated with $f(u) = u^p + g(u)$, where $g(u)$ is an unknown function.

**Problem.** Assume that for $\alpha \gg 1$

$$\lambda(1, \alpha) \approx \lambda_0(1, \alpha)$$

in some sense. Then can we conclude $g(u) \equiv 0$?

To solve this problem, we assume the following conditions on $g$.

**(B.2)** $g(u)$ is $C^1$ function for $u \geq 0$ with compact support.

We note that $\eta_1(x) = \eta_2(x)$ nearly exponentially for $x \gg 1$ implies that

$$\eta_1(x) = \eta_2(x) + o(x^{-N}) \quad (x \to \infty)$$

for any $N \in \mathbb{N}$.

**Theorem 3.1** \cite{16}. \textit{Let} $N = 1$ \textit{and consider} (1.1). \textit{Let} $p > 1$ \textit{be a given constant and assume that} $f(u) = u^p + g(u)$ \textit{satisfies} (A.1)-(A.3) \textit{and} (B.2), where $g(u)$ is unknown. \textit{Suppose} $\lambda(1, \alpha) = \lambda_0(1, \alpha)$ \textit{nearly exponentially}. \textit{Then} $g(u) \equiv 0$.

The proof of Theorem 3.1 relies on the fact that the equation (1.1) is ODE, and we treat it in $L^1$-framework with the aid of the time map.

Now we give the brief sketch of the proof of Theorem 3.1. Without loss of generality, we assume that $\text{supp } g \subset [a, b]$ $(0 \leq a < b)$. $C$ denotes arbitrary positive constants independent of $\lambda \gg 1$.

We know that $(\lambda, u_\lambda) \in \mathbb{R}_+ \times C^2(\bar{I})$ : the solution of (1.1) for given $\lambda > \pi^2$. Therefore, $\alpha = \|u_\lambda\|_1$. We write $\lambda = \lambda(\alpha)$ for simplicity. Let

$$G(u) := \int_0^u g(s) ds.$$
For two functions $X(\lambda)$ and $Y(\lambda)$,

$$X(\lambda) \sim Y(\lambda)$$

implies

$$C^{-1}Y(\lambda) \leq X(\lambda) \leq CY(\lambda) \quad (\lambda \gg 1).$$

(3.2)

It is well known that for $\lambda \gg 1$,

$$\|u_\lambda\|^{p-1}_{\infty} = \lambda \left(1 + O(e^{-C\sqrt{\lambda}})\right).$$

(3.3)

We know that for $\lambda > \pi^2$

$$u_\lambda(t) = u_\lambda(1 - t), \quad 0 \leq t \leq 1,$$

(3.4)

$$u_\lambda\left(\frac{1}{2}\right) = \max_{0 \leq t \leq 1} u_\lambda(t) = \|u_\lambda\|_{\infty},$$

(3.5)

$${\lambda} u_\lambda'(t) > 0, \quad 0 \leq t < \frac{1}{2}.$$  

(3.6)

For $\lambda > \pi^2$ and $0 \leq s \leq 1$, let

$$L_\lambda(s) := 1 - s^2 - \frac{2}{p+1}(1 - s^{p+1}),$$

(3.7)

$$M_\lambda(s) := 1 - s^2 - \frac{2}{p+1} \frac{\|u_\lambda\|_{\infty}}{\lambda} (1 - s^{p+1})$$

$$- \frac{2}{\lambda \|u_\lambda\|^{2}_{\infty}} \left( G(\|u_\lambda\|_{\infty}) - G(\|u_\lambda\|_{\infty}s) \right),$$

(3.8)

$$U_\lambda := \frac{2(\|u_\lambda\|_{\infty} - \lambda)}{(p+1)\lambda} \int_{0}^{1} \frac{(1-s)(1-s^{p+1})}{\sqrt{M_\lambda(s)}\sqrt{L_\lambda(s)}(\sqrt{M_\lambda(s)} + \sqrt{L_\lambda(s)})} ds,$$

$$V_\lambda := \frac{2}{\lambda \|u_\lambda\|^{2}_{\infty}} \int_{0}^{1} \frac{(1-s)(G(\|u_\lambda\|_{\infty}) - G(\|u_\lambda\|_{\infty}s))}{\sqrt{M_\lambda(s)}\sqrt{L_\lambda(s)}(\sqrt{M_\lambda(s)} + \sqrt{L_\lambda(s)})} ds.$$

Lemma 3.2. For $\lambda \gg 1$

$$\|u_\lambda\|_{\infty} - \|u_\lambda\|_{1} = \frac{1}{\sqrt{\lambda}} \|u_\lambda\|_{\infty}(C(1) + U_\lambda + V_\lambda),$$

(3.9)

where $C(1)$ is a constant determined explicitly.

Lemma 3.3. For $\lambda \gg 1$

$$|U_\lambda| \leq C\sqrt{\lambda} e^{-C\sqrt{\lambda}}.$$  

(3.10)
Proposition 3.4. Assume that $V_{\lambda} = 0$ for $\lambda \gg 1$. That is,
\[ \|u_{\lambda}\|_{\infty} - \|u_{\lambda}\|_{1} = \frac{1}{\sqrt{\lambda}} \|u_{\lambda}\|_{\infty}(C(1) + U_{\lambda}). \] (3.11)

Then for $\alpha \gg 1$,
\[ \lambda(\alpha) = \alpha^{p-1} + C_{1}\alpha^{(p-1)/2} + \sum_{k=0}^{N} a_{k}\alpha^{k(1-p)/2} + o(\alpha^{N(1-p)/2}), \] (3.12)
where $C_{1}, \{a_{j}\}_{j=0}^{N}$ are constants determined explicitly.

Theorem 3.1 follows from Lemma 3.6. Thus the proof is complete.

4 Direct problems

We consider the semilinear non-autonomous logistic equation of population dynamics
\[ -u''(t) + k(t)u(t)^{p} = \lambda u(t), \quad t \in I := (-1/2, 1/2), \] (4.1)
\[ u(t) > 0 \quad t \in I, \] (4.2)
\[ u(-1/2) = u(1/2) = 0, \] (4.3)
where $p > 1$ is a given constant, and $\lambda > 0$ is a parameter. We assume that $k(t) \in C^{2}(\bar{I})$ satisfies the following conditions.
\[ k(t) > 0, \quad k(t) = k(-t), \quad t \in \bar{I}, \] (4.4)
\[ k'(t) \geq 0, \quad 0 \leq t \leq 1/2. \] (4.5)
The local and global structure of the bifurcation diagrams of (4.1)-(4.3) have been investigated by many authors in $L^\infty$-framework. Especially, the following basic properties are well known.

(a) For each $\lambda > \pi^2$, there exists a unique solution $u_\lambda \in C^2(\bar{I})$ such that $(\lambda, u_\lambda)$ satisfies (4.1)-(4.3).

(b) The set $\{ (\lambda, u_\lambda) : \lambda > \pi^2 \}$ gives all the solutions of (1.1)-(1.3) and is a continuous unbounded curve in $\mathbb{R}_+ \times C(\bar{I})$ emanating from $(\pi^2, 0)$.

(c) $\pi^2 < \mu < \lambda$ holds if and only if $u_\mu < u_\lambda$ in $I$.

For a given $\alpha > 0$, we denote by $(\lambda(q, \alpha), u_\alpha) \in \{ \lambda > \pi^2 \} \times C^2(\bar{I})$ the solution pair of (4.1)-(4.3) with $\| k^{1/(p-1)} u_\alpha \|_q = \alpha$, which uniquely exists by (c) above. We call the graph $\lambda = \lambda(q, \alpha)$ $(\alpha > 0)$ the $L^q$-bifurcation diagram of (4.1)-(4.3). Then we know that

(d) $\lambda(q, \alpha)$ is increasing for $\alpha > 0$ and $\lambda(q, \alpha) \to \infty$ as $\alpha \to \infty$.

We assume the following condition.

(H) Assume that $k(t)$ satisfies (1.4) and (1.5). Furthermore, $K'(t)/K(t)$ and $K''(t)/K(t)$ are non-increasing for $0 \leq t \leq 1/2$, where $K(t) := k(t)^{-1/(p-1)}$.

Comparing to the autonomous case, however, there are no works which obtain precise asymptotic formula in non-autonomous case. By the terms which come from $k, k', k''$ and $u'$, the tools for autonomous case are not useful any more in non-autonomous problems. To overcome this difficulty, we adopt a new parameter $\| k^{1/(p-1)} u_\alpha \|_q = \alpha$ to parameterize the bifurcation curve $\lambda(q, \alpha)$. By the new idea above, the tools for autonomous problems can be available to our non-autonomous case.

**Theorem 4.1** [15]. Let $p > 1$ and $q \geq 1$ be fixed. Assume that $k$ is a given function which satisfies (H). Then as $\alpha \to \infty$,

$$\lambda(q, \alpha) \geq \alpha^{p-1} + C_1 \alpha^{(p-1)/2} + a_0 + m_0 - r_{p,q} + o(1), \quad (4.6)$$

$$\lambda(q, \alpha) \leq \alpha^{p-1} + C_1 \alpha^{(p-1)/2} + a_0 + M_0 + o(1), \quad (4.7)$$

where $C_1, C_2, C(q), a_0, M_0, M_1, m_0, r_{p,q}, w_{p,q}$ are constants determined explicitly.

The proof of Theorem 4.1 depends on the precise calculation of the time map.

**References**


