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On the influence of delay to the strength of oscillation criteria for neutral second order half-linear differential equations

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Abstract

In the paper we study the second order neutral delay half-linear differential equation

$$\Big[r(t)\Phi(z'(t))\Big]' + q(t)\Phi(x(\sigma(t))) = 0,$$

where $\Phi(t) = |t|^{p-2}t$, $p \geq 2$ and $z(t) = x(t) + b(t)x(\tau(t))$. We summarize three different methods available to derive oscillation criteria for this equation and compare them on the particular example of Euler type equation and proportional delay. We show that for this particular equation the comparison method produces better results if the delay is significant, whereas Riccati equation method produces sharper results if the delay argument is close to the classical argument. We also point out some recent development in this area.

Keywords: half-linear differential equation, oscillation criteria, Riccati technique, delay equation, neutral equation, Euler type equation
MSC: 34K11, 34K40

1 Introduction

Consider the second order half-linear neutral differential equation

$$\Big[r(t)\Phi(z'(t))\Big]' + c(t)\Phi(x(\sigma(t))) = 0, \quad z(t) = x(t) + b(t)x(\tau(t)), \tag{1}$$

where $\Phi(t) = |t|^{p-2}t$ is the power type nonlinearity and $p \geq 2$, which ensures that the function $\Phi(\cdot)$ is a convex function on $(0, \infty)$.

Under the solution of (1) we understand any differentiable function $x(t)$ which does not identically equal zero eventually, such that $r(t)\Phi(z'(t))$ is differentiable and (1) holds for large t.

The solution of equation (1) is said to be oscillatory if it has infinitely many zeros tending to infinity. Equation (1) is said to be oscillatory if all its solutions are oscillatory. In the opposite case, i.e., if there exists an eventually positive solution of (1), equation (1) is said to be nonoscillatory.

There are many oscillation criteria for equation (1) obtained by an application of essentially one of three main methods:

(i) using comparison with a half-linear second order equation based on *apriori bound* for the quotient $x(t)/z(t)$ (see e.g. [2, 8, 18]),

(ii) using the *comparison method* (comparison with linear first order delay differential equation, see e.g. [4–7, 9, 11, 13, 15]) and

(iii) using the *Riccati type substitution* (see e.g. [10, 14, 17]).

The underlying principle of all methods is to replace the equation (1) by a simpler object. The special focus is usually devoted to the second term, since the most disturbing thing in equation (1) is the presence of both $z(t)$ and $x(t)$ in one equation and it is easier to eliminate x than z.

It is typical that all new criteria published in the literature are compared against the criteria obtained by the same method. The aim of this paper is to collect available methods which can be used to derive oscillation criteria for equation (1) and compare them together. On a test of Euler type equation we will show that one of the methods produces sharper results for small delays and another one for large delays. We will also modify the currently used approach in (i) and derive a result which is based on the comparison with second order delay differential equation rather than second order ordinary differential equation.

We will use the following assumptions on the coefficients and parameters. The coefficients r and b are subject of usual conditions $r \in C^1([t_0, \infty), \mathbb{R}^+)$, $b \in C^1([t_0, \infty), \mathbb{R}_0^+)$ and the coefficient c is positive $c \in C([t_0, \infty), \mathbb{R}^+)$. Further we suppose that the deviating arguments are unbounded, increasing and sufficiently smooth functions which satisfy the commutative law: $\tau \in C^2([t_0, \infty), \mathbb{R})$, $\tau'(t) > 0$, $\lim_{t\to\infty} \tau(t) = \infty$, $\sigma \in C^1([t_0, \infty), \mathbb{R})$, $\sigma'(t) > 0$, $\lim_{t\to\infty} \sigma(t) = \infty$ and $\sigma(\tau(t)) = \tau(\sigma(t))$ (for the weaker assumption $\sigma(\tau(t)) \geq \tau(\sigma(t))$ see [10]). Depending on the method which will be used to handle the problem, we will use some additional conditions, such as $b(t) \leq b_0$ or $b(t) < 1$. For simplicity we will also suppose that both σ and τ are delays and that the retardation in the differential term is smaller than the retardation in the second term, i.e. we will suppose $\sigma(t) \leq \tau(t) \leq t$.

In the paper the number $q = \frac{p}{p-1}$ is the conjugate number to the number p. Given a positive parameter φ further define the function Q as minimum of $c(t)$ and $\varphi c(\tau(t))$

$$Q(t;\varphi) = \min\{c(t), \varphi c(\tau(t))\} \tag{2}$$

Finally, we will use the assumption

$$\int^{\infty} r^{1-q}(t)\,\mathrm{d}t = \infty \tag{3}$$

which (together with nonnegativity of $c(t)$) ensures that all eventually positive solutions satisfy $z'(t) > 0$ eventually, see Lemma 1 below. Thus there is only one family of solutions which has to be eliminated in order to ensure oscillation. Note that the opposite case to (3) is also handled frequently in the literature and since in that case we have to eliminate two types of solutions (with $z'(t) > 0$ and $z'(t) < 0$ eventually), the resulting oscillation criteria consist of two (similar but relatively independent) conditions – each condition eliminates one of the families.

2 Monotonocity lemma

The following lemma ensures that the eventually positive solutions satisfy $z'(t) > 0$ for large t. The proof utilizes the condition (3) and can be found in many papers dealing with oscillation criteria for (1), see e.g. the first part of the proof of [10, Theorem 1].

Lemma 1. *If $x(t)$ is an eventually nonoscillatory solution of* (1), *then the corresponding function* $z(t) = x(t) + b(t)x(\tau(t))$ *satisfies*

$$z(t) > 0, \quad z'(t) > 0, \quad \left(r(t)\Phi(z'(t))\right)' < 0$$

eventually.

3 Method of apriori bound

The first idea how to remove the simultaneous presence of both $x(t)$ and $z(t)$ in (1) is to replace $x(\sigma(t))$ by a lower bound in term of $z(\sigma(t))$.

If $x(t)$ is eventually nonegative, then $x(\tau(t))$ is also eventually nonegative and since $z(t)$ is increasing and clearly $z(t) \geq x(t)$, we have

$$z(t) = x(t) + b(t)x(\tau(t)) \leq x(t) + b(t)z(\tau(t)) \leq x(t) + b(t)z(t).$$

Thus if $b(t) < 1$, the resulting inequality can be utilized to obtain the apriori bound for the quotient $x(t)/z(t)$

$$0 < 1 - b(t) \leq \frac{x(t)}{z(t)}. \tag{4}$$

This approach is summarized in the following lemma. The special case $\eta(t) = \sigma(t)$ has been used in most papers dealing with the method of apriori bound. The idea to replace σ by a smaller function η (and thus increase the delay $t - \sigma(t)$) is new but very simple and allows to improve results obtained by an application of a the so-called Myshkis-type criterion from Theorem 1 below.

Lemma 2. *Let $\eta(t)$ be continuous function such that $\eta(t) \leq \sigma(t)$ and $\lim_{t\to\infty} \eta(t) = \infty$. Suppose that*

$$b(t) < 1 \tag{5}$$

holds eventually and suppose that the inequality

$$\left[r(t)\Phi(z'(t))\right]' + c(t)\left(1 - b(\sigma(t))\right)^{p-1}\Phi(z(\eta(t))) \leq 0 \tag{6}$$

does not have an eventually positive solution. Then (1) *is oscillatory.*

Proof. It is sufficient to show that if $x(t)$ is an eventually positive solution of (1), then the corresponding function $z(t)$ satisfies (6) eventually. If $\eta \equiv \sigma$, then (6) follows immediately from (1) and inequality (4). The fact that $\sigma(t)$ can be replaced by $\eta(t) \leq \sigma(t)$ follows from the monotonicity of the function $z(t)$, see Lemma 1. □

Inequality (6) is the second order delay differential inequality which does not contain delay in the differential term and is not neutral anymore. If $p = 2$, then this inequality is handled by the following theorem of Koplatadze.

Theorem 1 ([12, Theorem 2]). *Let*

$$\liminf_{t\to\infty} \int_{\eta(t)}^{t} \eta(s)c(s)\mathrm{d}s > \frac{1}{e}.$$

Then the inequality

$$x''(t)\operatorname{sgn} x(\eta(t)) + c(t)|x(\eta(t))| \le 0$$

is oscillatory.

Note that the constant $\frac{1}{e}$ is optimal and cannot be improved in general, but in some particular cases a refinement of Koplatadze's results is possible, see [16]. As a consequence of Theorem 1, equation

$$x''(t) + \frac{\beta}{t^2}x(\lambda t) = 0$$

is oscillatory if

$$\beta > \frac{1}{e\lambda \ln \frac{1}{\lambda}} \tag{7}$$

holds. You may see that the right hand side of this inequality becomes unbounded if λ approaches 1. This undesired effect can be eliminated by suitable choice of the function η from Lemma 2, see (24) below.

Theorem 1 cannot be applied in the general case $p \ge 2$. In this case it is possible to use another apriory bound

$$k\frac{\sigma(t)}{t} \le \frac{z(\sigma(t))}{z(t)} \quad \text{for arbitrary } k \in (0,1) \text{ and large } t \tag{8}$$

or

$$\frac{\sigma(t)}{t} \le \frac{z(\sigma(t))}{z(t)} \quad \text{for large } t \tag{9}$$

which holds if

$$\int^{\infty} c(s)\Bigl(1 - b(\sigma(s))\Bigr)^{p-1}(\sigma(s))^{p-1}\,\mathrm{d}s = \infty, \tag{10}$$

see [8, Theorem 13] for details.

Thus (6) does not have an eventually positive solution if the corresponding equation is oscillatory which is true if either

$$\Bigl[r(t)\Phi(z'(t))\Bigr]' + kc(t)\Bigl(1 - b(\sigma(t))\Bigr)^{p-1}\left(\frac{\sigma(t)}{t}\right)^{p-1}\Phi(z(t)) = 0 \tag{11}$$

is oscillatory for some $k \in (0,1)$, or if (10) holds and

$$\Bigl[r(t)\Phi(z'(t))\Bigr]' + c(t)\Bigl(1 - b(\sigma(t))\Bigr)^{p-1}\left(\frac{\sigma(t)}{t}\right)^{p-1}\Phi(z(t)) = 0 \tag{12}$$

is oscillatory.

Various oscillation criteria for half-linear ordinary differential equations (12) and (11) are in details discussed in the book [3] which covers main directions in oscillation theory of the half-linear differential equations.

4 Comparison method

The main idea of the comparison method is to compare equation (1) with certain first order delay differential inequality. Consider the original equation (1) and the same equation shifted from t to $\tau(t)$. This gives in the second term expressions involving $x(\sigma(t))$ and $x(\sigma(\tau(t)))$ which can be combined into $z(\sigma(t))$. This allows to eliminate $x(t)$ in equation (1) and introduce $z(t)$ instead. More precisely, we consider the equation (1) and the equation

$$\frac{1}{\tau'(t)}\Big[r(\tau(t))\Phi(z'(\tau(t)))\Big]' + c(\tau(t))\Phi(x(\sigma(\tau(t)))) = 0 \tag{13}$$

which arises from (1) by shifting from t to $\tau(t)$. Then we take a suitable linear combination of both equations and use a series of estimates which allow to compare the resulting equation with certain first order linear differential inequality. Note that the main steps to accomplish the desired result are inequality

$$c(t)x^{p-1}(\sigma(t)) + c(\tau(t))b_0^{p-1}x^{p-1}(\sigma(\tau(t))) \geq \min\Big\{c(t), c(\tau(t))\Big\}\Big(x^{p-1}(\sigma(t)) + b_0^{p-1}x^{p-1}(\sigma(\tau(t)))\Big) \tag{14}$$

followed by the assumption on commutativity between τ and σ and by inequality

$$x^{p-1}(\sigma(t)) + b_0^{p-1}x^{p-1}(\tau(\sigma(t))) \geq 2^{p-2}\Big(x(\sigma(t)) + b_0 x(\tau(\sigma(t)))\Big)^{p-1} \geq 2^{p-2}z^{p-1}(\sigma(t)), \tag{15}$$

where b_0 is a constant upper bound of the function $b(t)$.

A closer examination of the published results shows, that these inequalities are in some sense weak points of the comparison method and posses some improvement. See [10] for detailed discussion and also for more general version of these inequalities and see also [8] for application of these ideas to the equation (1) and the comparison method. One of the main results of [8] states the following.

Theorem 2 ([8, Corollary (9), statement (ii)]). *Suppose that there exists a number b_0 such that $b(t) \leq b_0$. Equation (1) is oscillatory if there exists a number $\varphi > 0$ and a function $\eta(t)$ satisfying $\eta(t) \leq \sigma(t)$ and $\lim_{t\to\infty}\eta(t) = \infty$ such that $\eta(t) < \tau(t) \leq t$ and for every T there exists t_1 such that*

$$\liminf_{t\to\infty}\int_{\tau^{-1}(\eta(t))}^{t} Q_\eta^*(s;\varphi,t_1)\,\mathrm{d}s > \frac{1}{e}\left(1 + \left(\frac{\varphi}{\tau_0}\right)^{q-1} b_0\right)^{p-1}, \tag{16}$$

where

$$Q_\eta^*(t;\varphi,t_1) := Q(t;\varphi)\left[\int_{t_1}^{\eta(t)} r^{1-q}(s)\,\mathrm{d}s\right]^{p-1}.$$

Observe the presence of the function $\eta(t)$ in Theorem 2. An absolute majority of the recent papers does not consider function like η in the comparison method and deal just with the special case $\eta(t) = \sigma(t)$. However, the influence of $\eta(t)$ to the value of the limes inferior on the left hand side of the inequality (16) is ambivalent: bigger η gives bigger Q^* and thus has a positive influence on the condition (16), but bigger η also makes the interval of integration in (16) shorter and this has a negative influence on the condition (16). As a consequence, some of the oscillation criteria obtained by comparison method which produce poor results if $\sigma(t)$ is close to t or $\tau(t)$ can be improved by a suitable choice of $\eta(t) < \sigma(t)$, see the examples with some discussion in [10, Example 2 and Figure 1], [9, Example 10 and Figure 2]. As far as the author knows, the only papers dealing with comparison method which consider the η-like function are [8, 15].

5 Riccati equation method

The Riccati equation method is based on the Riccati type substitution

$$\omega(t) = \rho(t)\frac{r(t)(z'(t))^{p-1}}{z^{p-1}(\sigma(t))}. \tag{17}$$

which implies

$$\omega'(t) = \rho'(t)\frac{r(t)(z'(t))^{p-1}}{z^{p-1}(\sigma(t))} + \rho(t)\frac{\left(r(t)(z'(t))^{p-1}\right)'}{z^{p-1}(\sigma(t))} - (p-1)\rho(t)\frac{r(t)(z'(t))^{p-1}z'(\sigma(t))\sigma'(t)}{z^{p}(\sigma(t))}.$$

From $\sigma(t) \leq t$ and from the monotonicity of $r(t)\Phi(z'(t))$ we have

$$z'(\sigma(t)) \geq \left(\frac{r(t)}{r(\sigma(t))}\right)^{q-1} z'(t)$$

and combining these computations with (1) we get

$$\omega'(t) - \frac{\rho'(t)}{\rho(t)}\omega(t) + \frac{(p-1)\sigma'(t)}{\rho^{q-1}(t)r^{q-1}(\sigma(t))}\omega^{q}(t) \leq -\rho(t)\frac{c(t)x^{p-1}(\sigma(t))}{z^{p-1}(\sigma(t))}. \tag{18}$$

Note that to perform these steps it is necessary to suppose differentiability of $\sigma(t)$ and $\sigma(t) \leq t$.

Together with this computation we derive a variant of the last inequality which contains $x^{p-1}(\tau(\sigma(t)))$ instead of $x^{p-1}(\sigma(t))$. Then we will be able to use the same inequalities as in the comparison method to combine $x(\sigma(t))$ and $x(\tau(\sigma(t)))$ into $z(\sigma(t))$. To accomplish this task we define

$$v(t) = \rho(t)\frac{r(\tau(t))\left(z'(\tau(t))\right)^{p-1}}{z^{p-1}(\sigma(t))}, \tag{19}$$

use the obvious fact $v(t) > 0$ and differentiate

$$v'(t) = \rho'(t)\frac{r(\tau(t))(z'(\tau(t)))^{p-1}}{z^{p-1}(\sigma(t))} + \rho(t)\frac{\left(r(\tau(t))(z'(\tau(t)))^{p-1}\right)'}{z^{p-1}(\sigma(t))} - (p-1)\rho(t)\frac{r(\tau(t))(z'(\tau(t)))^{p-1}z'(\sigma(t))\sigma'(t)}{z^{p}(\sigma(t))}.$$

Using the monotonicity of $r(t)\Phi(z'(t))$ and $\sigma(t) \leq \tau(t)$ we have

$$z'(\sigma(t)) \geq \left(\frac{r(\tau(t))}{r(\sigma(t))}\right)^{q-1} z'(\tau(t))$$

and hence from the above computations and from (13) we get

$$v'(t) - \frac{\rho'(t)}{\rho(t)}v(t) + \frac{(p-1)\sigma'(t)}{\rho^{q-1}(t)r^{q-1}(\sigma(t))}v^{q}(t) \leq -\rho(t)\frac{\tau'(t)c(\tau(t))x^{p-1}(\sigma(\tau(t)))}{z^{p-1}(\sigma(t))}. \tag{20}$$

Note that these steps require $\sigma(t) \leq \tau(t)$ and differentiability of both $\tau(t)$ and $\sigma(t)$.

Let $l > 1$ and $l^* = l/(l-1) > 1$ be mutually conjugate numbers. Using linear combination of (18), (20) with coefficients l^{p-2}, $(l^*)^{p-2}$ and using essentially the same estimates as in the comparison method we can obtain inequality

$$\begin{aligned} l^{p-2}\omega'(t) + (l^*)^{p-2}\frac{\left[b(\sigma(t))\right]^{p-1}\varphi}{\tau'(t)}v'(t) - l^{p-2}\left[\frac{\rho'(t)}{\rho(t)}\omega(t) - \frac{(p-1)\sigma'(t)}{\rho^{q-1}(t)r^{q-1}(\sigma(t))}\omega^{q}(t)\right] \\ - (l^*)^{p-2}\frac{\left[b(\sigma(t))\right]^{p-1}\varphi}{\tau'(t)}\left[\frac{\rho'(t)}{\rho(t)}v(t) - \frac{(p-1)\sigma'(t)}{\rho^{q-1}(t)r^{q-1}(\sigma(t))}v^{q}(t)\right] \leq -\rho(t)Q(t) \end{aligned}$$

which can be studied using just slight modifications of classical methods. You can see [10, Theorem 1] for details.

From the technical point of view, the problem is much simpler if we have a constant upper bound for the expressions $b(t)$ and $\frac{1}{\tau'(t)}$, see [10, Corollary 1]. The following theorem is a variant of [10, Corollary 1] which is more suitable for comparison with other methods.

Theorem 3. *Suppose that* (3), $\sigma(t) \le t$ *and* $\sigma(t) \le \tau(t)$ *are satisfied and there exist constants* $b_0 \ge 0$ *and* $\tau_0 > 0$ *such that* $b(t) \le b_0 < \infty$ *and* $\tau'(t) \ge \tau_0$. *If there exist a positive number* φ *and a positive function* $\rho(t)$ *such that*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \rho(s)Q(s) - \frac{1}{p^p}\frac{\rho(s)r(\sigma(s))}{(\sigma'(s))^{p-1}}\left(1+\frac{b_0\varphi^{q-1}}{\tau_0^{q-1}}\right)^{p-1}\left(\frac{\rho'(s)}{\rho(s)}\right)_+^p \mathrm{d}s = \infty, \tag{21}$$

then (1) *is oscillatory.*

Proof. Follows from [10, Corollary 1], Really, taking constant function $\varphi(t)$ in the condition (20) of [10, Corollary 1] we get

$$\limsup_{t\to\infty} \int_{t_0}^{t} \rho(s)Q(s) - \frac{1}{p^p}\frac{\rho(s)r(\sigma(s))}{(\sigma'(s))^{p-1}}\left[l^{p-2}+(l^*)^{p-2}\frac{b_0^{p-1}\varphi}{\tau_0}\right]\left(\frac{\rho'(s)}{\rho(s)}\right)_+^p \mathrm{d}s = \infty, \tag{22}$$

(where l, l^* are mutually conjugate numbers) as a sufficient condition for oscillation of (1). Taking $l = 1+\frac{b_0\varphi^{q-1}}{\tau_0^{q-1}}$ and $l^* = \frac{l}{l-1} = 1+\frac{\tau_0^{q-1}}{b_0\varphi^{q-1}}$ (which gives a minimum for the function inside brackets with respect to l variable, see [8, Lemma 1]) we see that (22) takes the form (21). □

6 Comparison across available methods for Euler type equation

Let us test the strength of the above introduced methods on an example of Euler type equation. This equation is suitable for testing oscillation criteria, since it is conditionally oscillatory. As a well known particular case,

$$x'' + \frac{\gamma}{t^2}x = 0$$

is oscillatory if and only if $\gamma > \frac{1}{4}$.

Now let us consider the half-linear extension of Euler equation with proportional delay and with neutral term also with proportional delay, i.e. we consider equation in the form

$$\left(\Phi(z'(t))\right)' + \frac{\beta}{t^p}\Phi(x(\lambda_2 t)) = 0 \tag{23}$$

where

$$z(t) = x(t) + b_0 x(\lambda_1 t),$$

and $\lambda_2 < \lambda_1 < 1$. Hence $\sigma(t) = \lambda_2 t$, $\sigma'(t) = \lambda_2$, $\tau(t) = \lambda_1 t$, $\tau'(t) = \lambda_1$, $\tau_0 = \lambda_1$, $c(t) = \frac{\beta}{t^p}$, $c(\tau(t)) = \frac{\beta}{\lambda_1^p t^p}$. We choose the parameter φ such that $c(\tau) = \varphi c(\tau(t))$ and thus we loose nothing when taking minimum in (2). Consequently, $\varphi = \lambda_1^p$ and $Q(t;\varphi) = \frac{\beta}{t^p}$.

Method 1: Apriori bound. Using method of apriori bound we compare the equation with second order ordinary differential equation. As a result of this comparison, equation (23) is oscillatory if

$$\left[\Phi(z'(t))\right]' + k\frac{\beta}{t^p}\left(1-b_0\right)^{p-1}\lambda_2^{p-1}\Phi(z(t)) = 0$$

is oscillatory for some $k \in (0,1)$, which is true if

$$\beta > \left(\frac{p-1}{p}\right)^p \frac{1}{\lambda_2^{p-1}(1-b_0)^{p-1}}.$$

Method1b: Apriori bound in the linear case. If $p = 2$ and the equation is linear, we can also utilize Lemma 2 and Theorem 1. Thus

$$z''(t) + \frac{\beta}{t^2}x(\lambda_2 t) = 0$$

is oscillatory if

$$z''(t) + \frac{\beta}{t^2}(1-b_0)z(\lambda t) = 0 \quad \text{for some } \lambda \le \lambda_2$$

is oscillatory which is guaranteed (see (7)) by the condition

$$\beta > \min_{\lambda\in(0,\lambda_2]} \frac{1}{(1-b_0)e\lambda \ln\frac{1}{\lambda}}. \tag{24}$$

Method 2: Riccati method. Equation (23) has been examined in [10, Example 1] and it turns out that (1) is oscillatory if

$$\beta > \left(\frac{p-1}{p}\right)^p \frac{(1+b_0\lambda_1)^{p-1}}{\lambda_2^{p-1}}. \tag{25}$$

This result has been obtained using Riccati equation method from Theorem 3 with the choice $\rho(t) = t^{p-1}$. Note that this example shows, that an improvement of the classical approach based on (14) and (15) has to be revisited, if we wish to obtain the sharp oscillation constant in the limiting case when the delays shrink to zero and equation (1) becomes second order ordinary differential equation. Note also that the fact that (25) sharp in some sense can be verified by the fact that if we formally put $\lambda_1 = \lambda_2 = 1$ and $b_0 = 0$ in (25), the condition (25) gives

$$\beta > \left(\frac{p-1}{p}\right)^p$$

which is known to be sharp and nonimprovable oscillation condition for (23) if $\lambda_1 = \lambda_2 = 1$ and $b_0 = 0$.

Method 3: Comparison method (comparison with first order equation). Let us investigate equation (23) from the point of view of the comparison method, i.e. we will use Theorem 2. The choice $\eta(t) = \lambda t$, $\lambda \le \lambda_2$ and direct computation shows

$$Q^*(t;\varphi,t_1) = \frac{\beta}{t^p}\left[\lambda t - t_1\right]^{p-1}$$

and

$$\int_{\tau^{-1}\eta(t)}^t Q^*(t;\varphi,t_1)\,\mathrm{d}t = \int_{\frac{\lambda}{\lambda_1}t}^t \frac{\beta}{t}\left[\lambda - \frac{t_1}{t}\right]^{p-1} \mathrm{d}t > \beta\lambda^{p-1}\ln\frac{\lambda_1}{\lambda}.$$

Thus the comparison method gives the result that equation (1) is oscillatory if

$$\beta > \frac{1}{e}(1+b_0\lambda_1)^{p-1} \min_{\lambda\in(0,\lambda_2]} \frac{1}{\lambda^{p-1}} \frac{1}{\ln\frac{\lambda_1}{\lambda}}.$$

Existence of nonoscillatory solution. Motivated by the classical ordinary differential equation we may test the conditions under which equation (23) has a nonoscillatory solution $x(t) = t^\alpha$ for some α. Direct substitution into (23) reveals, that (23) has a nonoscillatory solution if α satisfies

$$\alpha^{p-1}(\alpha-1)(p-1) + \beta\Phi\left(\frac{\lambda_2^\alpha}{1+b_0\lambda_1^\alpha}\right) = 0.$$

Given p, λ_1, λ_2 and b_0 we may look for maximal value of β for which this equation is satisfied for some $\alpha \in \mathbb{R}$. Thus (23) is not oscillatory if

$$\beta \leq \max_{\alpha\in(0,1)} \alpha^{p-1}(1-\alpha)(p-1)\Phi\left(\frac{1+b_0\lambda_1^\alpha}{\lambda_2^\alpha}\right).$$

On the following graphs we illustrate these bounds as functions of λ_2. In order to allow both variants of method of apriori bound, we will consider the linear case $p = 2$. Thus Methods 1 and 1b reffer to the method of apriory bound with comparison to second order ordinary and delay differential equation, respectively. Further Methods 2 and 3 reffer to the method of Riccati equation and comparison method. The dotted curve which branches off the curve for Method 3 denotes the result of comparison method without considering the case $\eta \neq \sigma$ in Theorem 2, i.e. without the min operator in (24), which is often considered in the literature.

Observe, that if b_0 is small, then estimate based on Method 1 is reasonably good, but becomes worse as b_0 increases. Note also that we obtain better results from Riccati method if λ_2 is large. On the contrary, if λ_2 is small, better bound for the oscillation constant β can be obtained from the comparison method. Note also that the curves for Methods 1b and 3 end up with constant parts, the other curves are decreasing.

The following three pictures demonstrate the fact that no method beats the other ones and that the mutual relationship of the obtained curves is rich. However these pictures (as well as exploring more similar portaits) give the general impression that

- apriory method (Methods 1 and 1b) wastes of b_0 is not small enough,
- for large delay (large difference $t-\sigma(t)$, i.e. $\sigma(t) \ll t$) it is more convenient to use the methods based on Myshkis-type oscillation criteria for delay differential equations, i.e. Method 1b (if b_0 is small) and Method 3
- for small delay (small difference $t-\sigma(t)$) it is more convenient to use the methods based on oscillation criteria for ordinary differential equations, i.e. Method 1 (if b_0 is small) and Method 2.

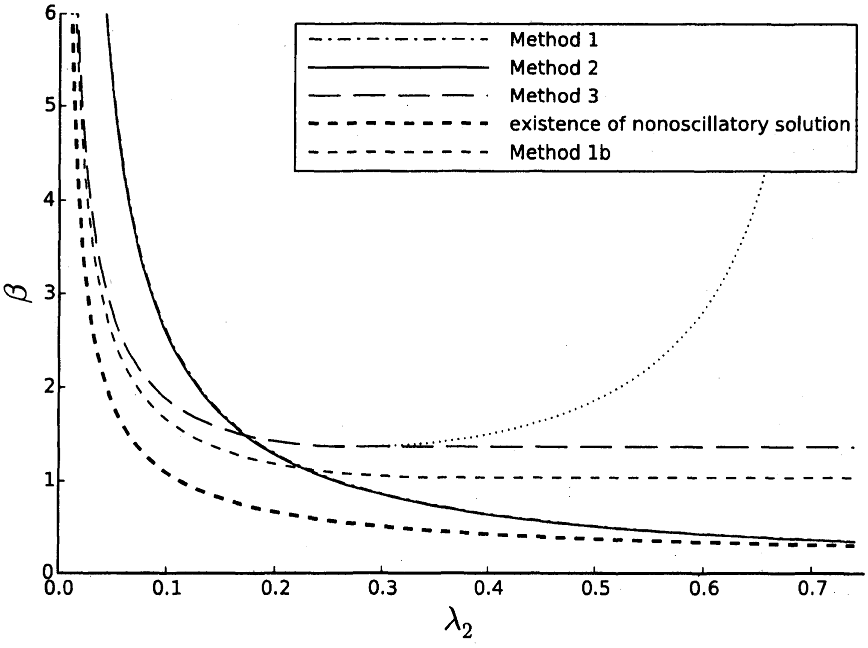

Figure 1: $b_0 = 0.1$, $\lambda_1 = 0.75$, $p = 2$, Method 1 almost coincides with Method 2.

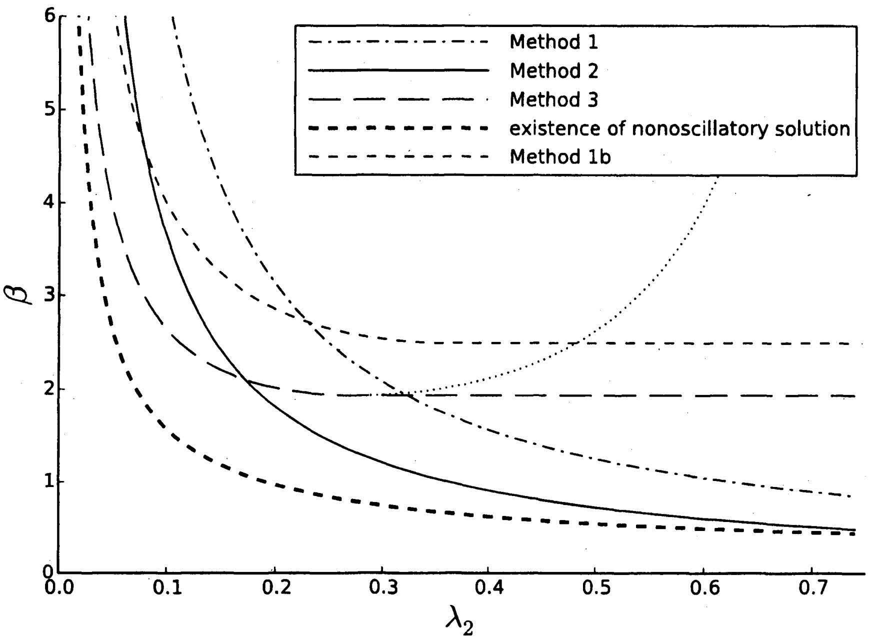

Figure 2: $b_0 = 0.6$, $\lambda_1 = 0.75$, $p = 2$

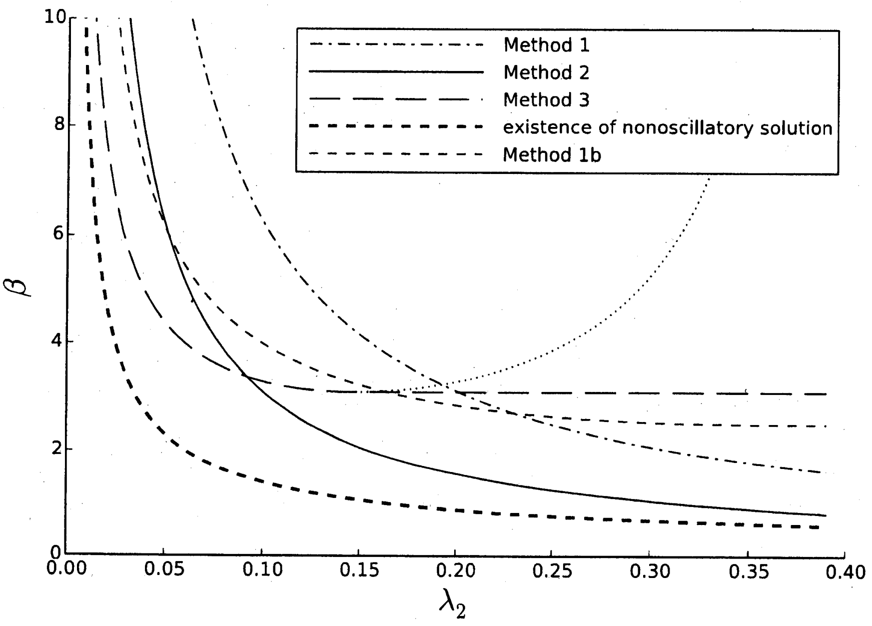

Figure 3: $b_0 = 0.6$, $\lambda_1 = 0.4$, $p = 2$

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