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Kyoto University
On the asymptotic expansion of the Kashaev invariant and the twisted Reidemeister torsion of two-bridge knots

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1 Introduction

This note is a survey of the joint work [8] with Tomotada Ohtsuki. In [2, 3], Kashaev defined the Kashaev invariant \( \langle L \rangle_N \in \mathbb{C} \) for a link \( L \) for \( N = 2, 3, \cdots \) by using the quantum dilogarithm at \( q = e^{2\pi \sqrt{-1}/N} \). In [4], he conjectured that, for any hyperbolic link \( L \), \( \frac{2\pi}{N} \log |\langle L \rangle_N| \) goes to the hyperbolic volume of \( S^3 - L \) as \( N \to \infty \). In [6], Ohtsuki proposed the following refined conjecture:

**Conjecture 1 ([6]).** For any hyperbolic knot \( K \), the asymptotic expansions of the Kashaev invariant of \( K \) is presented by the following form,

\[
\langle K \rangle_N = e^{N \zeta(K)} N^{3/2} \omega(K) \cdot \left( 1 + \sum_{i=1}^{d} \kappa_i(K) \cdot \left( \frac{2\pi \sqrt{-1}}{N} \right)^i + O\left( \frac{1}{N^{d+1}} \right) \right),
\]

(1)

for any \( d \), where \( \omega(K) \) and \( \kappa_i(K) \)'s are some scalars only depending on \( K \). Here \( \zeta(K) = \frac{1}{2\pi \sqrt{-1}} (cs(S^3 - K) + \sqrt{-1} \vol(S^3 - K)) \), where “\( cs \)” and “\( \vol \)” denote the Chern-Simons invariant and the hyperbolic volume.

It is shown in [6, 9, 7] that, for any hyperbolic knot \( K \) with up to 7 crossings, Conjecture 1 holds. Moreover, the following is conjectured for \( \omega(K) \) of (1):

**Conjecture 2.** For any hyperbolic knot \( K \),

\[
2\pi \sqrt{-1} \omega(K)^2 = \pm \tau(K),
\]

where \( \tau(K) \) is the twisted Reidemeister torsion associated with the holonomy representation of the hyperbolic structure of the complement of \( K \).

For the figure-eight knot, this conjecture was shown by Andersen and Hansen [1] and H. Murakami [5]. We show

**Theorem 1 ([8]).** For any hyperbolic knot \( K \) with up to 7 crossings, Conjecture 2 holds.
2 Results

Let us review a parameterized knot diagram of an open knot, where an open knot is a 1-tangle whose closure is a knot. We parameterize edges of an open knot diagram by parameters in $\mathbb{C} \cup \{\infty\}$. We parameterize edges adjacent to unbounded regions by 1. We parameterize edges next to the terminal edges by 0 or $\infty$; we parameterize such an edge by $\infty$ (resp. 0) if it is connected to the terminal edge by an under-path (resp. an over-path). We parameterize the other edges in such a way that the parameters belong to $\mathbb{C} - \{0\}$, and satisfy the hyperbolicity equations

$$
\frac{u'}{u} x \frac{v'}{v} = \left(1 - \frac{x}{u}\right) \left(1 - \frac{v'}{x}\right) = \left(1 - \frac{x}{u}\right) \left(1 - \frac{v}{x}\right).
$$

We consider a hyperbolic two-bridge knot $K$. Any open two-bridge knot can be presented by a plat closure of a 3-braid of a product of copies of $\sigma_1$ and $\sigma_2^{-1}$, i.e., any open two-bridge knot diagram $D$ (or its mirror image) can be obtained by gluing copies of the following tangle diagrams, which we call elementary diagrams.

From the hyperbolicity equations among parameters of the resulting tangle diagram, the values of $x_i$ are recursively determined by

$$
x_{i+1} = \begin{cases} 
x_i + 1 - \frac{x_i}{x_{i-1}} & \text{if the strand of } x_i \text{ is between } \sigma_1 \text{ and } \sigma_1, \\
x_i + \frac{(x_i - 1)^2}{1 - \frac{x_i}{x_{i+1}}} & \text{otherwise.}
\end{cases}
$$

It is known that a hyperbolic structure of the complement of $K$ is obtained from a parametrized diagram ([11], [13]). Calculating the monodromy representation, from the definition of $\tau(K)$, we can obtain a reformulation of $\tau(K)$. Explicitly, we define $\Phi(D)$ to be the composition of $\Phi$ of elementary diagrams whose values are given as follows,

$$
\Phi\left( \begin{array}{c} \infty \\ x_1 \\ \end{array} \right) = x_1(x_1-1) \begin{pmatrix} 1 & 2x_1 & 0 \end{pmatrix},
$$

$$
\Phi\left( \begin{array}{c} x_i \\ x_{i+1} \\ \end{array} \right) = x_{i+1} \begin{pmatrix} 1 & 2x_{i+1} & 1 \\ 0 & -x_{i+1} & -1 \\ 0 & 0 & 1 \end{pmatrix},
$$

$$
\Phi\left( \begin{array}{c} x_i \\ x_{i+1} \\ \end{array} \right) = x_{i+1} \begin{pmatrix} 1 & 0 & 0 \\ -1 & -x_{i+1} & 0 \\ 1 & 2x_{i+1} & 1 \end{pmatrix}.
$$
\[
\Phi \left( \begin{array}{ccc}
1 & x_{m-1} & 1 \\
0 & 0 & 1
\end{array} \right) = \frac{x_{m-1}^3}{(x_{m-1} - 1)^3} \left( \begin{array}{cc}
1 & -1 \\
2 & 2
\end{array} \right),
\]
\[
\Phi \left( \begin{array}{ccc}
1 & x_{m-1} & 1 \\
1 & 0 & 1
\end{array} \right) = \frac{x_{m-1}^3}{(x_{m-1} - 1)^3} \left( \begin{array}{cc}
2 & -1 \\
1 & 1
\end{array} \right).
\]

Then, we have that \( \frac{2}{\tau(K)} = \Phi(D) \).

Let us review the definition of the Kashaev invariant. Let \( K \) be an oriented knot and \( N \geq 2 \). We put \( q = \exp(2\pi \sqrt{-1}/N) \), \( (x)_n = (1 - x)(1 - x^2) \cdots (1 - x^n) \) and \( \mathcal{N} = \{0, 1, \ldots, N - 1\} \).

For \( i, j, k, l \in \mathcal{N} \), we put
\[
R_{kl}^{ij} = \frac{Nq^{-\frac{1}{2}+i-k}\theta_{kl}^{ij}}{(q)_{[i-j]}(\overline{q})_{[j-l]}(q)_{[l-k-1]}(\overline{q})_{[k-i]}},
\]
\[
\overline{R}_{kl}^{ij} = \frac{Nq^{\frac{1}{2}+j-l}\theta_{kl}^{ij}}{(\overline{q})_{[i-j]}(q)_{[j-l]}(\overline{q})_{[l-k-1]}(q)_{[k-i]}},
\]
where \([m] \in \mathcal{N}\) is the residue of \( m \) modulo \( N \), and we put
\[
\theta_{kl}^{ij} = \begin{cases} 
1 & \text{if } [i - j] + [j - l] + [l - k - 1] + [k - i] = N - 1, \\
0 & \text{otherwise}.
\end{cases}
\]

Let \( D \) be an 1-tangle diagram of an open knot whose closure is the knot \( K \). A labeling is an assignment of an element of \( \mathcal{N} \) to each edge of \( D \). We define the weights of labeled elementary tangle diagrams by
\[
W \left( \begin{array}{c}
i \\
k
\end{array} \right) = R_{kl}^{ij}, 
W \left( \begin{array}{c}
\cap \\
k
\end{array} \right) = q^{-1/2}\delta_{k,l-1}, 
W \left( \begin{array}{c}
\cup \\
k
\end{array} \right) = \delta_{k,l},
\]
\[
W \left( \begin{array}{c}
i \\
l
\end{array} \right) = \overline{R}_{kl}^{ij}, 
W \left( \begin{array}{c}
\cap \\
l
\end{array} \right) = q^{1/2}\delta_{i,j+1}, 
W \left( \begin{array}{c}
\cup \\
l
\end{array} \right) = \delta_{i,j}.
\]

Then, the Kashaev invariant \( \langle K \rangle_N \) of \( K \) is defined by
\[
\langle K \rangle_N = \sum_{\text{labelings}} \prod_{\text{crossings of } D} W(\text{crossings}) \prod_{\text{critical points of } D} W(\text{critical points}) \in \mathbb{C}.
\]

We define the potential function for an open knot diagram parametrized by hyperbolicity parameters. We consider an angle consisting of two adjacent edges at a crossing, and associate such an angle with the value
\[
\xymatrix@R=10pt@C=10pt{ \times^x & \times^y \ar[ll]^x \ar[rr]^y & \\
\times^y & Li_2 \left( \frac{x}{y} \right) - Li_2(1) \ar[ll]^x \ar[rr]^y & \\
\times^x & Li_2(1) - Li_2 \left( \frac{y}{x} \right) \ar[ll]^x \ar[rr]^y & }
\]
where $\text{Li}_2(z) = -\int_0^z \frac{\log(1-t)}{t} \, dt$. We define the potential function $V$ to be the sum of such values for all angles except for the constant terms. We remark that the equations
\[
\frac{\partial}{\partial x_i} V = 0 \quad \text{for all } i
\]
give the hyperbolicity equations and so, a solution of the hyperbolicity equations gives a critical point of $V$. Furthermore, it is known that
\[
\log(q)_n \sim -\frac{N}{2\pi \sqrt{-1}} \text{Li}_2(e^{2\pi \sqrt{-1} \frac{n}{N}}),
\]
So, from the definition of the potential function, formally, we obtain the following approximation:
\[
\langle K \rangle_N \sim \sum_{i_1, \ldots, i_m} \exp \left( \frac{N}{2\pi \sqrt{-1}} V(e^{2\pi \sqrt{-1} i_1}, \ldots, e^{2\pi \sqrt{-1} i_m}) \right).
\]
Putting $\frac{i}{N} = t_1, \ldots, \frac{i_m}{N} = t_m$ and using the Poisson summation formula formally,
\[
\langle K \rangle_N \sim N^m \int \exp \left( \frac{N}{2\pi \sqrt{-1}} V(x_1, \ldots, x_m) \right) dx_1 \cdots dx_m.
\]
Moreover, putting $x_i = e^{2\pi \sqrt{-1} t_i}$, we obtain
\[
\langle K \rangle_N \sim N^m \int \exp \left( \frac{N}{2\pi \sqrt{-1}} V(x_1, \ldots, x_m) \right) dx_1 \cdots dx_m.
\]
By using the saddle point method formally and more calculations of the expansions, we obtain
\[
\langle K \rangle_N \sim e^{N \zeta(K)} \cdot N^{3/2} \cdot \omega(K),
\]
where $\zeta(K) = \frac{1}{2\pi \sqrt{-1}} V(x_{1;c}, \ldots, x_{m;c})$ for a critical point $(x_{1;c}, \ldots, x_{m;c})$ of $V$ and $\omega(K)$ can be written in terms of the Hessian of $V$ at the critical point $(x_{1;c}, \ldots, x_{m;c})$.

Moreover, we define $\Psi(D)$ to be the composition of $\Psi$ of elementary diagrams whose values are given as follows,
\[ \Psi \left( \begin{array}{c} 1 \\ x_{m-1} \\ 0 \\ 0 \\ 1 \end{array} \right) = \left( \frac{1}{1-x_{m-1}} \right) \frac{1}{1-x_{m-1}}, \psi \left( \begin{array}{c} 1 \\ x_{m-1} \\ 0 \\ 0 \\ 1 \end{array} \right) = \left( \frac{1}{x_{m-1}-1} \right) \frac{1}{1}. \]

Noting that \( \omega(K)^2 \) can be presented in terms of the Hessian of the potential function defined from a parametrized open diagram, it follows that \( \frac{1}{\sqrt{-1} \omega(K)^2} = \Psi(D) \).

Showing that the values of \( \Phi(D) \) and \( \Psi(D) \) satisfy the same recursion formula, we prove Theorem 1.

3 Example

In this section, we explain our results for the $5_2$ knot $K$, which is presented by the following diagram $D$:

From (2), the hyperbolicity equations are presented by

\[ (1-x_1)(1-\frac{1}{x_1}) = 1 - \frac{x_2}{x_1}, \quad (1-\frac{x_2}{x_1})(1-\frac{1}{x_2}) = 1 - x_2. \]

Hence,

\[ x_1^3 - 2x_1^2 + 3x_1 - 1 = 0. \]

Corresponding to the holonomy representation of the hyperbolic structure of the knot complement, we choose a solution

\[ x_1 = 0.784920145... + \sqrt{-1} \cdot 1.307141278... , \]

which gives the complex hyperbolic volume by

\[ \varsigma(K) = \frac{1}{2\pi\sqrt{-1}} V(x_1, x_2) = 0.450109610... - \sqrt{-1} \cdot 0.4813049796... . \]

Then, from the definition of \( \Phi(D) \), we obtain

\[ \frac{2}{\tau(K)} = x_1(x_1 - 1) \begin{pmatrix} 1 & 2x_1 & 0 \\ -1 & -x_2 & 0 \\ 1 & 2x_2 & 1 \end{pmatrix} \begin{pmatrix} x_2^3 \\ (x_2 - 1)^3 \\ 1 \end{pmatrix} \left( \frac{1}{2} \right), \quad (4) \]

\[ = -0.6323164993... + \sqrt{-1} \cdot 2.2345852998... , \quad (5) \]
and, hence, the value of the twisted Reidemeister torsion of $K$ is given by
\[ \tau(K) = -0.2344867659... - \sqrt{-1} \cdot 0.8286683659... . \] (6)

Let us confirm that the above value is also obtained from [12], by transforming the Reidemeister torsion associated with the longitude (of [12]) to the Reidemeister torsion associated with the meridian (the above value) as mentioned in [5].

The knot group $\pi_1(K)$ of $K$ is presented by $\pi_1(K) = \langle a, b \mid aw^2 = w^2b \rangle$, where $w = ab^{-1}a^{-1}b$. The meridian longitude system $(\mu, \lambda)$ is presented in $\pi_1(K)$ by
\[ \mu = a, \quad \lambda = (ab^{-1}a^{-1}b)^2(ba^{-1}b^{-1}a)^2. \]

A non-abelian representation $\rho: \pi_1(K) \rightarrow \text{SL}_2\mathbb{C}$ is parametrized by two parameters $u$ and $s$ as follows:
\[ \rho(a) = \begin{pmatrix} \sqrt{s} & 1 \\ 0 & \frac{1}{\sqrt{s}} \end{pmatrix}, \quad \rho(b) = \begin{pmatrix} \sqrt{s} & 1 \\ -\sqrt{s}u & \frac{1}{\sqrt{s}} \end{pmatrix}, \]
where $s$ and $u$ satisfies the Riley’s equation $\phi_K(s, u) = 0$. The Riley’s polynomial $\phi_K(s, u)$ [10] is given by
\[ \phi_K(s, u) = -\frac{1}{s^2}(-2s + 3s^2 - 2s^3 + u - 3su + 6s^2u - 3s^3u + s^4u - 2su^2 + 3s^2u^2 - 2s^3u^2 + s^2u^3). \]

The holonomy representation $\rho_0$ corresponds to the case $s = 1$ and $\phi_K(1, u) = 1 - 2u + u^2 - u^3$. By [12], the Reidemeister torsion $T_{\lambda}^{\rho_0}(K)$ associated with the longitude is given by
\[ T_{\lambda}^{\rho_0}(K) = -\frac{(2 + u)(2 + 7u)}{u^3(4 + u^2)}. \]

Let $l_{1,1}(s, u)$ be the $(1, 1)$-entry of $\rho(\lambda)$. As mentioned in [5], we can transform $T_{\lambda}^{\rho_0}(K)$ to the Reidemeister torsion $\tau(K)$ associated with the meridian by the formula
\[ \pm \tau(K) = 2 \left( \frac{\partial l_{1,1}}{\partial s} + \frac{\partial l_{1,1}}{\partial u} \frac{du}{ds} \right) \bigg|_{s=1} \frac{1}{T_{\lambda}^{\rho_0}(K)}. \]

Then, choosing the solution $u = 1 - x_1 = 0.21508 - \sqrt{-1} \cdot 1.30714$ of $\phi_K(1, u) = 0$ (see [8, Appendix D]), we obtain
\[ \pm \tau(K) = \frac{2u^4(2 + u^2)(4 + u^2)(2 + 4u^2 + u^4)}{(2 + u)(2 + 7u)} = -0.234487 - \sqrt{-1} \cdot 0.828668, \]
which coincides with (6). Moreover, from the definition of $\Psi(D)$,
\[ \frac{1}{\sqrt{-1} \omega(K)^2} = (1 - \frac{x_1}{x_1 - 1}) \cdot \frac{x_2}{x_1} \cdot \left( \frac{x_1(x_2 - 1)}{(x_1 - 1)x_2} \frac{1}{x_2} \frac{x_1 - 1}{x_2 - 1} \right) \cdot (\frac{1}{1 - x_2} 1) = -0.632316... + \sqrt{-1} \cdot 2.23459..., \]
which agrees with (5). On the other hand, in [6], it is rigorously shown that
\[ \langle K \rangle_N \sim e^{N \sigma(K)} \cdot N^{3/2} \cdot \omega(K). \] (7)

Hence, we confirm Conjecture 2 for $K$. 

References


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